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ЗАДАЧИ И УПРАЖНЕНИЯ ПО МАТЕМАТИЧЕСКОМУ АНАЛИЗУ

Под редакцией
Б. П. ДЕМИДОВИЧА

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PROBLEMS IN MATHEMATICAL ANALYSIS

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TO THE READER

MIR Publishers would be glad to have your opinion of the translation and the design of this book.

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PREFACE

This collection of problems and exercises in mathematical analysis covers the maximum requirements of general courses in higher mathematics for higher technical schools. It contains over 3,000 problems sequentially arranged in Chapters I to X covering all branches of higher mathematics (with the exception of analytical geometry) given in college courses. Particular attention is given to the most important sections of the course that require established skills (the finding of limits, differentiation techniques, the graphing of functions, integration techniques, the applications of definite integrals, series, the solution of differential equations).

Since some institutes have extended courses of mathematics, the authors have included problems on field theory, the Fourier method, and approximate calculations. Experience shows that the number of problems given in this book not only fully satisfies the requirements of the student, as far as practical mastering of the various sections of the course goes, but also enables the instructor to supply a varied choice of problems in each section and to select problems for tests and examinations.

Each chapter begins with a brief theoretical introduction that covers the basic definitions and formulas of that section of the course. Here the most important typical problems are worked out in full. We believe that this will greatly simplify the work of the student. Answers are given to all computational problems; one asterisk indicates that hints to the solution are given in the answers, two asterisks, that the solution is given. The problems are frequently illustrated by drawings.

This collection of problems is the result of many years of teaching higher mathematics in the technical schools of the Soviet Union. It includes, in addition to original problems and examples, a large number of commonly used problems.

Chapter I

INTRODUCTION TO ANALYSIS

Sec. 1. Functions

1°. **Real numbers.** Rational and irrational numbers are collectively known as *real numbers* The *absolute value* of a real number a is understood to be the nonnegative number $|a|$ defined by the conditions: $|a| = a$ if $a \geqslant 0$, and $|a| = -a$ if $a < 0$. The following inequality holds for all real numbers a and b:

$$|a+b| \leqslant |a| + |b|.$$

2°. **Definition of a function.** If to every value*) of a variable x, which belongs to some collection (set) E, there corresponds one and only one finite value of the quantity y, then y is said to be a *function* (single-valued) of x or a *dependent variable* defined on the set E. x is the *argument* or *independent variable* The fact that y is a function of x is expressed in brief form by the notation $y = f(x)$ or $y = F(x)$, and the like

If to every value of x belonging to some set E there corresponds one or several values of the variable y, then y is called a *multiple-valued function* of x defined on E. From now on we shall use the word "function" only in the meaning of a *single-valued* function, if not otherwise stated

3° **The domain of definition of a function.** The collection of values of x for which the given function is defined is called the *domain of definition* (or the *domain*) of this function. In the simplest cases, the domain of a function is either a *closed interval* $[a, b]$, which is the set of real numbers x that satisfy the inequalities $a \leqslant x \leqslant b$, or an *open interval* (a, b), which is the set of real numbers that satisfy the inequalities $a < x < b$. Also possible is a more complex structure of the domain of definition of a function (see, for instance, Problem 21)

Example 1. Determine the domain of definition of the function

$$y = \frac{1}{\sqrt{x^2-1}}.$$

Solution. The function is defined if

$$x^2 - 1 > 0,$$

that is, if $|x| > 1$. Thus, the domain of the function is a set of two intervals: $-\infty < x < -1$ and $1 < x < +\infty$

4°. **Inverse functions.** If the equation $y = f(x)$ may be solved uniquely for the variable x, that is, if there is a function $x = g(y)$ such that $y \equiv f[g(y)]$,

*) Henceforth all values will be considered as real, if not otherwise stated.

then the function $x=g(y)$, or, in standard notation, $y=g(x)$, is the *inverse* of $y=f(x)$. Obviously, $g[f(x)]\equiv x$, that is, the function $f(x)$ is the *inverse* of $g(x)$ (and vice versa).

In the general case, the equation $y=f(x)$ defines a multiple-valued inverse function $x=f^{-1}(y)$ such that $y\equiv f[f^{-1}(y)]$ for all y that are values of the function $f(x)$

Example 2. Determine the inverse of the function

$$y=1-2^{-x}. \qquad (1)$$

Solution. Solving equation (1) for x, we have

$$2^{-x}=1-y$$

and

$$x=-\frac{\log(1-y)}{\log 2}\,^{*)}. \qquad (2)$$

Obviously, the domain of definition of the function (2) is $-\infty<y<1$.

5°. Composite and implicit functions. A function y of x defined by a series of equalities $y=f(u)$, where $u=\varphi(x)$, etc., is called a *composite function*, or a *function of a function*.

A function defined by an equation not solved for the dependent variable is called an *implicit function*. For example, the equation $x^3+y^3=1$ defines y as an implicit function of x.

6°. The graph of a function. A set of points (x, y) in an xy-plane, whose coordinates are connected by the equation $y=f(x)$, is called the ***graph*** of the given function.

1.** Prove that if a and b are real numbers then

$$||a|-|b||\leqslant|a-b|\leqslant|a|+|b|.$$

2. Prove the following equalities:

a) $|ab|=|a|\cdot|b|$; c) $\left|\frac{a}{b}\right|=\frac{|a|}{|b|}\ (b\neq 0)$;

b) $|a|^2=a^2$; d) $\sqrt{a^2}=|a|$.

3. Solve the inequalities:

a) $|x-1|<3$; c) $|2x+1|<1$;

b) $|x+1|>2$; d) $|x-1|<|x+1|$.

4. Find $f(-1)$, $f(0)$, $f(1)$, $f(2)$, $f(3)$, $f(4)$, if $f(x)=x^3-6x^2+11x-6$.

5. Find $f(0)$, $f\left(-\frac{3}{4}\right)$, $f(-x)$, $f\left(\frac{1}{x}\right)$, $\frac{1}{f(x)}$, if $f(x)=\sqrt{1+x^2}$.

6. $f(x)=\arccos(\log x)$. Find $f\left(\frac{1}{10}\right)$, $f(1)$, $f(10)$.

7. The function $f(x)$ is linear. Find this function, if $f(-1)=2$ and $f(2)=-3$.

---

*) Log x is the logarithm of the number x to the base 10.

8. Find the rational integral function $f(x)$ of degree two, if $f(0)=1$, $f(1)=0$ and $f(3)=5$.

9. Given that $f(4)=-2$, $f(5)=6$. Approximate the value of $f(4,3)$ if we consider the function $f(x)$ on the interval $4\leqslant x\leqslant 5$ linear *(linear interpolation of a function)*.

10. Write the function

$$f(x)=\begin{cases}0, & \text{if } x\leqslant 0,\\ x, & \text{if } x>0\end{cases}$$

as a single formula using the absolute-value sign.

Determine the domains of definition of the following functions:

11. a) $y=\sqrt{x+1}$;
    b) $y=\sqrt[3]{x+1}$.

12. $y=\frac{1}{4-x^2}$.

13. a) $y=\sqrt{x^2-2}$;
    b) $y=x\sqrt{x^2-2}$.

14**. $y=\sqrt{2+x-x^2}$.

15. $y=\sqrt{-x}+\frac{1}{\sqrt{2+x}}$.

16. $y=\sqrt{x-x^3}$.

17. $y=\log\frac{2+x}{2-x}$.

18. $y=\log\frac{x^2-3x+2}{x+1}$.

19. $y=\arccos\frac{2x}{1+x}$.

20. $y=\arcsin\left(\log\frac{x}{10}\right)$.

21. Determine the domain of definition of the function

$$y=\sqrt{\sin 2x}.$$

22. $f(x)=2x^4-3x^3-5x^2+6x-10$. Find

$$\varphi(x)=\frac{1}{2}[f(x)+f(-x)] \quad\text{and}\quad \psi(x)=\frac{1}{2}[f(x)-f(-x)].$$

23. A function $f(x)$ defined in a symmetric region $-l<x<l$ is called *even* if $f(-x)=f(x)$ and *odd* if $f(-x)=-f(x)$.

Determine which of the following functions are even and which are odd:

a) $f(x)=\frac{1}{2}(a^x+a^{-x})$;

b) $f(x)=\sqrt{1+x+x^2}-\sqrt{1-x+x^2}$;

c) $f(x)=\sqrt[3]{(x+1)^2}+\sqrt[3]{(x-1)^2}$;

d) $f(x)=\log\frac{1+x}{1-x}$;

e) $f(x)=\log(x+\sqrt{1+x^2})$.

24. Prove that any function $f(x)$ defined in the interval $-l<x<l$ may be represented in the form of the sum of an even function and an odd function.

25. Prove that the product of two even functions or of two odd functions is an even function, and that the product of an even function by an odd function is an odd function.

26. A function $f(x)$ is called *periodic* if there exists a positive number T (the *period of the function*) such that $f(x+T)\equiv f(x)$ for all values of x within the domain of definition of $f(x)$.

Determine which of the following functions are periodic, and for the periodic functions find their least period T:

a) $f(x)=10\sin 3x$, d) $f(x)=\sin^2 x$;

b) $f(x)=a\sin\lambda x+b\cos\lambda x$; e) $f(x)=\sin(\sqrt{x})$.

c) $f(x)=\sqrt{\tan x}$;

27. Express the length of the segment $y=MN$ and the area S of the figure AMN as a function of $x=AM$ (Fig. 1). Construct the graphs of these functions.

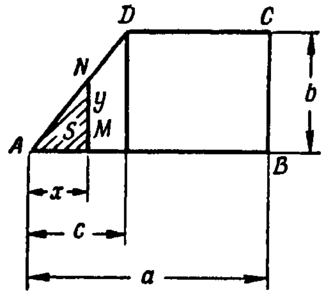

Fig. 1

28. The linear density (that is, mass per unit length) of a rod $AB=l$ (Fig. 2) on the segments $AC=l_1$, $CD=l_2$ and $DB=l_3$ $(l_1+l_2+l_3=l)$ is equal to q_1, q_2 and q_3, respectively. Express the mass m of a variable segment $AM=x$ of this rod as a function of x. Construct the graph of this function.

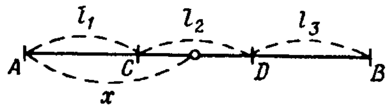

Fig. 2

29. Find $\varphi[\psi(x)]$ and $\psi[\varphi(x)]$, if $\varphi(x)=x^2$ and $\psi(x)=2^x$.

30. Find $f\{f[f(x)]\}$, if $f(x)=\frac{1}{1-x}$.

31. Find $f(x+1)$, if $f(x-1)=x^2$.

32. Let $f(n)$ be the sum of n terms of an arithmetic progression. Show that

$$f(n+3)-3f(n+2)+3f(n+1)-f(n)=0.$$

33. Show that if

$$f(x)=kx+b$$

and the numbers x_1, x_2, x_3 form an arithmetic progression, then the numbers $f(x_1)$, $f(x_2)$ and $f(x_3)$ likewise form such a progression.

34. Prove that if $f(x)$ is an exponential function, that is, $f(x)=a^x\ (a>0)$, and the numbers x_1, x_2, x_3 form an arithmetic progression, then the numbers $f(x_1)$, $f(x_2)$ and $f(x_3)$ form a geometric progression.

35. Let

$$f(x)=\log\frac{1+x}{1-x}.$$

Show that

$$f(x)+f(y)=f\left(\frac{x+y}{1+xy}\right).$$

36. Let $\varphi(x)=\frac{1}{2}(a^x+a^{-x})$ and $\psi(x)=\frac{1}{2}(a^x-a^{-x})$. Show that

$$\varphi(x+y)=\varphi(x)\varphi(y)+\psi(x)\psi(y)$$

and

$$\psi(x+y)=\varphi(x)\psi(y)+\varphi(y)\psi(x).$$

37. Find $f(-1)$, $f(0)$, $f(1)$ if

$$f(x)=\begin{cases}\arcsin x \text{ for } -1\leqslant x\leqslant 0,\\ \arctan x \text{ for } 0<x<+\infty.\end{cases}$$

38. Determine the roots (zeros) of the region of positivity and of the region of negativity of the function y if:

a) $y=1+x$;
b) $y=2+x-x^2$;
c) $y=1-x+x^2$;
d) $y=x^3-3x$;
e) $y=\log\frac{2x}{1+x}$.

39. Find the inverse of the function y if:

a) $y=2x+3$;
b) $y=x^2-1$;
c) $y=\sqrt[3]{1-x^3}$;
d) $y=\log\frac{x}{2}$;
e) $y=\arctan 3x$.

In what regions will these inverse functions be defined?

40. Find the inverse of the function

$$y=\begin{cases}x, \text{ if } x\leqslant 0,\\ x^2, \text{ if } x>0.\end{cases}$$

41. Write the given functions as a series of equalities each member of which contains a simple elementary function (power, exponential, trigonometric, and the like):

a) $y=(2x-5)^{10}$;
b) $y=2^{\cos x}$;
c) $y=\log\tan\frac{x}{2}$;
d) $y=\arcsin(3^{-x^2})$.

42. Write as a single equation the composite functions represented as a series of equalities:

a) $y=u^2$, $u=\sin x$;

b) $y=\arctan u$, $u=\sqrt{v}$, $v=\log x$;

c) $y=\begin{cases} 2u, & \text{if } u\leqslant 0, \\ 0, & \text{if } u>0; \end{cases}$

$u=x^2-1$.

43. Write, explicitly, functions of y defined by the equations:

a) $x^2-\arccos y=\pi$;

b) $10^x+10^y=10$;

c) $x+|y|=2y$.

Find the domains of definition of the given implicit functions.

Sec. 2. Graphs of Elementary Functions

Graphs of functions $y=f(x)$ are mainly constructed by marking a sufficiently dense net of points $M_i(x_i, y_i)$, where $y_i=f(x_i)$ $(i=0, 1, 2, \ldots)$ and by connecting the points with a line that takes account of intermediate points. Calculations are best done by a slide rule.

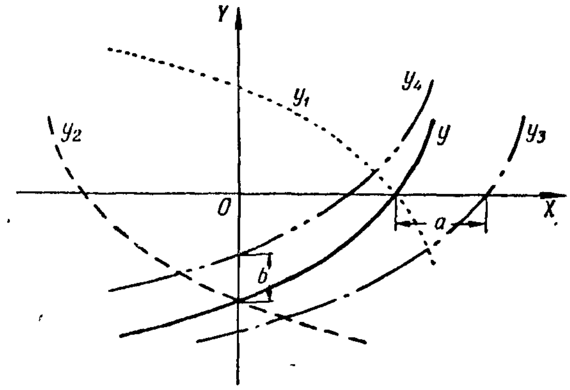

Fig. 3

Graphs of the basic elementary functions (see Appendix VI) are readily learned through their construction. Proceeding from the graph of

$$y=f(x), \qquad (\Gamma)$$

we get the graphs of the following functions by means of simple geometric constructions:

1) $y_1=-f(x)$ is the mirror image of the graph Γ about the x-axis;

2) $y_2=f(-x)$ is the mirror image of the graph Γ about the y-axis;

3) $y_3 = f(x-a)$ is the Γ graph displaced along the x-axis by an amount a;
4) $y_4 = b + f(x)$ is the Γ graph displaced along the y-axis by an amount b (Fig. 3).

Example. Construct the graph of the function

$$y = \sin\left(x - \frac{\pi}{4}\right).$$

Solution. The desired line is a sine curve $y = \sin x$ displaced along the x-axis to the right by an amount $\frac{\pi}{4}$ (Fig. 4)

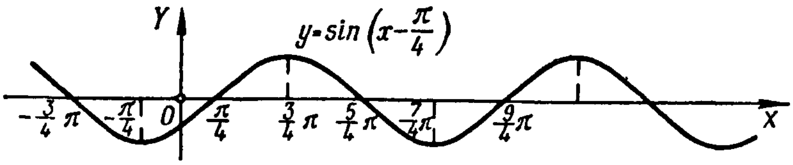

Fig. 4

Construct the graphs of the following linear functions (*straight lines*):

44. $y = kx$, if $k = 0, 1, 2, 1/2, -1, -2$.
45. $y = x + b$, if $b = 0, 1, 2, -1, -2$.
46. $y = 1.5x + 2$.

Construct the graphs of rational integral functions of degree two (*parabolas*).

47. $y = ax^2$, if $a = 1, 2, 1/2, -1, -2, 0$.
48. $y = x^2 + c$, if $c = 0, 1, 2, -1$.
49. $y = (x - x_0)^2$, if $x_0 = 0, 1, 2, -1$.
50. $y = y_0 + (x-1)^2$, if $y_0 = 0, 1, 2, -1$.
51*. $y = ax^2 + bx + c$, if: 1) $a = 1$, $b = -2$, $c = 3$; 2) $a = -2$, $b = 6$, $c = 0$.
52. $y = 2 + x - x^2$. Find the points of intersection of this parabola with the x-axis.

Construct the graphs of the following rational integral functions of degree above two:

53*. $y = x^3$ (*cubic parabola*).
54. $y = 2 + (x-1)^3$.
55. $y = x^3 - 3x + 2$.
56. $y = x^4$.
57. $y = 2x^2 - x^4$.

Construct the graphs of the following linear fractional functions (*hyperbolas*):

58*. $y = \frac{1}{x}$.

59. $y=\frac{1}{1-x}$.

60. $y=\frac{x-2}{x+2}$.

61*. $y=y_0+\frac{m}{x-x_0}$, if $x_0=1$, $y_0=-1$, $m=6$.

62*. $y=\frac{2x-3}{3x+2}$.

Construct the graphs of the fractional rational functions:

63. $y=x+\frac{1}{x}$.

64. $y=\frac{x^2}{x+1}$.

65*. $y=\frac{1}{x^2}$.

66. $y=\frac{1}{x^3}$.

67*. $y=\frac{10}{x^2+1}$ *(Witch of Agnesi)*.

68. $y=\frac{2x}{x^2+1}$ *(Newton's serpentine)*.

69. $y=x+\frac{1}{x^2}$.

70. $y=x^2+\frac{1}{x}$ *(trident of Newton)*.

Construct the graphs of the irrational functions:

71*. $y=\sqrt{x}$.

72. $y=\sqrt[3]{x}$.

73*. $y=\sqrt[3]{x^2}$ *(Niele's parabola)*.

74. $y=\pm x\sqrt{x}$ *(semicubical parabola)*.

75*. $y=\pm\frac{3}{5}\sqrt{25-x^2}$ *(ellipse)*.

76. $y=\pm\sqrt{x^2-1}$ *(hyperbola)*.

77. $y=\frac{1}{\sqrt{1-x^2}}$.

78*. $y=\pm x\sqrt{\frac{x}{4-x}}$ *(cissoid of Diocles)*.

79. $y=\pm x\sqrt{25-x^2}$.

Construct the graphs of the trigonometric functions:

80*. $y=\sin x$.

81*. $y=\cos x$.

82*. $y=\tan x$.

83*. $y=\cot x$.

84*. $y=\sec x$.

85*. $y=\operatorname{cosec} x$.

86. $y=A\sin x$, if $A=1$, 10, $1/2$, -2.

87*. $y=\sin nx$, if $n=1$, 2, 3, $1/2$.

88. $y=\sin(x-\varphi)$, if $\varphi=0$, $\frac{\pi}{2}$, $\frac{3\pi}{2}$, π, $-\frac{\pi}{4}$.

89*. $y=5\sin(2x-3)$.

90*. $y=a\sin x+b\cos x$, if $a=6$, $b=-8$.

91. $y=\sin x+\cos x$.

92*. $y=\cos^2 x$.

93*. $y=x+\sin x$.

94*. $y=x\sin x$.

95. $y=\tan^2 x$.

96. $y=1-2\cos x$.

97. $y=\sin x-\frac{1}{3}\sin 3x$.

98. $y=\cos x+\frac{1}{2}\cos 2x$.

99*. $y=\cos\frac{\pi}{x}$.

100. $y=\pm\sqrt{\sin x}$.

Construct the graphs of the exponential and logarithmic functions:

101. $y=a^x$, if $a=2$, $\frac{1}{2}$, e ($e=2, 718\ldots$) *).

102*. $y=\log_a x$, if $a=10$, 2, $\frac{1}{2}$, e.

103*. $y=\sinh x$, where $\sinh x=1/2\,(e^x-e^{-x})$.

104*. $y=\cosh x$, where $\cosh x=1/2\,(e^x+e^{-x})$.

105*. $y=\tanh x$, where $\tanh x=\frac{\sinh x}{\cosh x}$.

106. $y=10^{\frac{1}{x}}$.

107*. $y=e^{-x^2}$ (*probability curve*).

108. $y=2^{-\frac{1}{x^2}}$.

109. $y=\log x^2$.

110. $y=\log^2 x$.

111. $y=\log(\log x)$.

112. $y=\frac{1}{\log x}$.

113. $y=\log\frac{1}{x}$.

114. $y=\log(-x)$.

115. $y=\log_2(1+x)$.

116. $y=\log(\cos x)$.

117. $y=2^{-x}\sin x$.

Construct the graphs of the inverse trigonometric functions:

118*. $y=\arcsin x$.

119*. $y=\arccos x$.

120*. $y=\arctan x$.

121*. $y=\operatorname{arccot} x$.

122. $y=\arcsin\frac{1}{x}$.

123. $y=\arccos\frac{1}{x}$.

124. $y=x+\operatorname{arccot} x$.

Construct the graphs of the functions:

125. $y=|x|$.

126. $y=\frac{1}{2}(x+|x|)$.

127. a) $y=x|x|$; b) $y=\log_{\sqrt{2}}|x|$.

128. a) $y=\sin x+|\sin x|$; b) $y=\sin x-|\sin x|$.

129. $y=\begin{cases} 3-x^2 & \text{when } |x|\leqslant 1. \\ \frac{2}{|x|} & \text{when } |x|>1. \end{cases}$

---

*) About the number e see p. 22 for more details.

130. a) $y=[x]$, b) $y=x-[x]$, where $[x]$ is the integral part of the number x, that is, the greatest integer less than or equal to x.

Construct the graphs of the following functions in the polar coordinate system (r, φ) $(r \geqslant 0)$:

131. $r=1$.

132*. $r=\frac{\varphi}{2}$ *(spiral of Archimedes)*.

133*. $r=e^{\varphi}$ *(logarithmic spiral)*.

134*. $r=\frac{\pi}{\varphi}$ *(hyperbolic spiral)*.

135. $r=2\cos\varphi$ *(circle)*.

136. $r=\frac{1}{\sin\varphi}$ *(straight line)*.

137. $r=\sec^2\frac{\varphi}{2}$ *(parabola)*.

138*. $r=10\sin 3\varphi$ *(three-leafed rose)*

139*. $r=a(1+\cos\varphi)$ $(a>0)$ *(cardioid)*.

140*. $r^2=a^2\cos 2\varphi$ $(a>0)$ *(lemniscate)*.

Construct the graphs of the functions represented parametrically:

141*. $x=t^3$, $y=t^2$ *(semicubical parabola)*.

142*. $x=10\cos t$, $y=\sin t$ *(ellipse)*.

143*. $x=10\cos^3 t$, $y=10\sin^3 t$ *(astroid)*.

144*. $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$ *(involute of a circle)*.

145*. $x=\frac{at}{1+t^3}$, $y=\frac{at^2}{1+t^3}$ *(folium of Descartes)*.

146. $x=\frac{a}{\sqrt{1+t^2}}$, $y=\frac{at}{\sqrt{1+t^2}}$ *(semicircle)*.

147. $x=2^t+2^{-t}$, $y=2^t-2^{-t}$ *(branch of a hyperbola)*.

148. $x=2\cos^2 t$, $y=2\sin^2 t$ *(segment of a straight line)*.

149. $x=t-t^2$, $y=t^2-t^3$.

150. $x=a(2\cos t-\cos 2t)$, $y=a(2\sin t-\sin 2t)$ *(cardioid)*.

Construct the graphs of the following functions defined implicitly:

151*. $x^2+y^2=25$ *(circle)*.

152. $xy=12$ *(hyperbola)*.

153*. $y^2=2x$ *(parabola)*.

154. $\frac{x^2}{100}+\frac{y^2}{64}=1$ *(ellipse)*.

155. $y^2=x^2(100-x^2)$.

156*. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ *(astroid)*.

157*. $x+y=10\log y$.

158. $x^2=\cos y$.

159*. $\sqrt{x^2+y^2}=e^{\arctan\frac{y}{x}}$ *(logarithmic spiral).*

160*. $x^3+y^3-3xy=0$ *(folium of Descartes).*

161. Derive the conversion formula from the Celsius scale (C) to the Fahrenheit scale (F) if it is known that 0°C corresponds to 32°F and 100°C corresponds to 212°F.

Construct the graph of the function obtained.

162. Inscribed in a triangle (base $b=10$, altitude $h=6$) is a rectangle (Fig. 5). Express the area of the rectangle y as a function of the base x.

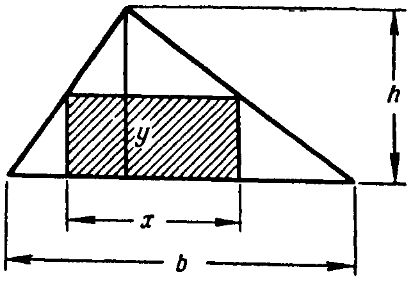

Fig. 5

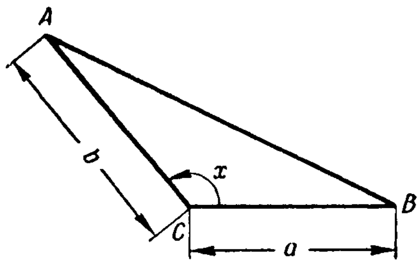

Fig 6

Construct the graph of this function and find its greatest value.

163. Given a triangle ACB with $BC=a$, $AC=b$ and a variable angle $\measuredangle ACB=x$ (Fig. 6).

Express $y=$ area $\triangle ABC$ as a function of x. Plot the graph of this function and find its greatest value.

164. Give a graphic solution of the equations:

a) $2x^2-5x+2=0$; d) $10^{-x}=x$;
b) $x^3+x-1=0$; e) $x=1+0.5\sin x$;
c) $\log x=0.1x$; f) $\cot x=x$ $(0<x<\pi)$.

165. Solve the systems of equations graphically:

a) $xy=10$, $x+y=7$;
b) $xy=6$, $x^2+y^2=13$;
c) $x^2-x+y=4$, $y^2-2x=0$;
d) $x^2+y=10$, $x+y^2=6$;
e) $y=\sin x$, $y=\cos x$ $(0<x<2\pi)$.

Sec. 3. Limits

1°. The limit of a sequence. The number a is the ***limit of a sequence*** $x_1, x_2, \ldots, x_n, \ldots$, or

$$\lim_{n \to \infty} x_n = a,$$

if for any $\varepsilon > 0$ there is a number $N = N(\varepsilon)$ such that
$|x_n - a| < \varepsilon$ when $n > N$.

Example 1. Show that

$$\lim_{n \to \infty} \frac{2n+1}{n+1} = 2. \qquad (1)$$

Solution. Form the difference

$$\frac{2n+1}{n+1} - 2 = -\frac{1}{n+1}.$$

Evaluating the absolute value of this difference, we have:

$$\left|\frac{2n+1}{n+1} - 2\right| = \frac{1}{n+1} < \varepsilon, \qquad (2)$$

if

$$n > \frac{1}{\varepsilon} - 1 = N(\varepsilon).$$

Thus, for every positive number ε there will be a number $N = \frac{1}{\varepsilon} - 1$ such that for $n > N$ we will have inequality (2) Consequently, the number 2 is the limit of the sequence $x_n = (2n+1)/(n+1)$, hence, formula (1) is true.

2°. The limit of a function. We say that a function $f(x) \to A$ as $x \to a$ (A and a are numbers), or

$$\lim_{x \to a} f(x) = A,$$

if for every $\varepsilon > 0$ we have $\delta = \delta(\varepsilon) > 0$ such that
$|f(x) - A| < \varepsilon$ for $0 < |x - a| < \delta$.
Similarly,

$$\lim_{x \to \infty} f(x) = A,$$

if $|f(x) - A| < \varepsilon$ for $|x| > N(\varepsilon)$.

The following conventional notation is also used:

$$\lim_{x \to a} f(x) = \infty,$$

which means that $|f(x)| > E$ for $0 < |x - a| < \delta(E)$, where E is an arbitrary positive number

3°. One-sided limits. If $x < a$ and $x \to a$, then we write conventionally $x \to a - 0$; similarly, if $x > a$ and $x \to a$, then we write $x \to a + 0$. The numbers

$$f(a-0) = \lim_{x \to a-0} f(x) \text{ and } f(a+0) = \lim_{x \to a+0} f(x)$$

are called, respectively, the *limit on the left* of the function $f(x)$ at the point a and the *limit on the right* of the function $f(x)$ at the point a (if these numbers exist).

For the existence of the limit of a function $f(x)$ as $x \to a$, it is necessary and sufficient to have the following equality:

$$f(a-0)=f(a+0).$$

If the limits $\lim\limits_{x \to a} f_1(x)$ and $\lim\limits_{x \to a} f_2(x)$ exist, then the following theorems hold:

1) $\lim\limits_{x \to a} [f_1(x)+f_2(x)] = \lim\limits_{x \to a} f_1(x) + \lim\limits_{x \to a} f_2(x)$;

2) $\lim\limits_{x \to a} [f_1(x) f_2(x)] = \lim\limits_{x \to a} f_1(x) \cdot \lim\limits_{x \to a} f_2(x)$;

3) $\lim\limits_{x \to a} [f_1(x)/f_2(x)] = \lim\limits_{x \to a} f_1(x)/ \lim\limits_{x \to a} f_2(x) \quad (\lim\limits_{x \to a} f_2(x) \neq 0)$.

The following two limits are frequently used:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

and

$$\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x = \lim_{\alpha \to 0} (1+\alpha)^{\frac{1}{\alpha}} = e = 2\ 71828\ \ldots$$

Example 2. Find the limits on the right and left of the function

$$f(x) = \arc\tan \frac{1}{x}$$

as $x \to 0$.

Solution. We have

$$f(+0) = \lim_{x \to +0} \left(\arc\tan \frac{1}{x}\right) = \frac{\pi}{2}$$

and

$$f(-0) = \lim_{x \to -0} \left(\arc\tan \frac{1}{x}\right) = -\frac{\pi}{2}.$$

Obviously, the function $f(x)$ in this case has no limit as $x \to 0$.

166. Prove that as $n \to \infty$ the limit of the sequence

$$1,\ \frac{1}{4},\ \ldots,\ \frac{1}{n^2},\ \ldots$$

is equal to zero. For which values of n will we have the inequality

$$\frac{1}{n^2} < \varepsilon$$

(ε is an arbitrary positive number)?

Calcula e numerically for a) $\varepsilon = 0.1$; b) $\varepsilon = 0.01$; c) $\varepsilon = 0.001$.

167. Prove that the limit of the sequence

$$x_n = \frac{n}{n+1} \quad (n = 1,\ 2,\ \ldots)$$

as $n \to \infty$ is unity. For which values of $n > N$ will we have the inequality

$$|x_n - 1| < \varepsilon$$

(ε is an arbitrary positive number)?

Find N for a) $\varepsilon = 0.1$; b) $\varepsilon = 0.01$; c) $\varepsilon = 0.001$.

168. Prove that

$$\lim_{x \to 2} x^2 = 4.$$

How should one choose, for a given positive number ε, some positive number δ so that the inequality

$$|x^2 - 4| < \varepsilon$$

should follow from

$$|x - 2| < \delta?$$

Compute δ for a) $\varepsilon = 0.1$; b) $\varepsilon = 0.01$; c) $\varepsilon = 0.001$.

169. Give the exact meaning of the following notations: a) $\lim_{x \to +0} \log x = -\infty$; b) $\lim_{x \to +\infty} 2^x = +\infty$; c) $\lim_{x \to \infty} f(x) = \infty$.

170. Find the limits of the sequences:

a) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, \frac{(-1)^{n-1}}{n}, \ldots$;

b) $\frac{2}{1}, \frac{4}{3}, \frac{6}{5}, \ldots, \frac{2n}{2n-1}, \ldots$;

c) $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$;

d) $0.2, 0.23, 0.233, 0.2333, \ldots$

Find the limits:

171. $\lim_{n \to \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \ldots + \frac{n-1}{n^2} \right)$.

172. $\lim_{n \to \infty} \frac{(n+1)(n+2)(n+3)}{n^3}$.

173. $\lim_{n \to \infty} \left[\frac{1+3+5+7+\ldots+(2n-1)}{n+1} - \frac{2n+1}{2} \right]$.

174. $\lim_{n \to \infty} \frac{n+(-1)^n}{n-(-1)^n}$.

175. $\lim_{n \to \infty} \frac{2^{n+1}+3^{n+1}}{2^n+3^n}$.

176. $\lim_{n \to \infty} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} \right)$.

177. $\lim_{n \to \infty} \left[1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots + \frac{(-1)^{n-1}}{3^{n-1}} \right]$.

178*. $\lim_{n \to \infty} \frac{1^2+2^2+3^2+\ldots+n^2}{n^3}$.

179. $\lim\limits_{n \to \infty} (\sqrt{n+1} - \sqrt{n})$.

180. $\lim\limits_{n \to \infty} \frac{n \sin n!}{n^2+1}$.

When seeking the limit of a ratio of two integral polynomials in x as $x \to \infty$, it is useful first to divide both terms of the ratio by x^n, where n is the highest degree of these polynomials.

A similar procedure is also possible in many cases for fractions containing irrational terms.

Example 1.

$$\lim_{x \to \infty} \frac{(2x-3)(3x+5)(4x-6)}{3x^3+x-1} = \lim_{x \to \infty} \frac{\left(2-\frac{3}{x}\right)\left(3+\frac{5}{x}\right)\left(4-\frac{6}{x}\right)}{3+\frac{1}{x^2}-\frac{1}{x^3}} = \frac{2 \cdot 3 \cdot 4}{3} = 8.$$

Example 2.

$$\lim_{x \to \infty} \frac{x}{\sqrt[3]{x^3+10}} = \lim_{x \to \infty} \frac{1}{\sqrt[3]{1+\frac{10}{x^3}}} = 1.$$

181. $\lim\limits_{x \to \infty} \frac{(x+1)^2}{x^2+1}$.

182. $\lim\limits_{x \to \infty} \frac{1000x}{x^2-1}$.

183. $\lim\limits_{x \to \infty} \frac{x^2-5x+1}{3x+7}$.

184. $\lim\limits_{x \to \infty} \frac{2x^2-x+3}{x^3-8x+5}$.

185. $\lim\limits_{x \to \infty} \frac{(2x+3)^3(3x-2)^2}{x^5+5}$.

186. $\lim\limits_{x \to \infty} \frac{2x^2-3x-4}{\sqrt{x^4+1}}$.

187. $\lim\limits_{x \to \infty} \frac{2x+3}{x+\sqrt[3]{x}}$.

188. $\lim\limits_{x \to \infty} \frac{x^2}{10+x\sqrt{x}}$.

189. $\lim\limits_{x \to \infty} \frac{\sqrt[3]{x^2+1}}{x+1}$.

190. $\lim\limits_{x \to +\infty} \frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$.

If $P(x)$ and $Q(x)$ are integral polynomials and $P(a) \neq 0$ or $Q(a) \neq 0$, then the limit of the rational fraction

$$\lim_{x \to a} \frac{P(x)}{Q(x)}$$

is obtained directly.

But if $P(a) = Q(a) = 0$, then it is advisable to cancel the binomial $x-a$ out of the fraction $\frac{P(x)}{Q(x)}$ once or several times.

Example 3.

$$\lim_{x \to 2} \frac{x^2-4}{x^2-3x+2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)(x-1)} = \lim_{x \to 2} \frac{x+2}{x-1} = 4.$$

191. $\lim\limits_{x \to -1} \frac{x^3+1}{x^2+1}$.

192. $\lim\limits_{x \to 5} \frac{x^2-5x+10}{x^2-25}$.

193. $\lim\limits_{x \to -1} \frac{x^2-1}{x^2+3x+2}$.

194. $\lim\limits_{x \to 2} \frac{x^2-2x}{x^2-4x+4}$.

195. $\lim\limits_{x \to 1} \frac{x^3-3x+2}{x^4-4x+3}$.

196. $\lim\limits_{x \to a} \frac{x^2-(a+1)x+a}{x^3-a^3}$.

197. $\lim\limits_{h \to 0} \frac{(x+h)^3-x^3}{h}$.

198. $\lim\limits_{x \to 1} \left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)$.

The expressions containing irrational terms are in many cases rationalized by introducing a new variable.

Example 4. Find

$$\lim_{x \to 0} \frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}.$$

Solution. Putting

$$1+x=y^6,$$

we have

$$\lim_{x \to 0} \frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}=\lim_{y \to 1} \frac{y^3-1}{y^2-1}=\lim_{y \to 1} \frac{y^2+y+1}{y+1}=\frac{3}{2}.$$

199. $\lim\limits_{x \to 1} \frac{\sqrt{x}-1}{x-1}$.

200. $\lim\limits_{x \to 64} \frac{\sqrt{x}-8}{\sqrt[3]{x}-4}$.

201. $\lim\limits_{x \to 1} \frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$.

202. $\lim\limits_{x \to 1} \frac{\sqrt[3]{x^2}-2\sqrt[3]{x}+1}{(x-1)^2}$.

Another way of finding the limit of an irrational expression is to transfer the irrational term from the numerator to the denominator, or vice versa, from the denominator to the numerator.

Example 5.

$$\lim_{x \to a} \frac{\sqrt{x}-\sqrt{a}}{x-a}=\lim_{x \to a} \frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})}=$$

$$=\lim_{x \to a} \frac{1}{\sqrt{x}+\sqrt{a}}=\frac{1}{2\sqrt{a}} \quad (a>0).$$

203. $\lim\limits_{x \to 7} \frac{2-\sqrt{x-3}}{x^2-49}$.

204. $\lim\limits_{x \to 8} \frac{x-8}{\sqrt[3]{x}-2}$.

205. $\lim\limits_{x \to 1} \frac{\sqrt{x}-1}{\sqrt[3]{x}-1}$.

206. $\lim\limits_{x \to 4} \frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}$.

207. $\lim\limits_{x \to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$.

208. $\lim\limits_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$.

209. $\lim\limits_{h \to 0} \dfrac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$.

210. $\lim\limits_{x \to 3} \dfrac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}$.

211. $\lim\limits_{x \to +\infty} (\sqrt{x+a}-\sqrt{x})$.

212. $\lim\limits_{x \to +\infty} [\sqrt{x(x+a)}-x]$.

213. $\lim\limits_{x \to +\infty} (\sqrt{x^2-5x+6}-x)$.

214. $\lim\limits_{x \to +\infty} x(\sqrt{x^2+1}-x)$.

215. $\lim\limits_{x \to \infty} (x+\sqrt[3]{1-x^3})$.

The formula

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

is frequently used when solving the following examples. It is taken for granted that $\lim\limits_{x \to a} \sin x = \sin a$ and $\lim\limits_{x \to a} \cos x = \cos a$.

Example 6.

$$\lim_{x \to 0} \frac{\sin 5x}{x} = \lim_{x \to 0} \left(\frac{\sin 5x}{5x} \cdot 5\right) = 1 \cdot 5 = 5.$$

216. a) $\lim\limits_{x \to 2} \dfrac{\sin x}{x}$;

b) $\lim\limits_{x \to \infty} \dfrac{\sin x}{x}$.

217. $\lim\limits_{x \to 0} \dfrac{\sin 3x}{x}$.

218. $\lim\limits_{x \to 0} \dfrac{\sin 5x}{\sin 2x}$.

219. $\lim\limits_{x \to 1} \dfrac{\sin \pi x}{\sin 3\pi x}$.

220. $\lim\limits_{n \to \infty} \left(n \sin \dfrac{\pi}{n}\right)$.

221. $\lim\limits_{x \to 0} \dfrac{1-\cos x}{x^2}$.

222. $\lim\limits_{x \to a} \dfrac{\sin x-\sin a}{x-a}$.

223. $\lim\limits_{x \to a} \dfrac{\cos x-\cos a}{x-a}$.

224. $\lim\limits_{x \to -2} \dfrac{\tan \pi x}{x+2}$.

225. $\lim\limits_{h \to 0} \dfrac{\sin (x+h)-\sin x}{h}$.

226. $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{\sin x-\cos x}{1-\tan x}$.

227. a) $\lim\limits_{x \to 0} x \sin \dfrac{1}{x}$;

b) $\lim\limits_{x \to \infty} x \sin \dfrac{1}{x}$.

228. $\lim\limits_{x \to 1} (1-x) \tan \dfrac{\pi x}{2}$.

229. $\lim\limits_{x \to 0} \cot 2x \cot \left(\dfrac{\pi}{2}-x\right)$.

230. $\lim\limits_{x \to \pi} \dfrac{1-\sin \frac{x}{2}}{\pi-x}$.

231. $\lim\limits_{x \to \frac{\pi}{3}} \dfrac{1-2\cos x}{\pi-3x}$.

232. $\lim\limits_{x \to 0} \dfrac{\cos mx-\cos nx}{x^2}$.

233. $\lim\limits_{x \to 0} \dfrac{\tan x-\sin x}{x^3}$.

234. $\lim\limits_{x \to 0} \dfrac{\arcsin x}{x}$.

235. $\lim\limits_{x \to 0} \dfrac{\arctan 2x}{\sin 3x}$.

236. $\lim\limits_{x \to 1} \dfrac{1-x^2}{\sin \pi x}$.

237. $\lim\limits_{x\to 0}\dfrac{x-\sin 2x}{x+\sin 3x}$.

238. $\lim\limits_{x\to 1}\dfrac{\cos\dfrac{\pi x}{2}}{1-\sqrt{x}}$.

239. $\lim\limits_{x\to 0}\dfrac{1-\sqrt{\cos x}}{x^2}$.

240. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}$.

When taking limits of the form

$$\lim_{x\to a}[\varphi(x)]^{\psi(x)}=C \qquad (3)$$

one should bear in mind that:

1) if there are final limits

$$\lim_{x\to a}\varphi(x)=A \text{ and } \lim_{x\to a}\psi(x)=B,$$

then $C=A^B$;

2) if $\lim\limits_{x\to a}\varphi(x)=A\neq 1$ and $\lim\limits_{x\to a}\psi(x)=\pm\infty$, then the problem of finding the limit of (3) is solved in straightforward fashion;

3) if $\lim\limits_{x\to a}\varphi(x)=1$ and $\lim\limits_{x\to a}\psi(x)=\infty$, then we put $\varphi(x)=1+\alpha(x)$, where $\alpha(x)\to 0$ as $x\to a$ and, hence,

$$C=\lim_{x\to a}\left\{[1+\alpha(x)]^{\frac{1}{\alpha(x)}}\right\}^{\alpha(x)\psi(x)}=e^{\lim\limits_{x\to a}\alpha(x)\psi(x)}=e^{\lim\limits_{x\to a}[\varphi(x)-1]\psi(x)},$$

where $e=2.718\ldots$ is Napier's number.

Example 7. Find

$$\lim_{x\to 0}\left(\frac{\sin 2x}{x}\right)^{1+x}.$$

Solution. Here,

$$\lim_{x\to 0}\left(\frac{\sin 2x}{x}\right)=2 \text{ and } \lim_{x\to 0}(1+x)=1;$$

hence,

$$\lim_{x\to 0}\left(\frac{\sin 2x}{x}\right)^{1+x}=2^1=2.$$

Example 8. Find

$$\lim_{x\to\infty}\left(\frac{x+1}{2x+1}\right)^{x^2}.$$

Solution. We have

$$\lim_{x\to\infty}\frac{x+1}{2x+1}=\lim_{x\to\infty}\frac{1+\frac{1}{x}}{2+\frac{1}{x}}=\frac{1}{2}$$

and

$$\lim_{x\to\infty}x^2=+\infty.$$

Therefore,

$$\lim_{x\to\infty}\left(\frac{x+1}{2x+1}\right)^{x^2}=0.$$

Example 9. Find

$$\lim_{x\to\infty}\left(\frac{x-1}{x+1}\right)^{x}.$$

Solution. We have

$$\lim_{x\to\infty}\frac{x-1}{x+1}=\lim_{x\to\infty}\frac{1-\frac{1}{x}}{1+\frac{1}{x}}=1.$$

Transforming, as indicated above, we have

$$\lim_{x\to\infty}\left(\frac{x-1}{x+1}\right)^{x}=\lim_{x\to\infty}\left[1+\left(\frac{x-1}{x+1}-1\right)\right]^{x}=$$

$$=\lim_{x\to\infty}\left\{\left[1+\left(\frac{-2}{x+1}\right)\right]^{\frac{x+1}{-2}}\right\}^{-\frac{2x}{1+x}}=e^{\lim\limits_{x\to\infty}\frac{-2x}{x+1}}=e^{-2}.$$

In this case it is easier to find the limit without resorting to the general procedure:

$$\lim_{x\to\infty}\left(\frac{x-1}{x+1}\right)^{x}=\lim_{x\to\infty}\frac{\left(1-\frac{1}{x}\right)^{x}}{\left(1+\frac{1}{x}\right)^{x}}=\frac{\lim\limits_{x\to\infty}\left[\left(1-\frac{1}{x}\right)^{-x}\right]^{-1}}{\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}}=\frac{e^{-1}}{e}=e^{-2}.$$

Generally, it is useful to remember that

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{x}=e^{k}$$

241. $\lim\limits_{x\to 0}\left(\frac{2+x}{3-x}\right)^{x}$.

242. $\lim\limits_{x\to 1}\left(\frac{x-1}{x^2-1}\right)^{x+1}$.

243. $\lim\limits_{x\to\infty}\left(\frac{1}{x^2}\right)^{\frac{2x}{x+1}}$.

244. $\lim\limits_{x\to 0}\left(\frac{x^2-2x+3}{x^2-3x+2}\right)^{\frac{\sin x}{x}}$.

245. $\lim\limits_{x\to\infty}\left(\frac{x^2+2}{2x^2+1}\right)^{x^2}$.

246. $\lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right)^{n}$.

247 $\lim\limits_{x\to\infty}\left(1+\frac{2}{x}\right)^{x}$.

248. $\lim\limits_{x\to\infty}\left(\frac{x}{x+1}\right)^{x}$.

249. $\lim\limits_{x\to\infty}\left(\frac{x-1}{x+3}\right)^{x+2}$.

250. $\lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}$.

251. $\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}}$.

252.** a) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x}}$;

b) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}}$.

When solving the problems that follow, it is useful to know that if the limit $\lim\limits_{x\to a} f(x)$ exists and is positive, then

$$\lim_{x\to a}[\ln f(x)] = \ln[\lim_{x\to a} f(x)].$$

Example 10. Prove that

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1. \qquad (*)$$

Solution. We have

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}[\ln(1+x)^{\frac{1}{x}}] = \ln[\lim_{x\to 0}(1+x)^{\frac{1}{x}}] = \ln e = 1.$$

Formula (*) is frequently used in the solution of problems.

253. $\lim\limits_{x\to\infty}[\ln(2x+1) - \ln(x+2)]$.

254. $\lim\limits_{x\to 0}\frac{\log(1+10x)}{x}$.

255. $\lim\limits_{x\to 0}\left(\frac{1}{x}\ln\sqrt{\frac{1+x}{1-x}}\right)$.

256. $\lim\limits_{x\to+\infty} x[\ln(x+1) - \ln x]$.

257. $\lim\limits_{x\to 0}\frac{\ln(\cos x)}{x^2}$.

258*. $\lim\limits_{x\to 0}\frac{e^x-1}{x}$.

259*. $\lim\limits_{x\to 0}\frac{a^x-1}{x}$ $(a>0)$.

260*. $\lim\limits_{n\to\infty} n(\sqrt[n]{a}-1)$ $(a>0)$.

261. $\lim\limits_{x\to 0}\frac{e^{ax}-e^{bx}}{x}$.

262. $\lim\limits_{x\to 0}\frac{1-e^{-x}}{\sin x}$.

263. a) $\lim\limits_{x\to 0}\frac{\sinh x}{x}$;

b) $\lim\limits_{x\to 0}\frac{\cosh x-1}{x^2}$

(see Problems 103 and 104).

Find the following limits that occur on one side:

264. a) $\lim\limits_{x\to-\infty}\frac{x}{\sqrt{x^2+1}}$;

b) $\lim\limits_{x\to+\infty}\frac{x}{\sqrt{x^2+1}}$.

265. a) $\lim\limits_{x\to-\infty}\tanh x$;

b) $\lim\limits_{x\to+\infty}\tanh x$,

where $\tanh x = \frac{e^x-e^{-x}}{e^x+e^{-x}}$.

266. a) $\lim\limits_{x\to-0}\frac{1}{1+e^{\frac{1}{x}}}$;

b) $\lim\limits_{x\to+0}\frac{1}{1+e^{\frac{1}{x}}}$.

267. a) $\lim\limits_{x\to-\infty}\frac{\ln(1+e^x)}{x}$;

b) $\lim\limits_{x\to+\infty}\frac{\ln(1+e^x)}{x}$.

268. a) $\lim\limits_{x\to-0}\frac{|\sin x|}{x}$;

b) $\lim\limits_{x\to+0}\frac{|\sin x|}{x}$.

269. a) $\lim\limits_{x\to 1-0}\frac{x-1}{|x-1|}$;

b) $\lim\limits_{x\to 1+0}\frac{x-1}{|x-1|}$.

270. a) $\lim\limits_{x\to 2-0}\frac{x}{x-2}$;

b) $\lim\limits_{x\to 2+0}\frac{x}{x-2}$.

Construct the graphs of the following functions:

271**. $y=\lim\limits_{n\to\infty}(\cos^{2n}x)$.

272*. $y=\lim\limits_{n\to\infty}\frac{x}{1+x^n}$ $(x\geqslant 0)$.

273. $y=\lim\limits_{\alpha\to 0}\sqrt{x^2+\alpha^2}$.

274. $y=\lim\limits_{n\to\infty}(\arctan nx)$.

275. $y=\lim\limits_{n\to\infty}\sqrt[n]{1+x^n}$ $(x\geqslant 0)$.

276. Transform the following mixed periodic fraction into a common fraction:

$$\alpha=0.13555...$$

Regard it as the limit of the corresponding finite fraction.

277. What will happen to the roots of the quadratic equation

$$ax^2+bx+c=0,$$

if the coefficient a approaches zero while the coefficients b and c are constant, and $b\neq 0$?

278. Find the limit of the interior angle of a regular n-gon as $n\to\infty$.

279. Find the limit of the perimeters of regular n-gons inscribed in a circle of radius R and circumscribed about it as $n\to\infty$.

280. Find the limit of the sum of the lengths of the ordinates of the curve

$$y=e^{-x}\cos\pi x,$$

drawn at the points $x=0, 1, 2, \ldots, n$, as $n\to\infty$.

281. Find the limit of the sum of the areas of the squares constructed on the ordinates of the curve

$$y=2^{1-x}$$

as on bases, where $x=1, 2, 3, \ldots, n$, provided that $n\to\infty$.

282. Find the limit of the perimeter of a broken line $M_0M_1\ldots M_n$ inscribed in a logarithmic spiral

$$r=e^{-\varphi}$$

(as $n \to \infty$), if the vertices of this broken line have, respectively, the polar angles

$$\varphi_0 = 0, \ \varphi_1 = \frac{\pi}{2}, \ \ldots, \ \varphi_n = \frac{n\pi}{2}.$$

283. A segment $AB = a$ (Fig. 7) is divided into n equal parts, each part serving as the base of an isosceles triangle with base angles $\alpha = 45°$. Show that the limit of the perimeter of the broken line thus formed differs from the length of AB despite the fact that in the limit the broken line "geometrically merges with the segment AB".

Fig. 7

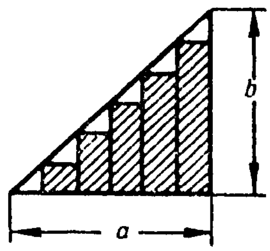

Fig 8

284. The point C_1 divides a segment $AB = l$ in half; the point C_2 divides a segment AC_1 in half; the point C_3 divides a segment C_2C_1 in half; the point C_4 divides C_2C_3 in half, and so on. Determine the limiting position of the point C_n when $n \to \infty$.

285. The side a of a right triangle is divided into n equal parts, on each of which is constructed an inscribed rectangle (Fig. 8). Determine the limit of the area of the step-like figure thus formed if $n \to \infty$.

286. Find the constants k and b from the equation

$$\lim_{x \to \infty}\left(kx + b - \frac{x^3+1}{x^2+1}\right) = 0. \qquad (1)$$

What is the geometric meaning of (1)?

287*. A certain chemical process proceeds in such fashion that the increase in quantity of a substance during each interval of time τ out of the infinite sequence of intervals $(i\tau, (i+1)\tau)$ $(i = 0, 1, 2, \ldots)$ is proportional to the quantity of the substance available at the commencement of each interval and to the length of the interval. Assuming that the quantity of substance at the initial time is Q_0, determine the quantity of substance $Q_t^{(n)}$ after the elapse of time t if the increase takes place each nth part of the time interval $\tau = \frac{t}{n}$.

Find $Q_t = \lim_{n \to \infty} Q_t^{(n)}$.

Sec. 4. Infinitely Small and Large Quantities

1°. Infinitely small quantities (infinitesimals). If

$$\lim_{x\to a} \alpha(x) = 0,$$

i.e., if $|\alpha(x)| < \varepsilon$ when $0 < |x-a| < \delta(\varepsilon)$, then the function $\alpha(x)$ is an *infinitesimal* as $x \longrightarrow a$. In similar fashion we define the infinitesimal $\alpha(x)$ as $x \longrightarrow \infty$.

The sum and product of a limited number of infinitesimals as $x \longrightarrow a$ are also infinitesimals as $x \to a$.

If $\alpha(x)$ and $\beta(x)$ are infinitesimals as $x \longrightarrow a$ and

$$\lim_{x\to a} \frac{\alpha(x)}{\beta(x)} = C,$$

where C is some number different from zero, then the functions $\alpha(x)$ and $\beta(x)$ are called *infinitesimals of the same order;* but if $C = 0$, then we say that the function $\alpha(x)$ is an *infinitesimal of higher order* than $\beta(x)$. The function $\alpha(x)$ is called an *infinitesimal of order n* compared with the function $\beta(x)$ if

$$\lim_{x\to a} \frac{\alpha(x)}{[\beta(x)]^n} = C,$$

where $0 < |C| < +\infty$.

If

$$\lim_{x\to a} \frac{\alpha(x)}{\beta(x)} = 1,$$

then the functions $\alpha(x)$ and $\beta(x)$ are called *equivalent functions* as $x \longrightarrow a$:

$$\alpha(x) \sim \beta(x).$$

For example, for $x \longrightarrow 0$ we have

$$\sin x \sim x; \quad \tan x \sim x; \quad \ln(1+x) \sim x$$

and so forth.

The sum of two infinitesimals of different orders is equivalent to the term whose order is lower.

The limit of a ratio of two infinitesimals remains unchanged if the terms of the ratio are replaced by equivalent quantities. By virtue of this theorem, when taking the limit of a fraction

$$\lim_{x\to a} \frac{\alpha(x)}{\beta(x)},$$

where $\alpha(x) \longrightarrow 0$ and $\beta(x) \longrightarrow 0$ as $x \longrightarrow a$, we can subtract from (or add to) the numerator or denominator infinitesimals of higher orders chosen so that the resultant quantities should be equivalent to the original quantities.

Example 1.

$$\lim_{x\to 0} \frac{\sqrt[3]{x^3+2x^4}}{\ln(1+2x)} = \lim_{x\to 0} \frac{\sqrt[3]{x^3}}{2x} = \frac{1}{2}.$$

2°. Infinitely large quantities (infinites). If for an arbitrarily large number N there exists a $\delta(N)$ such that when $0 < |x-a| < \delta(N)$ we have the inequality

$$|f(x)| > N,$$

then the function $f(x)$ is called an *infinite* as $x \longrightarrow a$.

The definition of an infinite $f(x)$ as $x \to \infty$ is analogous. As in the case of infinitesimals, we introduce the concept of infinites of different orders.

288. Prove that the function

$$f(x) = \frac{\sin x}{x}$$

is an infinitesimal as $x \to \infty$. For what values of x is the inequality

$$|f(x)| < \varepsilon$$

fulfilled if ε is an arbitrary number?

Calculate for: a) $\varepsilon = 0.1$; b) $\varepsilon = 0.01$; c) $\varepsilon = 0.001$.

289. Prove that the function

$$f(x) = 1 - x^2$$

is an infinitesimal for $x \to 1$. For what values of x is the inequality

$$|f(x)| < \varepsilon$$

fulfilled if ε is an arbitrary positive number? Calculate numerically for: a) $\varepsilon = 0.1$; b) $\varepsilon = 0.01$; c) $\varepsilon = 0.001$.

290. Prove that the function

$$f(x) = \frac{1}{x-2}$$

is an infinite for $x \to 2$. In what neighbourhoods of $|x-2| < \delta$ is the inequality

$$|f(x)| > N$$

fulfilled if N is an arbitrary positive number?

Find δ if a) $N = 10$; b) $N = 100$; c) $N = 1000$.

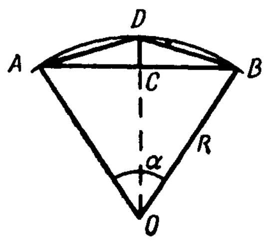

Fig. 9

291. Determine the order of smallness of: a) the surface of a sphere, b) the volume of a sphere if the radius of the sphere r is an infinitesimal of order one. What will the orders be of the radius of the sphere and the volume of the sphere with respect to its surface?

292. Let the central angle α of a circular sector ABO (Fig. 9) with radius R tend to zero. Determine the orders of the infinitesimals relative to the infinitesimal α: a) of the chord AB; b) of the line CD; c) of the area of ΔABD.

293. For $x \to 0$ determine the orders of smallness relative to x of the functions:

a) $\frac{2x}{1+x}$;

b) $\sqrt{x+\sqrt{x}}$;

c) $\sqrt[3]{x^2}-\sqrt{x^3}$;

d) $1-\cos x$;

e) $\tan x-\sin x$.

294. Prove that the length of an infinitesimal arc of a circle of constant radius is equivalent to the length of its chord.

295. Can we say that an infinitesimally small segment and an infinitesimally small semicircle constructed on this segment as a diameter are equivalent?

Using the theorem of the ratio of two infinitesimals, find

296. $\lim\limits_{x\to 0}\frac{\sin 3x\cdot\sin 5x}{(x-x^3)^2}$.

297. $\lim\limits_{x\to 0}\frac{\arcsin\frac{x}{\sqrt{1-x^2}}}{\ln(1-x)}$.

298. $\lim\limits_{x\to 1}\frac{\ln x}{1-x}$.

299. $\lim\limits_{x\to 0}\frac{\cos x-\cos 2x}{1-\cos x}$.

300. Prove that when $x \to 0$ the quantities $\frac{x}{2}$ and $\sqrt{1+x}-1$ are equivalent. Using this result, demonstrate that when $|x|$ is small we have the approximate equality

$$\sqrt{1+x}\approx 1+\frac{x}{2}. \qquad (1)$$

Applying formula (1), approximate the following:

a) $\sqrt{1.06}$; b) $\sqrt{0.97}$; c) $\sqrt{10}$; d) $\sqrt{120}$

and compare the values obtained with tabular data.

301. Prove that when $x \to 0$ we have the following approximate equalities accurate to terms of order x^2:

a) $\frac{1}{1+x}\approx 1-x$;

b) $\sqrt{a^2+x}\approx a+\frac{x}{2a}$ $(a>0)$;

c) $(1+x)^n\approx 1+nx$ (n is a positive integer);

d) $\log(1+x)=Mx$,

where $M=\log e=0.43429\ldots$

Using these formulas, approximate:

1) $\frac{1}{1.02}$; 2) $\frac{1}{0.97}$; 3) $\frac{1}{105}$; 4) $\sqrt{15}$; 5) 1.04^3; 6) 0.93^4; 7) $\log 1.1$.

Compare the values obtained with tabular data.

302. Show that for $x \to \infty$ the rational integral function

$$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n \quad (a_0 \neq 0)$$

is an infinitely large quantity equivalent to the term of highest degree a_0x^n.

303. Let $x \to \infty$. Taking x to be an infinite of the first order, determine the order of growth of the functions:

a) $x^2 - 100x - 1{,}000$;

b) $\frac{x^5}{x+2}$;

c) $\sqrt{x+\sqrt{x}}$;

d) $\sqrt[3]{x-2x^2}$.

Sec. 5. Continuity of Functions

1°. Definition of continuity. A function $f(x)$ is *continuous* when $x = \xi$ (or "at the point ξ"), if: 1) this function is defined at the point ξ, that is, there exists a number $f(\xi)$; 2) there exists a finite limit $\lim\limits_{x \to \xi} f(x)$; 3) this limit is equal to the value of the function at the point ξ, i.e.,

$$\lim_{x \to \xi} f(x) = f(\xi). \quad (1)$$

Putting

$$x = \xi + \Delta\xi,$$

where $\Delta\xi \to 0$, condition (1) may be rewritten as

$$\lim_{\Delta\xi \to 0} \Delta f(\xi) = \lim_{\Delta\xi \to 0} [f(\xi + \Delta\xi) - f(\xi)] = 0, \quad (2)$$

or the function $f(x)$ is continuous at the point ξ if (and only if) at this point to an infinitesimal increment in the argument there corresponds an infinitesimal increment in the function.

If a function is continuous at every point of some region (interval, etc.), then it is said to be *continuous in this region*.

Example 1. Prove that the function

$$y = \sin x$$

is continuous for every value of the argument x.

Solution. We have

$$\Delta y = \sin(x + \Delta x) - \sin x = 2\sin\frac{\Delta x}{2}\cos\left(x + \frac{\Delta x}{2}\right) = \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \cos\left(x + \frac{\Delta x}{2}\right) \cdot \Delta x.$$

Since

$$\lim_{\Delta x \to 0} \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} = 1 \text{ and } \left|\cos\left(x + \frac{\Delta x}{2}\right)\right| \leqslant 1,$$

it follows that for any x we have

$$\lim_{\Delta x \to 0} \Delta y = 0.$$

Hence, the function $\sin x$ is continuous when $-\infty < x < +\infty$.

2°. **Points of discontinuity of a function.** We say that a function $f(x)$ has a *discontinuity* at $x=x_0$ (or at the point x_0) within the domain of definition of the function or on the boundary of this domain if there is a break in the continuity of the function at this point.

Example 2. The function $f(x)=\frac{1}{(1-x)^2}$ (Fig. 10 *a*) is discontinuous when $x=1$. This function is not defined at the point $x=1$, and no matter

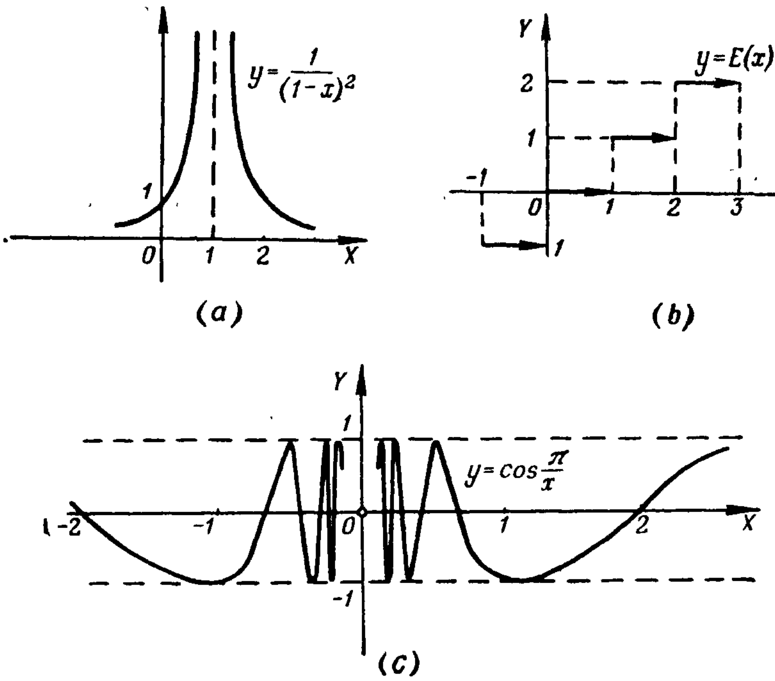

Fig. 10

how we choose the number $f(1)$, the redefined function $f(x)$ will not be continuous for $x=1$.

If the function $f(x)$ has *finite* limits:

$$\lim_{x\to x_0-0} f(x)=f(x_0-0) \quad \text{and} \quad \lim_{x\to x_0+0} f(x)=f(x_0+0),$$

and not all three numbers $f(x_0)$, $f(x_0-0)$, $f(x_0+0)$ are equal, then x_0 is called a *discontinuity of the first kind*. In particular, if

$$f(x_0-0)=f(x_0+0),$$

then x_0 is called a *removable discontinuity*.

For continuity of a function $f(x)$ at a point x_0, it is necessary and sufficient that

$$f(x_0)=f(x_0-0)=f(x_0+0).$$

Example 3. The function $f(x)=\frac{\sin x}{|x|}$ has a discontinuity of the first kind at $x=0$. Indeed, here,

$$f(+0)=\lim_{x\to+0}\frac{\sin x}{x}=+1$$

and

$$f(-0)=\lim_{x\to-0}\frac{\sin x}{-x}=-1.$$

Example 4. The function $y=E(x)$, where $E(x)$ denotes the integral part of the number x [i.e., $E(x)$ is an integer that satisfies the equality $x=E(x)+q$, where $0\leqslant q<1$], is discontinuous (Fig. 10*b*) at every integral point: $x=0$, ±1, ±2, ..., and all the discontinuities are of the first kind.

Indeed, if n is an integer, then $E(n-0)=n-1$ and $E(n+0)=n$. At all other points this function is, obviously, continuous.

Discontinuities of a function that are not of the first kind are called *discontinuities of the second kind.*

Infinite discontinuities also belong to discontinuities of the second kind. These are points x_0 such that at least one of the one-sided limits, $f(x_0-0)$ or $f(x_0+0)$, is equal to ∞ (see Example 2).

Example 5. The function $y=\cos\frac{\pi}{x}$ (Fig. 10*c*) at the point $x=0$ has a discontinuity of the second kind, since both one-sided limits are nonexistent here:

$$\lim_{x\to-0}\cos\frac{\pi}{x} \quad\text{and}\quad \lim_{x\to+0}\cos\frac{\pi}{x}.$$

3°. Properties of continuous functions. When testing functions for continuity, bear in mind the following theorems:

1) the sum and product of a limited number of functions continuous in some region is a function that is continuous in this region;

2) the quotient of two functions continuous in some region is a continuous function for all values of the argument of this region that do not make the divisor zero;

3) if a function $f(x)$ is continuous in an interval (a, b), and a set of its values is contained in the interval (A, B), and a function $\varphi(x)$ is continuous in (A, B), then the composite function $\varphi[f(x)]$ is continuous in (a, b).

A function $f(x)$ continuous in an interval $[a, b]$ has the following properties:

1) $f(x)$ is bounded on $[a, b]$, i.e., there is some number M such that $|f(x)|\leqslant M$ when $a\leqslant x\leqslant b$;

2) $f(x)$ has a minimum and a maximum value on $[a, b]$;

3) $f(x)$ takes on all intermediate values between the two given values; that is, if $f(\alpha)=A$ and $f(\beta)=B$ $(a\leqslant\alpha<\beta\leqslant b)$, then no matter what the number C between A and B, there will be at least one value $x=\gamma$ $(\alpha<\gamma<\beta)$ such that $f(\gamma)=C$.

In particular, if $f(\alpha)f(\beta)<0$, then the equation

$$f(x)=0$$

has at least one real root in the interval (α, β).

304. Show that the function $y=x^2$ is continuous for any value of the argument x.

305. Prove that the rational integral function

$$P(x)=a_0x^n+a_1x^{n-1}+\ldots+a_n$$

is continuous for any value of x.

306. Prove that the rational fractional function

$$R(x)=\frac{a_0x^n+a_1x^{n-1}+\ldots+a_n}{b_0x^m+b_1x^{m-1}+\ldots+b_m}$$

is continuous for all values of x except those that make the denominator zero.

307*. Prove that the function $y=\sqrt{x}$ is continuous for $x\geqslant 0$.

308. Prove that if the function $f(x)$ is continuous and non-negative in the interval (a, b), then the function

$$F(x)=\sqrt{f(x)}$$

is likewise continuous in this interval.

309*. Prove that the function $y=\cos x$ is continuous for any x.

310. For what values of x are the functions a) $\tan x$ and b) $\cot x$ continuous?

311*. Show that the function $y=|x|$ is continuous. Plot the graph of this function.

312. Prove that the absolute value of a continuous function is a continuous function.

313. A function is defined by the formulas

$$f(x)=\begin{cases}\dfrac{x^2-4}{x-2} & \text{for } x\neq 2,\\ A & \text{for } x=2.\end{cases}$$

How should one choose the value of the function $A=f(2)$ so that the thus redefined function $f(x)$ is continuous for $x=2$? Plot the graph of the function $y=f(x)$.

314. The right side of the equation

$$f(x)=1-x\sin\frac{1}{x}$$

is meaningless for $x=0$. How should one choose the value $f(0)$ so that $f(x)$ is continuous for $x=0$?

315. The function

$$f(x)=\operatorname{arc\,tan}\frac{1}{x-2}$$

is meaningless for $x=2$. Is it possible to define the value of $f(2)$ in such a way that the redefined function should be continuous for $x=2$?

316. The function $f(x)$ is not defined for $x=0$. Define $f(0)$ so that $f(x)$ is continuous for $x=0$, if:

a) $f(x)=\frac{(1+x)^n-1}{x}$ (n is a positive integer);

b) $f(x)=\frac{1-\cos x}{x^2}$;

c) $f(x)=\frac{\ln(1+x)-\ln(1-x)}{x}$;

d) $f(x)=\frac{e^x-e^{-x}}{x}$;

e) $f(x)=x^2\sin\frac{1}{x}$;

f) $f(x)=x\cot x$.

Investigate the following functions for continuity:

317. $y=\frac{x^2}{x-2}$.

318. $y=\frac{1+x^3}{1+x}$.

319. $y=\frac{\sqrt{7+x}-3}{x^2-4}$.

320. $y=\frac{x}{|x|}$.

321. a) $y=\sin\frac{\pi}{x}$;

b) $y=x\sin\frac{\pi}{x}$.

322. $y=\frac{x}{\sin x}$.

323. $y=\ln(\cos x)$.

324. $y=\ln\left|\tan\frac{x}{2}\right|$.

325. $y=\arctan\frac{1}{x}$.

326. $y=(1+x)\arctan\frac{1}{1-x^2}$.

327. $y=e^{\frac{1}{x+1}}$.

328. $y=e^{-\frac{1}{x^2}}$.

329. $y=\frac{1}{1+e^{\frac{1}{1-x}}}$.

330. $y=\begin{cases} x^2 & \text{for } x\leqslant 3, \\ 2x+1 & \text{for } x>3. \end{cases}$ Plot the graph of this function.

331. Prove that the Dirichlet function $\chi(x)$, which is zero for irrational x and unity for rational x, is discontinuous for every value of x.

Investigate the following functions for continuity and construct their graphs:

332. $y=\lim\limits_{n\to\infty}\frac{1}{1+x^n}$ $(x\geqslant 0)$.

333. $y=\lim\limits_{n\to\infty}(x\arctan nx)$.

334. a) $y = \operatorname{sgn} x$, b) $y = x \operatorname{sgn} x$, c) $y = \operatorname{sgn}(\sin x)$, where the function sgn x is defined by the formulas:

$$\operatorname{sgn} x = \begin{cases} +1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

335. a) $y = x - E(x)$, b) $y = xE(x)$, where $E(x)$ is the integral part of the number x.

336. Give an example to show that the sum of two discontinuous functions may be a continuous function.

337*. Let α be a regular positive fraction tending to zero $(0 < \alpha < 1)$. Can we put the limit of α into the equality

$$E(1 + \alpha) = E(1 - \alpha) + 1,$$

which is true for all values of α?

338. Show that the equation

$$x^3 - 3x + 1 = 0$$

has a real root in the interval (1,2). Approximate this root.

339. Prove that any polynomial $P(x)$ of odd power has at least one real root.

340. Prove that the equation

$$\tan x = x$$

has an infinite number of real roots.

Chapter II

DIFFERENTIATION OF FUNCTIONS

Sec. 1. Calculating Derivatives Directly

1°. Increment of the argument and increment of the function. If x and x_1 are values of the argument x, and $y=f(x)$ and $y_1=f(x_1)$ are corresponding values of the function $y=f(x)$, then

$$\Delta x = x_1 - x$$

is called the *increment of the argument* x in the interval (x, x_1), and

$$\Delta y = y_1 - y$$

or

$$\Delta y = f(x_1) - f(x) = f(x+\Delta x) - f(x) \qquad (1)$$

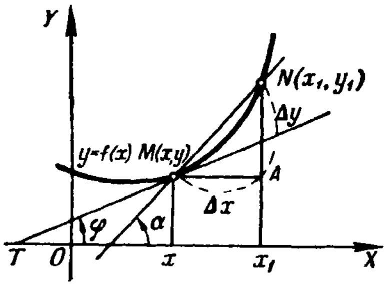

Fig. 11

is called the *increment of the function* y in the same interval (x, x_1) (Fig. 11, where $\Delta x = MA$ and $\Delta y = AN$). The ratio

$$\frac{\Delta y}{\Delta x} = \tan \alpha$$

is the slope of the secant MN of the graph of the function $y=f(x)$ (Fig. 11) and is called the *mean rate of change* of the function y over the interval $(x, x+\Delta x)$.

Example 1. For the function

$$y = x^2 - 5x + 6$$

calculate Δx and Δy, corresponding to a change in the argument:

a) from $x=1$ to $x=1.1$;
b) from $x=3$ to $x=2$.

Solution. We have

a) $\Delta x=1.1-1=0.1$,
$\Delta y=(1.1^2-5\cdot 1.1+6)-(1^2-5\cdot 1+6)=-0.29$;
b) $\Delta x=2-3=-1$,
$\Delta y=(2^2-5\cdot 2+6)-(3^2-5\cdot 3+6)=0$.

Example 2. In the case of the hyperbola $y=\frac{1}{x}$, find the slope of the secant passing through the points $M\left(3, \frac{1}{3}\right)$ and $N\left(10, \frac{1}{10}\right)$.

Solution. Here, $\Delta x=10-3=7$ and $\Delta y=\frac{1}{10}-\frac{1}{3}=-\frac{7}{30}$. Hence, $k=\frac{\Delta y}{\Delta x}=-\frac{1}{30}$.

2°. **The derivative.** *The derivative* $y'=\frac{dy}{dx}$ of a function $y=f(x)$ with respect to the argument x is the limit of the ratio $\frac{\Delta y}{\Delta x}$ when Δx approaches zero; that is,

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}.$$

The magnitude of the derivative yields the *slope* of the tangent MT to the graph of the function $y=f(x)$ at the point x (Fig. 11):

$$y'=\tan\varphi.$$

Finding the derivative y' is usually called *differentiation of the function*. The derivative $y'=f'(x)$ is the *rate of change of the function* at the point x.

Example 3. Find the derivative of the function

$$y=x^2.$$

Solution. From formula (1) we have

$$\Delta y=(x+\Delta x)^2-x^2=2x\Delta x+(\Delta x)^2$$

and

$$\frac{\Delta y}{\Delta x}=2x+\Delta x.$$

Hence,

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}(2x+\Delta x)=2x.$$

3°. **One-sided derivatives.** The expressions

$$f'_-(x)=\lim_{\Delta x\to -0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

and

$$f'_+(x)=\lim_{\Delta x\to +0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

are called, respectively, the *left-hand* or *right-hand derivative* of the function $f(x)$ at the point x. For $f'(x)$ to exist, it is necessary and sufficient that

$$f'_-(x)=f'_+(x).$$

Example 4 Find $f'_-(0)$ and $f'_+(0)$ of the function

$$f(x)=|x|.$$

Solution. By the definition we have

$$f'_-(0)=\lim_{\Delta x\to -0}\frac{|\Delta x|}{\Delta x}=-1,$$

$$f'_+(0)=\lim_{\Delta x\to +0}\frac{|\Delta x|}{\Delta x}=1.$$

4°. Infinite derivative. If at some point we have

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\infty,$$

then we say that the continuous function $f(x)$ has an infinite derivative at x. In this case, the tangent to the graph of the function $y=f(x)$ is perpendicular to the x-axis.

Example 5. Find $f'(0)$ of the function

$$y=\sqrt[3]{x}.$$

Solution. We have

$$f'(0)=\lim_{\Delta x\to 0}\frac{\sqrt[3]{\Delta x}}{\Delta x}=\lim_{\Delta x\to 0}\frac{1}{\sqrt[3]{\Delta x^2}}=\infty.$$

341. Find the increment of the function $y=x^2$ that corresponds to a change in argument:

a) from $x=1$ to $x_1=2$;
b) from $x=1$ to $x_1=1.1$;
c) from $x=1$ to $x_1=1+h$.

342. Find Δy of the function $y=\sqrt[3]{x}$ if:

a) $x=0$, $\Delta x=0.001$;
b) $x=8$, $\Delta x=-9$;
c) $x=a$, $\Delta x=h$.

343. Why can we, for the function $y=2x+3$, determine the increment Δy if all we know is the corresponding increment $\Delta x=5$, while for the function $y=x^2$ this cannot be done?

344. Find the increment Δy and the ratio $\frac{\Delta y}{\Delta x}$ for the functions:

a) $y=\frac{1}{(x^2-2)^2}$ for $x=1$ and $\Delta x=0.4$;
b) $y=\sqrt{x}$ for $x=0$ and $\Delta x=0.0001$;
c) $y=\log x$ for $x=100{,}000$ and $\Delta x=-90{,}000$.

345. Find Δy and $\frac{\Delta y}{\Delta x}$ which correspond to a change in argument from x to $x+\Delta x$ for the functions:

a) $y=ax+b$; d) $y=\sqrt{x}$;
b) $y=x^3$; e) $y=2^x$;
c) $y=\frac{1}{x^2}$; f) $y=\ln x$.

346. Find the slope of the secant to the parabola

$$y=2x-x^2,$$

if the abscissas of the points of intersection are equal:

a) $x_1=1,\ x_2=2$;
b) $x_1=1,\ x_2=0.9$;
c) $x_1=1,\ x_2=1+h$.

To what limit does the slope of the secant tend in the latter case if $h\to 0$?

347. What is the mean rate of change of the function $y=x^3$ in the interval $1\leqslant x\leqslant 4$?

348. The law of motion of a point is $s=2t^2+3t+5$, where the distance s is given in centimetres and the time t is in seconds. What is the average velocity of the point over the interval of time from $t=1$ to $t=5$?

349. Find the mean rise of the curve $y=2^x$ in the interval $1\leqslant x\leqslant 5$.

350. Find the mean rise of the curve $y=f(x)$ in the interval $[x,\ x+\Delta x]$.

351. What is to be understood by the rise of the curve $y=f(x)$ at a given point x?

352. Define: a) the mean rate of rotation; b) the instantaneous rate of rotation.

353. A hot body placed in a medium of lower temperature cools off. What is to be understood by: a) the mean rate of cooling; b) the rate of cooling at a given instant?

354. What is to be understood by the rate of reaction of a substance in a chemical reaction?

355. Let $m=f(x)$ be the mass of a non-homogeneous rod over the interval $[0,\ x]$. What is to be understood by: a) the mean linear density of the rod on the interval $[x,\ x+\Delta x]$; b) the linear density of the rod at a point x?

356. Find the ratio $\frac{\Delta y}{\Delta x}$ of the function $y=\frac{1}{x}$ at the point $x=2$, if: a) $\Delta x=1$; b) $\Delta x=0.1$; c) $\Delta x=0.01$. What is the derivative y' when $x=2$?

357.** Find the derivative of the function $y=\tan x$.

358. Find $y'=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$ of the functions:

a) $y=x^2$; c) $y=\sqrt{x}$;
b) $y=\frac{1}{x^2}$; d) $y=\cot x$.

359. Calculate $f'(8)$, if $f(x)=\sqrt[3]{x}$.

360. Find $f'(0)$, $f'(1)$, $f'(2)$, if $f(x)=x(x-1)^2(x-2)^3$.

361. At what points does the derivative of the function $f(x)=x^3$ coincide numerically with the value of the function itself, that is, $f(x)=f'(x)$?

362. The law of motion of a point is $s=5t^2$, where the distance s is in metres and the time t is in seconds. Find the speed at $t=3$.

363. Find the slope of the tangent to the curve $y=0.1x^3$ drawn at a point with abscissa $x=2$.

364. Find the slope of the tangent to the curve $y=\sin x$ at the point $(\pi, 0)$.

365. Find the value of the derivative of the function $f(x)=\frac{1}{x}$ at the point $x=x_0$ $(x_0\neq 0)$.

366*. What are the slopes of the tangents to the curves $y=\frac{1}{x}$ and $y=x^2$ at the point of their intersection? Find the angle between these tangents.

367.** Show that the following functions do not have finite derivatives at the indicated points:

a) $y=\sqrt[3]{x^2}$ at $x=0$;
b) $y=\sqrt[5]{x-1}$ at $x=1$;
c) $y=|\cos x|$ at $x=\frac{2k+1}{2}\pi$, $k=0, \pm 1, \pm 2, \ldots$

Sec. 2. Tabular Differentiation

1°. **Basic rules for finding a derivative.** If c is a constant and $u=\varphi(x)$, $v=\psi(x)$ are functions that have derivatives, then

1) $(c)'=0$;
2) $(x)'=1$;
3) $(u\pm v)'=u'\pm v'$;
4) $(cu)'=cu'$;
5) $(uv)'=u'v+v'u$;
6) $\left(\frac{u}{v}\right)'=\frac{vu'-v'u}{v^2}$ $(v\neq 0)$;
7) $\left(\frac{c}{v}\right)'=\frac{-cv'}{v^2}$ $(v\neq 0)$.

2°. Table of derivatives of basic functions

I. $(x^n)' = nx^{n-1}$.

II. $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$ $(x > 0)$.

III. $(\sin x)' = \cos x$.

IV. $(\cos x)' = -\sin x$.

V. $(\tan x)' = \frac{1}{\cos^2 x}$.

VI. $(\cot x)' = \frac{-1}{\sin^2 x}$.

VII. $(\text{arc}\sin x)' = \frac{1}{\sqrt{1-x^2}}$ $(|x| < 1)$.

VIII. $(\text{arc}\cos x)' = \frac{-1}{\sqrt{1-x^2}}$ $(|x| < 1)$.

IX. $(\text{arc}\tan x)' = \frac{1}{1+x^2}$.

X. $(\text{arc}\cot x)' = \frac{-1}{x^2+1}$.

XI. $(a^x)' = a^x \ln a$.

XII. $(e^x)' = e^x$.

XIII. $(\ln x)' = \frac{1}{x}$ $(x > 0)$.

XIV. $(\log_a x)' = \frac{1}{x \ln a} = \frac{\log_a e}{x}$ $(x > 0,\ a > 0)$.

XV. $(\sinh x)' = \cosh x$.

XVI. $(\cosh x)' = \sinh x$.

XVII. $(\tanh x)' = \frac{1}{\cosh^2 x}$.

XVIII. $(\coth x)' = \frac{-1}{\sinh^2 x}$.

XIX. $(\text{arc}\sinh x)' = \frac{1}{\sqrt{1+x^2}}$.

XX. $(\text{arc}\cosh x)' = \frac{1}{\sqrt{x^2-1}}$ $(|x| > 1)$.

XXI. $(\text{arc}\tanh x)' = \frac{1}{1-x^2}$ $(|x| < 1)$.

XXII. $(\text{arc}\coth x)' = \frac{-1}{x^2-1}$ $(|x| > 1)$.

3°. Rule for differentiating a composite function. If $y = f(u)$ and $u = \varphi(x)$, that is, $y = f[\varphi(x)]$, where the functions y and u have derivatives, then

$$y'_x = y'_u u'_x \tag{1}$$

or in other notations

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

This rule extends to a series of any finite number of differentiable functions.

Example 1. Find the derivative of the function

$$y=(x^2-2x+3)^5.$$

Solution. Putting $y=u^5$, where $u=(x^2-2x+3)$, by formula (1) we will have

$$y'=(u^5)'_u\,(x^2-2x+3)'_x=5u^4\,(2x-2)=10\,(x-1)\,(x^2-2x+3)^4.$$

Example 2. Find the derivative of the function

$$y=\sin^3 4x.$$

Solution. Putting

$$y=u^3;\quad u=\sin v;\quad v=4x,$$

we find

$$y'=3u^2\cdot\cos v\cdot 4=12\sin^2 4x\cos 4x.$$

Find the derivatives of the following functions (the rule for differentiating a composite function is not used in problems 368-408).

A. Algebraic Functions

368. $y=x^5-4x^3+2x-3.$

369. $y=\frac{1}{4}-\frac{1}{3}x+x^2-0.5x^4.$

370. $y=ax^2+bx+c.$

371. $y=\frac{-5x^3}{a}.$

372. $y=at^m+bt^{m+n}.$

373. $y=\frac{ax^6+b}{\sqrt{a^2+b^2}}.$

374. $y=\frac{\pi}{x}+\ln 2.$

375. $y=3x^{\frac{2}{3}}-2x^{\frac{5}{2}}+x^{-3}.$

376*. $y=x^2\sqrt[3]{x^2}.$

377. $y=\frac{a}{\sqrt[3]{x^2}}-\frac{b}{x\sqrt[3]{x}}.$

378. $y=\frac{a+bx}{c+dx}.$

379. $y=\frac{2x+3}{x^2-5x+5}.$

380. $y=\frac{2}{2x-1}-\frac{1}{x}.$

381. $y=\frac{1+\sqrt{z}}{1-\sqrt{z}}$

B. Inverse Circular and Trigonometric Functions

382. $y=5\sin x+3\cos x.$

383. $y=\tan x-\cot x.$

384. $y=\frac{\sin x+\cos x}{\sin x-\cos x}.$

385. $y=2t\sin t-(t^2-2)\cos t.$

386. $y=\operatorname{arc\,tan} x+\operatorname{arc\,cot} x.$

387. $y=x\cot x.$

388. $y=x\operatorname{arc\,sin} x.$

389. $y=\frac{(1+x^2)\operatorname{arc\,tan} x-x}{2}.$

C. Exponential and Logarithmic Functions

390. $y = x^7 \cdot e^x$.
391. $y = (x-1)e^x$.
392. $y = \frac{e^x}{x^2}$.
393. $y = \frac{x^5}{e^x}$.
394. $f(x) = e^x \cos x$.
395. $y = (x^2 - 2x + 2)e^x$.
396. $y = e^x \arcsin x$.
397. $y = \frac{x^2}{\ln x}$.
398. $y = x^3 \ln x - \frac{x^3}{3}$.
399. $y = \frac{1}{x} + 2\ln x - \frac{\ln x}{x}$.
400. $y = \ln x \log x - \ln a \log_a x$.

D. Hyperbolic and Inverse Hyperbolic Functions

401. $y = x \sinh x$.
402. $y = \frac{x^2}{\cosh x}$.
403. $y = \tanh x - x$.
404. $y = \frac{3 \coth x}{\ln x}$
405. $y = \arctan x - \operatorname{arc tanh} x$.
406. $y = \arcsin x \operatorname{arc sinh} x$.
407. $y = \frac{\operatorname{arc cosh} x}{x}$.
408. $y = \frac{\operatorname{arc coth} x}{1 - x^2}$.

E. Composite Functions

In problems 409 to 466, use the rule for differentiating a composite function with one intermediate argument.

Find the derivatives of the following functions:

409**. $y = (1 + 3x - 5x^2)^{30}$.

Solution. Denote $1 + 3x - 5x^2 = u$; then $y = u^{30}$. We have:

$$y'_u = 30u^{29}; \quad u'_x = 3 - 10x;$$

$$y'_x = 30u^{29} \cdot (3 - 10x) = 30(1 + 3x - 5x^2)^{29} \cdot (3 - 10x).$$

410. $y = \left(\frac{ax+b}{c}\right)^3$.
411. $f(y) = (2a + 3by)^2$.
412. $y = (3 + 2x^2)^4$.
413. $y = \frac{3}{56(2x-1)^7} - \frac{1}{24(2x-1)^6} - \frac{1}{40(2x-1)^5}$.
414. $y = \sqrt{1 - x^2}$.
415. $y = \sqrt[3]{a + bx^3}$.
416. $y = (a^{2/3} - x^{2/3})^{3/2}$.

417. $y=(3-2\sin x)^5$.

Solution. $y'=5(3-2\sin x)^4\cdot(3-2\sin x)'=5(3-2\sin x)^4(-2\cos x)=-10\cos x\,(3-2\sin x)^4$.

418. $y=\tan x-\frac{1}{3}\tan^3 x+\frac{1}{5}\tan^5 x$.

419. $y=\sqrt{\cot x}-\sqrt{\cot \alpha}$.

420. $y=2x+5\cos^3 x$.

421*. $x=\operatorname{cosec}^2 t+\sec^2 t$.

422. $f(x)=-\frac{1}{6(1-3\cos x)^2}$.

423. $y=\frac{1}{3\cos^3 x}-\frac{1}{\cos x}$.

424. $y=\sqrt{\frac{3\sin x-2\cos x}{5}}$.

425. $y=\sqrt[3]{\sin^2 x}+\frac{1}{\cos^3 x}$.

426. $y=\sqrt{1+\arcsin x}$.

427. $y=\sqrt{\arctan x}-(\arcsin x)^3$.

428. $y=\frac{1}{\arctan x}$.

429. $y=\sqrt{xe^x+x}$.

430. $y=\sqrt[3]{2e^x-2^x+1}+\ln^5 x$.

431. $y=\sin 3x+\cos\frac{x}{5}+\tan\sqrt{x}$.

Solution. $y'=\cos 3x\cdot(3x)'-\sin\frac{x}{5}\left(\frac{x}{5}\right)'+\frac{1}{\cos^2\sqrt{x}}(\sqrt{x})'=3\cos 3x-\frac{1}{5}\sin\frac{x}{5}+\frac{1}{2\sqrt{x}\cos^2\sqrt{x}}$.

432. $y=\sin(x^2-5x+1)+\tan\frac{a}{x}$.

433. $f(x)=\cos(\alpha x+\beta)$.

434. $f(t)=\sin t\sin(t+\varphi)$.

435. $y=\frac{1+\cos 2x}{1-\cos 2x}$.

436. $f(x)=a\cot\frac{x}{a}$.

437. $y=-\frac{1}{20}\cos(5x^2)-\frac{1}{4}\cos x^2$.

438. $y=\arcsin 2x$.

Solution. $y'=\frac{1}{\sqrt{1-(2x)^2}}\cdot(2x)'=\frac{2}{\sqrt{1-4x^2}}$.

439. $y=\arcsin\frac{1}{x^2}$.

440. $f(x)=\arccos\sqrt{x}$.

441. $y=\arctan\frac{1}{x}$.

442. $y=\operatorname{arc\,cot}\frac{1+x}{1-x}$.

443. $y = 5e^{-x^2}$.

444. $y = \frac{1}{5^{x^2}}$.

445. $y = x^2 10^{2x}$.

446. $f(t) = t \sin 2^t$.

447. $y = \arccos e^x$.

448. $y = \ln(2x+7)$.

449. $y = \log \sin x$.

450. $y = \ln(1-x^2)$.

451. $y = \ln^2 x - \ln(\ln x)$.

452. $y = \ln(e^x + 5\sin x - 4\arcsin x)$.

453. $y = \arctan(\ln x) + \ln(\arctan x)$.

454. $y = \sqrt{\ln x + 1} + \ln(\sqrt{x} + 1)$.

F. Miscellaneous Functions

455**. $y = \sin^3 5x \cos^2 \frac{x}{3}$.

456. $y = -\frac{11}{2(x-2)^2} - \frac{4}{x-2}$.

457. $y = -\frac{15}{4(x-3)^4} - \frac{10}{3(x-3)^3} - \frac{1}{2(x-3)^2}$.

458. $y = \frac{x^8}{8(1-x^2)^4}$.

459. $y = \frac{\sqrt{2x^2-2x+1}}{x}$.

460. $y = \frac{x}{a^2\sqrt{a^2+x^2}}$.

461. $y = \frac{x^3}{3\sqrt{(1+x^2)^3}}$.

462. $y = \frac{3}{2}\sqrt[3]{x^2} + \frac{18}{7}x\sqrt[6]{x} + \frac{9}{5}x\sqrt[3]{x^2} + \frac{6}{13}x^2\sqrt[6]{x}$.

463. $y = \frac{1}{8}\sqrt[3]{(1+x^3)^8} - \frac{1}{5}\sqrt[3]{(1+x^3)^5}$.

464. $y = \frac{4}{3}\sqrt[4]{\frac{x-1}{x+2}}$.

465. $y = x^4(a-2x^3)^2$.

466. $y = \left(\frac{a+bx^n}{a-bx^n}\right)^m$.

467. $y = \frac{9}{5(x+2)^5} - \frac{3}{(x+2)^4} + \frac{2}{(x+2)^3} - \frac{1}{2(x+2)^2}$.

468. $y = (a+x)\sqrt{a-x}$.

469. $y = \sqrt{(x+a)(x+b)(x+c)}$.

470. $z = \sqrt[3]{y + \sqrt{y}}$.

471. $f(t) = (2t+1)(3t+2)\sqrt[3]{3t+2}$.

472. $x=\frac{1}{\sqrt{2ay-y^2}}$.

473. $y=\ln(\sqrt{1+e^x}-1)-\ln(\sqrt{1+e^x}+1)$.

474. $y=\frac{1}{15}\cos^3 x(3\cos^2 x-5)$.

475. $y=\frac{(\tan^2 x-1)(\tan^4 x+10\tan^2 x+1)}{3\tan^3 x}$.

476. $y=\tan^5 5x$.

477. $y=\frac{1}{2}\sin(x^2)$.

478. $y=\sin^2(t^3)$.

479. $y=3\sin x\cos^2 x+\sin^3 x$.

480. $y=\frac{1}{3}\tan^3 x-\tan x+x$.

481. $y=-\frac{\cos x}{3\sin^3 x}+\frac{4}{3}\cot x$.

482. $y=\sqrt{\alpha\sin^2 x+\beta\cos^2 x}$.

483. $y=\arcsin x^2+\arccos x^2$.

484. $y=\frac{1}{2}(\arcsin x)^2\arccos x$.

485. $y=\arcsin\frac{x^2-1}{x^2}$.

486. $y=\arcsin\frac{x}{\sqrt{1+x^2}}$.

487. $y=\frac{\arccos x}{\sqrt{1-x^2}}$.

488. $y=\frac{1}{\sqrt{b}}\arcsin\left(x\sqrt{\frac{b}{a}}\right)$.

489. $y=\sqrt{a^2-x^2}+a\arcsin\frac{x}{a}$.

490. $y=x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}$.

491. $y=\arcsin(1-x)+\sqrt{2x-x^2}$.

492. $y=\left(x-\frac{1}{2}\right)\arcsin\sqrt{x}+\frac{1}{2}\sqrt{x-x^2}$.

493. $y=\ln(\arcsin 5x)$.

494. $y=\arcsin(\ln x)$.

495. $y=\arctan\frac{x\sin\alpha}{1-x\cos\alpha}$.

496. $y=\frac{2}{3}\arctan\frac{5\tan\frac{x}{2}+4}{3}$.

497. $y=3b^2\arctan\sqrt{\frac{x}{b-x}}-(3b+2x)\sqrt{bx-x^2}$.

498. $y=-\sqrt{2}\operatorname{arccot}\frac{\tan x}{\sqrt{2}}-x$.

499. $y=\sqrt{e^{ax}}$.

500. $y=e^{\sin^2 x}$.

501. $F(x)=(2ma^{mx}+b)^p$.

502. $F(t)=e^{\alpha t}\cos\beta t$.

503. $y=\frac{(\alpha\sin\beta x-\beta\cos\beta x)e^{\alpha x}}{\alpha^2+\beta^2}$.

504. $y=\frac{1}{10}e^{-x}(3\sin 3x-\cos 3x)$.

505. $y=x^n a^{-x^2}$.

506. $y=\sqrt{\cos x}\,a^{\sqrt{\cos x}}$.

507. $y=3^{\cot\frac{1}{x}}$.

508. $y=\ln(ax^2+bx+c)$.

509. $y=\ln(x+\sqrt{a^2+x^2})$.

510. $y=x-2\sqrt{x}+2\ln(1+\sqrt{x})$.

511. $y=\ln(a+x+\sqrt{2ax+x^2})$.

512. $y=\frac{1}{\ln^2 x}$.

513. $y=\ln\cos\frac{x-1}{x}$.

514*. $y=\ln\frac{(x-2)^5}{(x+1)^3}$.

515. $y=\ln\frac{(x-1)^3(x-2)}{x-3}$.

516. $y=-\frac{1}{2\sin^2 x}+\ln\tan x$.

517. $y=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln(x+\sqrt{x^2-a^2})$.

518. $y=\ln\ln(3-2x^3)$.

519. $y=5\ln^3(ax+b)$.

520. $y=\ln\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}$.

521. $y=\frac{m}{2}\ln(x^2-a^2)+\frac{n}{2a}\ln\frac{x-a}{x+a}$.

522. $y=x\cdot\sin\left(\ln x-\frac{\pi}{4}\right)$.

523. $y=\frac{1}{2}\ln\tan\frac{x}{2}-\frac{1}{2}\frac{\cos x}{\sin^2 x}$.

524. $f(x)=\sqrt{x^2+1}-\ln\frac{1+\sqrt{x^2+1}}{x}$.

525. $y=\frac{1}{3}\ln\frac{x^2-2x+1}{x^2+x+1}$.

526. $y=2^{\arcsin 3x}+(1-\arccos 3x)^2$.

527. $y=3^{\frac{\sin ax}{\cos bx}}+\frac{1}{3}\frac{\sin^3 ax}{\cos^3 bx}$.

528. $y=\frac{1}{\sqrt{3}}\ln\frac{\tan\frac{x}{2}+2-\sqrt{3}}{\tan\frac{x}{2}+2+\sqrt{3}}$.

529. $y=\arctan\ln x$.

530. $y=\ln\arcsin x+\frac{1}{2}\ln^2 x+\arcsin\ln x$.

531. $y=\arctan\ln\frac{1}{x}$.

532. $y=\frac{\sqrt{2}}{3}\arctan\frac{x}{\sqrt{2}}+\frac{1}{6}\ln\frac{x-1}{x+1}$.

533. $y=\ln\frac{1+\sqrt{\sin x}}{1-\sqrt{\sin x}}+2\arctan\sqrt{\sin x}$.

534. $y=\frac{3}{4}\ln\frac{x^2+1}{x^2-1}+\frac{1}{4}\ln\frac{x-1}{x+1}+\frac{1}{2}\arctan x$.

535. $f(x)=\frac{1}{2}\ln(1+x)-\frac{1}{6}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$.

536. $f(x)=\frac{x\arcsin x}{\sqrt{1-x^2}}+\ln\sqrt{1-x^2}$.

537. $y=\sinh^2 2x$.

538. $y=e^{\alpha x}\cosh\beta x$.

539. $y=\tanh^2 2x$.

540. $y=\ln\sinh 2x$.

541. $y=\operatorname{arc\,sinh}\frac{x^2}{a^2}$.

542. $y=\operatorname{arc\,cosh}\ln x$.

543. $y=\operatorname{arc\,tanh}(\tan x)$.

544. $y=\operatorname{arc\,coth}(\sec x)$.

545. $y=\operatorname{arc\,tanh}\frac{2x}{1+x^2}$.

546. $y=\frac{1}{2}(x^2-1)\operatorname{arc\,tanh}x+\frac{1}{2}x$.

547. $y=\left(\frac{1}{2}x^2+\frac{1}{4}\right)\operatorname{arc\,sinh}x-\frac{1}{4}x\sqrt{1+x^2}$.

548. Find y', if:

a) $y=|x|$;

b) $y=x|x|$.

Construct the graphs of the functions y and y'.

549. Find y' if

$$y=\ln|x| \quad (x\neq 0).$$

550. Find $f'(x)$ if

$$f(x)=\begin{cases}1-x & \text{for } x\leqslant 0,\\ e^{-x} & \text{for } x>0.\end{cases}$$

551. Calculate $f'(0)$ if

$$f(x)=e^{-x}\cos 3x.$$

Solution. $f'(x)=e^{-x}(-3\sin 3x)-e^{-x}\cos 3x$;

$f'(0)=e^0(-3\sin 0)-e^0\cos 0=-1$.

552. $f(x)=\ln(1+x)+\arcsin\frac{x}{2}$. Find $f'(1)$.

553. $y=\tan^2\frac{\pi x}{6}$. Find $\left(\frac{dy}{dx}\right)_{x=2}$.

554. Find $f'_+(0)$ and $f'_-(0)$ of the functions:

a) $f(x)=\sqrt{\sin(x^2)}$;

b) $f(x)=\arcsin\frac{a^2-x^2}{a^2+x^2}$;

c) $f(x)=\frac{x}{1+e^{\frac{1}{x}}}$, $x\neq 0$; $f(0)=0$;

d) $f(x)=x^2\sin\frac{1}{x}$, $x\neq 0$; $f(0)=0$;

e) $f(x)=x\sin\frac{1}{x}$ $x\neq 0$; $f(0)=0$

555. Find $f(0)+xf'(0)$ of the function $f(x)=e^{-x}$.

556. Find $f(3)+(x-3)f'(3)$ of the function $f(x)=\sqrt{1+x}$.

557. Given the functions $f(x)=\tan x$ and $\varphi(x)=\ln(1-x)$, find $\frac{f'(0)}{\varphi'(0)}$.

558. Given the functions $f(x)=1-x$ and $\varphi(x)=1-\sin\frac{\pi x}{2}$, find $\frac{\varphi'(1)}{f'(1)}$.

559. Prove that the derivative of an even function is an odd function, and the derivative of an odd function is an even function.

560. Prove that the derivative of a periodic function is also a periodic function.

561. Show that the function $y=xe^{-x}$ satisfies the equation $xy'=(1-x)y$.

562. Show that the function $y=xe^{-\frac{x^2}{2}}$ satisfies the equation $xy'=(1-x^2)y$.

563. Show that the function $y=\frac{1}{1+x+\ln x}$ satisfies the equation $xy'=y(y\ln x-1)$.

G. Logarithmic Derivative

A *logarithmic derivative* of a function $y=f(x)$ is the derivative of the logarithm of this function; that is,

$$(\ln y)'=\frac{y'}{y}=\frac{f'(x)}{f(x)}$$

Finding the derivative is sometimes simplified by first taking logs of the function.

Example. Find the derivative of the exponential function

$$y=u^v,$$

where $u=\varphi(x)$ and $v=\psi(x)$.

Solution. Taking logarithms we get

$$\ln y=v\ln u.$$

Differentiate both sides of this equation with respect *to* x:

$$(\ln y)'=v'\ln u+v(\ln u)',$$

or

$$\frac{1}{y}y'=v'\ln u+v\frac{1}{u}u',$$

whence

$$y'=y\left(v'\ln u+\frac{v}{u}u'\right),$$

or

$$y'=u^v\left(v'\ln u+\frac{v}{u}u'\right)$$

564. Find y', if

$$y=\sqrt[3]{x^2}\,\frac{1-x}{1+x^2}\sin^3 x\cos^2 x.$$

Solution. $\ln y=\frac{2}{3}\ln x+\ln(1-x)-\ln(1+x^2)+3\ln\sin x+2\ln\cos x;$

$$\frac{1}{y}y'=\frac{2}{3}\,\frac{1}{x}+\frac{(-1)}{1-x}-\frac{2x}{1+x^2}+3\frac{1}{\sin x}\cos x-\frac{2\sin x}{\cos x},$$

whence $y'=y\left(\frac{2}{3x}-\frac{1}{1-x}-\frac{2x}{1+x^2}+3\cot x-2\tan x\right).$

565. Find y', if $y=(\sin x)^x$.

Solution. $\ln y=x\ln\sin x$; $\frac{1}{y}y'=\ln\sin x+x\cot x$;

$y'=(\sin x)^x(\ln\sin x+x\cot x)$.

In the following problems find y' after first taking logs of the function $y=f(x)$:

566. $y=(x+1)(2x+1)(3x+1)$.

567. $y=\frac{(x+2)^2}{(x+1)^3(x+3)^4}$.

568. $y=\sqrt{\frac{x(x-1)}{x-2}}$.

569. $y=x\sqrt[3]{\frac{x^2}{x^2+1}}$.

570. $y=\frac{(x-2)^9}{\sqrt{(x-1)^5(x-3)^{11}}}$.

571. $y=\frac{\sqrt{x-1}}{\sqrt[3]{(x+2)^2}\sqrt{(x+3)^3}}$.

572. $y=x^x$.

573. $y=x^{x^2}$.

574. $y=\sqrt[x]{x}$.

575. $y=x^{\sqrt{x}}$.

576. $y=x^{x^x}$.

577. $y=x^{\sin x}$.

578. $y=(\cos x)^{\sin x}$.

579. $y=\left(1+\frac{1}{x}\right)^x$.

580. $y=(\arctan x)^x$.

Sec. 3. The Derivatives of Functions Not Represented Explicitly

1°. The derivative of an inverse function. If a function $y=f(x)$ has a derivative $y'_x\neq 0$, then the derivative of the inverse function $x=f^{-1}(y)$ is

$$x'_y=\frac{1}{y'_x}$$

or

$$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}.$$

Example 1. Find the derivative x'_y, if

$$y=x+\ln x.$$

Solution. We have $y'_x=1+\frac{1}{x}=\frac{x+1}{x}$; hence, $x'_y=\frac{x}{x+1}$.

2°. The derivatives of functions represented parametrically. If a function y is related to an argument x by means of a parameter t,

$$\begin{cases} x=\varphi(t), \\ y=\psi(t), \end{cases}$$

then

$$y'_x=\frac{y'_t}{x'_t},$$

or, in other notation,

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Example 2. Find $\frac{dy}{dx}$, if

$$\left.\begin{array}{l} x=a\cos t, \\ y=a\sin t \end{array}\right\}$$

Solution. We find $\frac{dx}{dt}=-a\sin t$ and $\frac{dy}{dt}=a\cos t$. Whence

$$\frac{dy}{dx}=-\frac{a\cos t}{a\sin t}=-\cot t.$$

3°. The derivative of an implicit function. If the relationship between x and y is given in implicit form,

$$F(x, y)=0, \qquad (1)$$

then to find the derivative $y'_x=y'$ in the simplest cases it is sufficient: 1) to calculate the derivative, with respect to x, of the left side of equation (1), taking y as a function of x; 2) to equate this derivative to zero, that is, to put

$$\frac{d}{dx}F(x, y)=0, \qquad (2)$$

and 3) to solve the resulting equation for y'.

Example 3. Find the derivative y'_x if

$$x^3+y^3-3axy=0. \qquad (3)$$

Solution. Forming the derivative of the left side of (3) and equating it to zero, we get

$$3x^2+3y^2y'-3a(y+xy')=0,$$

whence

$$y' = \frac{x^2 - ay}{ax - y^2}.$$

581. Find the derivative x'_y if

a) $y = 3x + x^3$;

b) $y = x - \frac{1}{2}\sin x$;

c) $y = 0.1x + e^{\frac{x}{2}}$.

In the following problems, find the derivative $y' = \frac{dy}{dx}$ of the functions y represented parametrically:

582. $\begin{cases} x = 2t - 1, \\ y = t^3. \end{cases}$

583. $\begin{cases} x = \frac{1}{t+1}, \\ y = \left(\frac{t}{t+1}\right)^2. \end{cases}$

584. $\begin{cases} x = \frac{2at}{1+t^2}, \\ y = \frac{a(1-t^2)}{1+t^2}. \end{cases}$

585. $\begin{cases} x = \frac{3at}{1+t^3}, \\ y = \frac{3at^2}{1+t^3}. \end{cases}$

586. $\begin{cases} x = \sqrt{t}, \\ y = \sqrt[3]{t}. \end{cases}$

587. $\begin{cases} x = \sqrt{t^2+1}, \\ y = \frac{t-1}{\sqrt{t^2+1}}. \end{cases}$

588. $\begin{cases} x = a(\cos t + t\sin t), \\ y = a(\sin t - t\cos t). \end{cases}$

589. $\begin{cases} x = a\cos^2 t, \\ y = b\sin^2 t. \end{cases}$

590. $\begin{cases} x = a\cos^3 t, \\ y = b\sin^3 t. \end{cases}$

591. $\begin{cases} x = \frac{\cos^3 t}{\sqrt{\cos 2t}}, \\ y = \frac{\sin^3 t}{\sqrt{\cos 2t}}. \end{cases}$

592. $\begin{cases} x = \arccos \frac{1}{\sqrt{1+t^2}}, \\ y = \arcsin \frac{t}{\sqrt{1+t^2}}. \end{cases}$

593. $\begin{cases} x = e^{-t}, \\ y = e^{2t}. \end{cases}$

594. $\begin{cases} x = a\left(\ln\tan\frac{t}{2} + \cos t - \sin t\right), \\ y = a(\sin t + \cos t). \end{cases}$

595. Calculate $\frac{dy}{dx}$ when $t = \frac{\pi}{2}$ if

$$\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t). \end{cases}$$

Solution. $\frac{dy}{dx} = \frac{a\sin t}{a(1-\cos t)} = \frac{\sin t}{1-\cos t}$

and

$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{2}}=\frac{\sin\frac{\pi}{2}}{1-\cos\frac{\pi}{2}}=1.$$

596. Find $\frac{dy}{dx}$ when $t=1$ if $\begin{cases} x=t\ln t, \\ y=\frac{\ln t}{t}. \end{cases}$

597. Find $\frac{dy}{dx}$ when $t=\frac{\pi}{4}$ if $\begin{cases} x=e^t\cos t, \\ y=e^t\sin t. \end{cases}$

598. Prove that a function y represented parametrically by the equations

$$\begin{cases} x=2t+3t^2, \\ y=t^2+2t^3, \end{cases}$$

satisfies the equation

$$y=\left(\frac{dy}{dx}\right)^2+2\left(\frac{dy}{dx}\right)^3.$$

599. When $x=2$ the following equation is true:

$$x^2=2x.$$

Does it follow from this that

$$(x^2)'=(2x)'$$

when $x=2$?

600. Let $y=\sqrt{a^2-x^2}$. Is it possible to perform term-by-term differentiation of

$$x^2+y^2=a^2?$$

In the examples that follow it is required to find the derivative $y'=\frac{dy}{dx}$ of the implicit functions y.

601. $2x-5y+10=0$.

602. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

603. $x^3+y^3=a^3$.

604. $x^3+x^2y+y^2=0$.

605. $\sqrt{x}+\sqrt{y}=\sqrt{a}$.

606. $\sqrt[3]{x^2}+\sqrt[3]{y^2}=\sqrt[3]{a^2}$.

607. $y^3=\frac{x-y}{x+y}$.

608. $y-0.3\sin y=x$.

609. $a\cos^2(x+y)=b$.

610. $\tan y=xy$.

611. $xy=\arctan\frac{x}{y}$.

612. $\arctan(x+y)=x$.

613. $e^y=x+y$.

614. $\ln x+e^{-\frac{y}{x}}=c$.

615. $\ln y+\frac{x}{y}=c$.

616. $\arctan\frac{y}{x}=\frac{1}{2}\ln(x^2+y^2)$.

617. $\sqrt{x^2+y^2}=c\arctan\frac{y}{x}$. **618.** $x^y=y^x$.

619. Find y' at the point $M(1,1)$, if

$$2y=1+xy^3.$$

Solution. Differentiating, we get $2y'=y^3+3xy^2y'$. Putting $x=1$ and $y=1$, we obtain $2y'=1+3y'$, whence $y'=-1$.

620. Find the derivatives y' of specified functions y at the indicated points:

a) $(x+y)^3=27(x-y)$ for $x=2$ and $y=1$;

b) $ye^y=e^{x+1}$ for $x=0$ and $y=1$;

c) $y^2=x+\ln\frac{y}{x}$ for $x=1$ and $y=1$.

Sec. 4. Geometrical and Mechanical Applications of the Derivative

1°. Equations of the tangent and the normal. From the geometric significance of a derivative it follows that the *equation of the tangent* to a curve $y=f(x)$ or $F(x, y)=0$ at a point $M(x_0, y_0)$ will be

$$y-y_0=y'_0(x-x_0),$$

where y'_0 is the value of the derivative y' at the point $M(x_0, y_0)$. The straight line passing through the point of tangency perpendicularly to the tangent is called the *normal to the curve*. For the normal we have the equation

$$x-x_0+y'_0(y-y_0)=0.$$

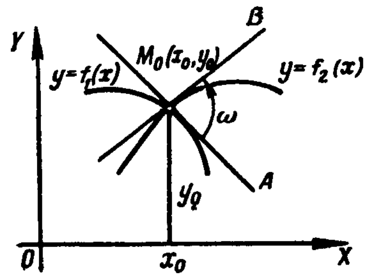

Fig. 12

2°. The angle between curves. The angle between the curves

$$y=f_1(x)$$

and

$$y=f_2(x)$$

at their common point $M_0(x_0, y_0)$ (Fig. 12) is the angle ω between the tangents M_0A and M_0B to these curves at the point M_0.

Using a familiar formula of analytic geometry, we get

$$\tan\omega=\frac{f'_2(x_0)-f'_1(x_0)}{1+f'_1(x_0)\cdot f'_2(x_0)}.$$

3°. Segments associated with the tangent and the normal in a rectangular coordinate system. The tangent and the normal determine the following four segments (Fig. 13):

$t=TM$ is the so-called *segment of the tangent*,
$S_t=TK$ is the *subtangent*,
$n=NM$ is the *segment of the normal*,
$S_n=KN$ is the *subnormal*.

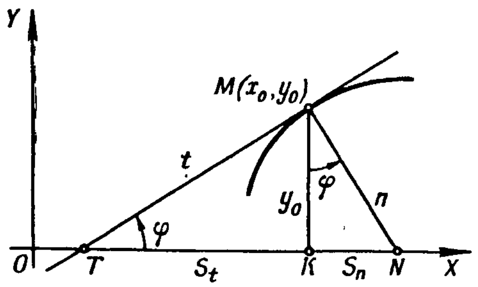

Fig. 13

Since $KM=|y_0|$ and $\tan\varphi=y_0'$, it follows that

$$t=TM=\left|\frac{y_0}{y_0'}\sqrt{1+(y_0')^2}\right|;\quad n=NM=\left|y_0\sqrt{1+(y_0')^2}\right|;$$

$$S_t=TK=\left|\frac{y_0}{y_0'}\right|;\quad S_n=\left|y_0y_0'\right|.$$

4°. Segments associated with the tangent and the normal in a polar system of coordinates. If a curve is given in polar coordinates by the equation $r=f(\varphi)$, then the angle μ formed by the tangent MT and the radius vector $r=OM$ (Fig. 14), is defined by the following formula:

$$\tan\mu=r\frac{d\varphi}{dr}=\frac{r}{r'}.$$

The tangent MT and the normal MN at the point M together with the radius vector of the point of tangency and with the perpendicular to the radius vector drawn through the pole O determine the following four segments (see Fig. 14):

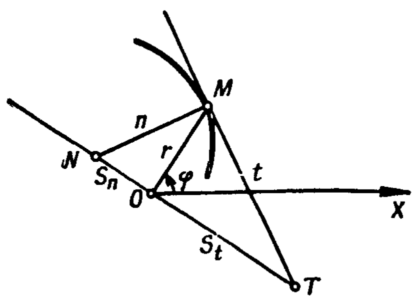

Fig. 14

$t=MT$ is the *segment of the polar tangent*,
$n=MN$ is the *segment of the polar normal*,
$S_t=OT$ is the *polar subtangent*,
$S_n=ON$ is the *polar subnormal*.

These segments are expressed by the following formulas:

$$t = MT = \frac{r}{|r'|}\sqrt{r^2+(r')^2}; \qquad S_t = OT = \frac{r^2}{|r'|};$$
$$n = MN = \sqrt{r^2+(r')^2}; \qquad S_n = ON = |r'|.$$

621. What angles φ are formed with the x-axis by the tangents to the curve $y = x - x^2$ at points with abscissas:

a) $x = 0$; b) $x = 1/2$; c) $x = 1$?

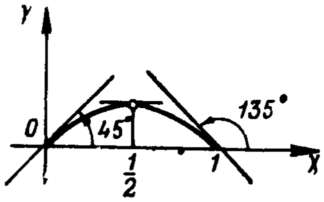

Fig. 15

Solution. We have $y' = 1 - 2x$. Whence a) $\tan\varphi = 1$, $\varphi = 45°$; b) $\tan\varphi = 0$, $\varphi = 0°$; c) $\tan\varphi = -1$, $\varphi = 135°$ (Fig. 15).

622. At what angles do the sine curves $y = \sin x$ and $y = \sin 2x$ intersect the axis of abscissas at the origin?

623. At what angle does the tangent curve $y = \tan x$ intersect the axis of abscissas at the origin?

624. At what angle does the curve $y = e^{0.5x}$ intersect the straight line $x = 2$?

625. Find the points at which the tangents to the curve $y = 3x^4 + 4x^3 - 12x^2 + 20$ are parallel to the x-axis.

626. At what point is the tangent to the parabola

$$y = x^2 - 7x + 3$$

parallel to the straight line $5x + y - 3 = 0$?

627. Find the equation of the parabola $y = x^2 + bx + c$ that is tangent to the straight line $x = y$ at the point (1,1).

628. Determine the slope of the tangent to the curve $x^3 + y^3 - xy - 7 = 0$ at the point (1,2).

629. At what point of the curve $y^2 = 2x^3$ is the tangent perpendicular to the straight line $4x - 3y + 2 = 0$?

630. Write the equation of the tangent and the normal to the parabola

$$y = \sqrt{x}$$

at the point with abscissa $x = 4$.

Solution. We have $y' = \frac{1}{2\sqrt{x}}$; whence the slope of the tangent is $k = [y']_{x=4} = \frac{1}{4}$. Since the point of tangency has coordinates $x = 4$, $y = 2$, it follows that the equation of the tangent is $y - 2 = 1/4\,(x - 4)$ or $x - 4y + 4 = 0$.

Since the slope of the normal must be perpendicular,

$$k_1 = -4;$$

whence the equation of the normal: $y - 2 = -4\,(x - 4)$ or $4x + y - 18 = 0$.

631. Write the equations of the tangent and the normal to the curve $y=x^3+2x^2-4x-3$ at the point $(-2,5)$.

632. Find the equations of the tangent and the normal to the curve

$$y=\sqrt[3]{x-1}$$

at the point (1,0).

633. Form the equations of the tangent and the normal to the curves at the indicated points:

a) $y=\tan 2x$ at the origin;

b) $y=\arcsin\frac{x-1}{2}$ at the point of intersection with the x-axis;

c) $y=\arccos 3x$ at the point of intersection with the y-axis;

d) $y=\ln x$ at the point of intersection with the x-axis;

e) $y=e^{1-x^2}$ at the points of intersection with the straight line $y=1$.

634. Write the equations of the tangent and the normal at the point (2,2) to the curve

$$x=\frac{1+t}{t^3},$$

$$y=\frac{3}{2t^2}+\frac{1}{2t}.$$

635. Write the equations of the tangent to the curve

$$x=t\cos t,\quad y=t\sin t$$

at the origin and at the point $t=\frac{\pi}{4}$.

636. Write the equations of the tangent and the normal to the curve $x^2+y^2+2x-6=0$ at the point with ordinate $y=3$.

637. Write the equation of the tangent to the curve $x^5+y^5-2xy=0$ at the point (1,1).

638. Write the equations of the tangents and the normals to the curve $y=(x-1)(x-2)(x-3)$ at the points of its intersection with the x-axis.

639. Write the equations of the tangent and the normal to the curve $y^4=4x^4+6xy$ at the point (1,2).

640*. Show that the segment of the tangent to the hyperbola $xy=a^2$ (the segment lies between the coordinate axes) is divided in two at the point of tangency.

641. Show that in the case of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ the segment of the tangent between the coordinate axes has a constant value equal to a.

642. Show that the normals to the involute of the circle

$$x = a(\cos t + t \sin t), \quad y = a(\sin t - t \cos t)$$

are tangents to the circle $x^2 + y^2 = a^2$.

643. Find the angle at which the parabolas $y = (x-2)^2$ and $y = -4 + 6x - x^2$ intersect.

644. At what angle do the parabolas $y = x^2$ and $y = x^3$ intersect?

645. Show that the curves $y = 4x^2 + 2x - 8$ and $y = x^3 - x + 10$ are tangent to each other at the point (3,34). Will we have the same thing at (—2,4)?

646. Show that the hyperbolas

$$xy = a^2; \quad x^2 - y = b^2$$

intersect at a right angle.

647. Given a parabola $y^2 = 4x$. At the point (1,2) evaluate the lengths of the segments of the subtangent, subnormal, tangent, and normal.

648. Find the length of the segment of the subtangent of the curve $y = 2^x$ at any point of it.

649. Show that in the equilateral hyperbola $x^2 - y^2 = a^2$ the length of the normal at any point is equal to the radius vector of this point.

650. Show that the length of the segment of the subnormal in the hyperbola $x^2 - y^2 = a^2$ at any point is equal to the abscissa of this point.

651. Show that the segments of the subtangents of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the circle $x^2 + y^2 = a^2$ at points with the same abscissas are equal. What procedure of construction of the tangent to the ellipse follows from this?

652. Find the length of the segment of the tangent, the normal, the subtangent, and the subnormal of the cycloid

$$\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t) \end{cases}$$

at an arbitrary point $t = t_0$.

653. Find the angle between the tangent and the radius vector of the point of tangency in the case of the logarithmic spiral

$$r = ae^{k\varphi}.$$

654. Find the angle between the tangent and the radius vector of the point of tangency in the case of the lemniscate $r^2 = a^2 \cos 2\varphi$.

655. Find the lengths of the segments of the polar subtangent, subnormal, tangent and normal, and also the angle between the tangent and the radius vector of the point of tangency in the case of the spiral of Archimedes

$$r = a\varphi$$

at a point with polar angle $\varphi = 2\pi$.

656. Find the lengths of the segments of the polar subtangent, subnormal, tangent, and normal, and also the angle between the tangent and the radius vector in the hyperbolic spiral $r = \frac{a}{\varphi}$ at an arbitrary point $\varphi = \varphi_0$; $r = r_0$.

657. The law of motion of a point on the x-axis is

$$x = 3t - t^3.$$

Find the velocity of the point at $t_0 = 0$, $t_1 = 1$, and $t_2 = 2$ (x is in centimetres and t is in seconds).

658. Moving along the x-axis are two points that have the following laws of motion: $x = 100 + 5t$ and $x = 1/2t^2$, where $t \geqslant 0$. With what speed are these points receding from each other at the time of encounter (x is in centimetres and t is in seconds)?

659. The end-points of a segment $AB = 5$ m are sliding along the coordinate axes OX and OY (Fig. 16). A is moving at 2 m/sec.

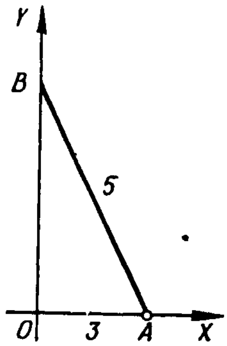

Fig. 16

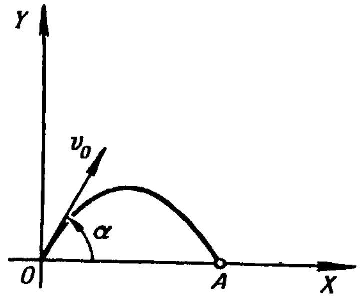

Fig. 17

What is the rate of motion of B when A is at a distance $OA = 3$ m from the origin?

660*. The law of motion of a material point thrown up at an angle α to the horizon with initial velocity v_0 (in the vertical plane OXY in Fig. 17) is given by the formulas (air resistance is disregarded):

$$x = v_0 t \cos \alpha, \qquad y = v_0 t \sin \alpha - \frac{gt^2}{2},$$

where t is the time and g is the acceleration of gravity. Find the trajectory of motion and the distance covered. Also determine the speed of motion and its direction.

661. A point is in motion along a hyperbola $y = \frac{10}{x}$ so that its abscissa x increases uniformly at a rate of 1 unit per second. What is the rate of change of its ordinate when the point passes through (5,2)?

662. At what point of the parabola $y^2 = 18x$ does the ordinate increase at twice the rate of the abscissa?

663. One side of a rectangle, $a = 10$ cm, is of constant length, while the other side, b, increases at a constant rate of 4 cm/sec. At what rate are the diagonal of the rectangle and its area increasing when $b = 30$ cm?

664. The radius of a sphere is increasing at a uniform rate of 5 cm/sec. At what rate are the area of the surface of the sphere and the volume of the sphere increasing when the radius becomes 50 cm?

665. A point is in motion along the spiral of Archimedes

$$r = a\varphi$$

($a = 10$ cm) so that the angular velocity of rotation of its radius vector is constant and equal to 6° per second. Determine the rate of elongation of the radius vector r when $r = 25$ cm.

666. A nonhomogeneous rod AB is 12 cm long. The mass of a part of it, AM, increases with the square of the distance of the moving point M from the end A and is 10 gm when $AM = 2$ cm. Find the mass of the entire rod AB and the linear density at any point M. What is the linear density of the rod at A and B?

Sec. 5. Derivatives of Higher Orders

1°. Definition of higher derivatives. A *derivative of the second order*, or the *second derivative*, of the function $y = f(x)$ is the derivative of its derivative; that is,

$$y'' = (y')'.$$

The second derivative may be denoted as

$$y'', \quad \text{or} \quad \frac{d^2y}{dx^2}, \quad \text{or} \quad f''(x).$$

If $x = f(t)$ is the law of rectilinear motion of a point, then $\frac{d^2x}{dt^2}$ is the acceleration of this motion.

Generally, the *n*th *derivative* of a function $y=f(x)$ is the derivative of a derivative of order $(n-1)$. For the *n*th derivative we use the notation

$$y^{(n)}, \quad \text{or} \quad \frac{d^n y}{dx^n}, \quad \text{or} \quad f^{(n)}(x).$$

Example 1. Find the second derivative of the function

$$y=\ln(1-x).$$

Solution. $y'=\frac{-1}{1-x}$; $y''=\left(\frac{-1}{1-x}\right)'=\frac{1}{(1-x)^2}$.

2°. **Leibniz rule.** If the functions $u=\varphi(x)$ and $v=\psi(x)$ have derivatives up to the *n*th order inclusive, then to evaluate the *n*th derivative of a product of these functions we can use the *Leibniz rule* (or formula):

$$(uv)^{(n)}=u^{(n)}v+n\cdot u^{(n-1)}v'+\frac{n(n-1)}{1\cdot 2}u^{(-2)}v''+\ldots+uv^{(n)}.$$

3°. **Higher-order derivatives of functions represented parametrically.** If

$$\begin{cases} x=\varphi(t), \\ y=\psi(t), \end{cases}$$

then the derivatives $y'_x=\frac{dy}{dx}$, $y''_{xx}=\frac{d^2y}{dx^2}$, ... can successively be calculated by the formulas

$$y'_x=\frac{y'_t}{x'_t}, \quad y''_{xx}=\left(y'_x\right)'_x=\frac{\left(y'_x\right)'_t}{x'_t}, \quad y'''_{xxx}=\frac{\left(y''_{xx}\right)'_t}{x'_t} \text{ and so forth.}$$

For a second derivative we have the formula

$$y''_{xx}=\frac{x'_t y''_{tt}-x''_{tt}y'_t}{(x'_t)^3}.$$

Example 2. Find y'', if

$$\begin{cases} x=a\cos t, \\ y=b\sin t. \end{cases}$$

Solution. We have

$$y'=\frac{(b\sin t)'_t}{(a\cos t)'_t}=\frac{b\cdot\cos t}{-a\sin t}=-\frac{b}{a}\cot t.$$

and

$$y''=\frac{\left(-\frac{b}{a}\cot t\right)'_t}{(a\cos t)'_t}=\frac{-\frac{b}{a}\cdot\frac{-1}{\sin^2 t}}{-a\sin t}=-\frac{b}{a^2\sin^3 t}.$$

A. Higher-Order Derivatives of Explicit Functions

In the examples that follow, find the second derivative of th given function.

667. $y = x^8 + 7x^6 - 5x + 4$.

668. $y = e^{x^2}$.

669. $y = \sin^2 x$.

670. $y = \ln \sqrt[3]{1 + x^2}$.

671. $y = \ln (x + \sqrt{a^2 + x^2})$.

672. $f(x) = (1 + x^2) \cdot \arctan x$.

673. $y = (\arcsin x)^2$.

674. $y = a \cosh \frac{x}{a}$.

675. Show that the function $y = \frac{x^2 + 2x + 2}{2}$ satisfies the differ ential equation $1 + y'^2 = 2yy''$.

676. Show that the function $y = \frac{1}{2} x^2 e^x$ satisfies the differen tial equation $y'' - 2y' + y = e^x$.

677. Show that the function $y = C_1 e^{-x} + C_2 e^{-2x}$ satisfies th equation $y'' + 3y' + 2y = 0$ for all constants C_1 and C_2.

678. Show that the function $y = e^{2x} \sin 5x$ satisfies the equa tion $y'' - 4y' + 29y = 0$.

679. Find y''', if $y = x^3 - 5x^2 + 7x - 2$.

680. Find $f'''(3)$, if $f(x) = (2x - 3)^5$.

681. Find y^{V} of the function $y = \ln(1 + x)$.

682. Find y^{VI} of the function $y = \sin 2x$.

683. Show that the function $y = e^{-x} \cos x$ satisfies the differ ential equation $y^{IV} + 4y = 0$.

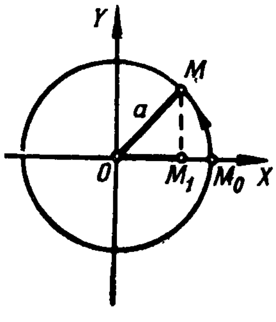

Fig. 18

684. Find $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0$ if $f(x) = e^x \sin x$.

685. The equation of motion of a poin along the x-axis is

$$x = 100 + 5t - 0.001t^3.$$

Find the velocity and the acceleration c the point for times $t_0 = 0$, $t_1 = 1$, an $t_2 = 10$.

686. A point M is in motion around circle $x^2 + y^2 = a^2$ with constant angula velocity ω. Find the law of motion of it projection M_1 on the x-axis if at time $t =$ the point is at $M_0(a, 0)$ (Fig. 18). Find the velocity and the ac celeration of motion of M_1.

What is the velocity and the acceleration of M_1 at the in tial time and when it passes through the origin?

What are the maximum values of the absolute velocity and th absolute acceleration of M_1?

687. Find the nth derivative of the function $y=(ax+b)^n$, where n is a natural number.

688. Find the nth derivatives of the functions:

a) $y=\frac{1}{1-x}$; and b) $y=\sqrt{x}$.

689. Find the nth derivative of the functions:

a) $y=\sin x$; e) $y=\frac{1}{1+x}$;

b) $y=\cos 2x$; f) $y=\frac{1+x}{1-x}$;

c) $y=e^{-3x}$; g) $y=\sin^2 x$;

d) $y=\ln(1+x)$; h) $y=\ln(ax+b)$.

690. Using the Leibniz rule, find $y^{(n)}$, if:

a) $y=x\cdot e^x$; d) $y=\frac{1+x}{\sqrt{x}}$;

b) $y=x^2\cdot e^{-2x}$; e) $y=x^3\ln x$.

c) $y=(1-x^2)\cos x$;

691. Find $f^{(n)}(0)$, if $f(x)=\ln\frac{1}{1-x}$

B. Higher-Order Derivatives of Functions Represented Parametrically and of Implicit Functions

In the following problems find $\frac{d^2y}{dx^2}$.

692. a) $\begin{cases} x=\ln t, \\ y=t^3; \end{cases}$ b) $\begin{cases} x=\arctan t, \\ y=\ln(1+t^2); \end{cases}$ c) $\begin{cases} x=\arcsin t \\ y=\sqrt{1-t^2}. \end{cases}$

693. a) $\begin{cases} x=a\cos t, \\ y=a\sin t; \end{cases}$ c) $\begin{cases} x=a(t-\sin t), \\ y=a(1-\cos t); \end{cases}$

b) $\begin{cases} x=a\cos^3 t, \\ y=a\sin^3 t; \end{cases}$ d) $\begin{cases} x=a(\sin t-t\cos t), \\ y=a(\cos t+t\sin t). \end{cases}$

694. a) $\begin{cases} x=\cos 2t, \\ y=\sin^2 t; \end{cases}$ **695.** a) $\begin{cases} x=\arctan t, \\ y=\frac{1}{2}t^2; \end{cases}$

b) $\begin{cases} x=e^{-at}, \\ y=e^{at}. \end{cases}$ b) $\begin{cases} x=\ln t, \\ y=\frac{1}{1-t}. \end{cases}$

696. Find $\frac{d^2x}{dy^2}$, if $\begin{cases} x=e^t\cos t, \\ y=e^t\sin t. \end{cases}$

697. Find $\frac{d^2y}{dx^2}$ for $t=0$, if $\begin{cases} x=\ln(1+t^2), \\ y=t^2. \end{cases}$

698. Show that y (as a function of x) defined by the equations $x=\sin t$, $y=ae^{t\sqrt{2}}+be^{-t\sqrt{2}}$ for any constants a and b satisfies the differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=2y.$$

In the following examples find $y'''=\frac{d^3y}{dx^3}$.

699. $\begin{cases} x=\sec t, \\ y=\tan t. \end{cases}$ **701.** $\begin{cases} x=e^{-t}, \\ y=t^3. \end{cases}$

700. $\begin{cases} x=e^{-t}\cos t, \\ y=e^{-t}\sin t. \end{cases}$ **702.** Find $\frac{d^ny}{dx^n}$, if $\begin{cases} x=\ln t, \\ y=t^m. \end{cases}$

703. Knowing the function $y=f(x)$, find the derivatives x'', x''' of the inverse function $x=f^{-1}(y)$.

704. Find y'', if $x^2+y^2=1$.

Solution. By the rule for differentiating a composite function we have $2x+2yy'=0$; whence $y'=-\frac{x}{y}$ and $y''=-\left(\frac{x}{y}\right)'_x=-\frac{y-xy'}{y^2}$. Substituting the value of y', we finally get:

$$y''=-\frac{y^2+x^2}{y^3}=-\frac{1}{y^3}.$$

In the following examples it is required to determine the derivative y'' of the function $y=f(x)$ represented implicitly.

705. $y^2=2px$.

706. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

707. $y=x+\arctan y$.

708. Having the equation $y=x+\ln y$, find $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{dy^2}$.

709. Find y'' at the point (1,1) if

$$x^2+5xy+y^2-2x+y-6=0.$$

710. Find y'' at (0,1) if

$$x^4-xy+y^4=1.$$

711. a) The function y is defined implicitly by the equation

$$x^2+2xy+y^2-4x+2y-2=0.$$

Find $\frac{d^3y}{dx^3}$ at the point (1,1).

b) Find $\frac{d^3y}{dx^3}$, if $x^2+y^2=a^2$.

Sec. 6. Differentials of First and Higher Orders

1°. **First-order differential.** The *differential (first-order) of a function* $y=f(x)$ is the principal part of its increment, which part is linear relative to the increment $\Delta x = dx$ of the independent variable x. The differential of a function is equal to the product of its derivative by the differential of the independent variable

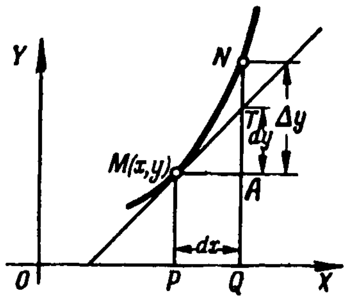

Fig. 19

$$dy = y'dx,$$

whence

$$y' = \frac{dy}{dx}.$$

If MN is an arc of the graph of the function $y=f(x)$ (Fig. 19), MT is the tangent at $M(x, y)$ and

$$PQ = \Delta x = dx,$$

then the increment in the ordinate of the tangent

$$AT = dy$$

and the segment $AN = \Delta y$.

Example 1. Find the increment and the differential of the function $y = 3x^2 - x$.

Solution. First method:

$$\Delta y = 3(x+\Delta x)^2 - (x+\Delta x) - 3x^2 + x$$

or

$$\Delta y = (6x-1)\,\Delta x + 3(\Delta x)^2.$$

Hence,

$$dy = (6x-1)\,\Delta x = (6x-1)\,dx.$$

Second method:

$$y' = 6x-1;\ dy = y'\,dx = (6x-1)\,dx.$$

Example 2. Calculate Δy and dy of the function $y = 3x^2 - x$ for $x=1$ and $\Delta x = 0.01$.

Solution. $\Delta y = (6x-1)\cdot\Delta x + 3(\Delta x)^2 = 5\cdot 0.01 + 3\cdot(0.01)^2 = 0.0503$

and

$$dy = (6x-1)\,\Delta x = 5\cdot 0.01 = 0.0500.$$

2°. Principal properties of differentials.

1) $dc = 0$, where $c = \text{const}$.
2) $dx = \Delta x$, where x is an independent variable.
3) $d(cu) = c\,du$.
4) $d(u \pm v) = du \pm dv$.
5) $d(uv) = u\,dv + v\,du$.
6) $d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2} \quad (v \neq 0)$.
7) $df(u) = f'(u)\,du$.

3°. Applying the differential to approximate calculations. If the increment Δx of the argument x is small in absolute value, then the differential dy of the function $y = f(x)$ and the increment Δy of the function are approximately equal:

$$\Delta y \approx dy,$$

that is,

$$f(x + \Delta x) - f(x) \approx f'(x)\,\Delta x,$$

whence

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx.$$

Example 3. By how much (approximately) does the side of a square change if its area increases from 9 m^2 to 9.1 m^2?

Solution. If x is the area of the square and y is its side, then

$$y = \sqrt{x}.$$

It is given that $x = 9$ and $\Delta x = 0.1$.

The increment Δy in the side of the square may be calculated approximately as follows:

$$\Delta y \approx dy = y'\,\Delta x = \frac{1}{2\sqrt{9}} \cdot 0.1 = 0.016 \text{ m}.$$

4°. Higher-order differentials. A *second-order differential* is the differential of a first-order differential:

$$d^2y = d(dy).$$

We similarly define the *differentials of the third* and higher orders.

If $y = f(x)$ and x is an independent variable, then

$$d^2y = y''(dx)^2,$$
$$d^3y = y'''(dx)^3,$$
$$\cdots\cdots\cdots$$
$$\cdots\cdots\cdots$$
$$d^ny = y^{(n)}(dx)^n.$$

But if $y = f(u)$, where $u = \varphi(x)$, then

$$d^2y = y''(du)^2 + y'\,d^2u,$$
$$d^3y = y'''(du)^3 + 3y''\,du \cdot d^2u + y'\,d^3u$$

and so forth. (Here the primes denote derivatives with respect to u).

712. Find the increment Δy and the differential dy of the function $y = 5x + x^2$ for $x = 2$ and $\Delta x = 0.001$.

713. Without calculating the derivative, find

$$d(1-x^3)$$

for $x=1$ and $\Delta x=-\frac{1}{3}$.

714. The area of a square S with side x is given by $S=x^2$. Find the increment and the differential of this function and explain the geometric significance of the latter.

715. Give a geometric interpretation of the increment and differential of the following functions:

a) the area of a circle, $S=\pi x^2$;

b) the volume of a cube, $v=x^3$.

716. Show that when $\Delta x \to 0$, the increment in the function $y=2^x$, corresponding to an increment Δx in x, is, for any x, equivalent to the expression $2^x \ln 2\,\Delta x$.

717. For what value of x is the differential of the function $y=x^2$ not equivalent to the increment in this function as $\Delta x \to 0$?

718. Has the function $y=|x|$ a differential for $x=0$?

719. Using the derivative, find the differential of the function $y=\cos x$ for $x=\frac{\pi}{6}$ and $\Delta x=\frac{\pi}{36}$.

720. Find the differential of the function

$$y=\frac{2}{\sqrt{x}}$$

for $x=9$ and $\Delta x=-0.01$.

721. Calculate the differential of the function

$$y=\tan x$$

for $x=\frac{\pi}{3}$ and $\Delta x=\frac{\pi}{180}$.

In the following problems find the differentials of the given functions for arbitrary values of the argument and its increment.

722. $y=\frac{1}{x^m}$.

723. $y=\frac{x}{1-x}$.

724. $y=\arcsin\frac{x}{a}$.

725. $y=\arctan\frac{x}{a}$.

726. $y=e^{-x^2}$.

727. $y=x\ln x-x$.

728. $y=\ln\frac{1-x}{1+x}$.

729. $r=\cot\varphi+\operatorname{cosec}\varphi$.

730. $s=\arctan e^t$.

731 Find dy if $x^2+2xy-y^2=a^2$.

Solution. Taking advantage of the invariancy of the form of a differential, we obtain $2x\,dx+2(y\,dx+x\,dy)-2y\,dy=0$

Whence

$$dy=-\frac{x+y}{x-y}dx.$$

In the following examples find the differentials of the functions defined implicitly.

732. $(x+y)^2\cdot(2x+y)^3=1$.

733. $y=e^{-\frac{x}{y}}$.

734. $\ln\sqrt{x^2+y^2}=\arctan\frac{y}{x}$.

735. Find dy at the point (1,2), if $y^3-y=6x^2$.

736. Find the approximate value of sin 31°.

Solution. Putting $x=\text{arc}\,30°=\frac{\pi}{6}$ and $\Delta x=\text{arc}\,1°=\frac{\pi}{180}$, from formula (1) (see 3°) we have $\sin 31°\approx\sin 30°+\frac{\pi}{180}\cos 30°=0.500+0.017\cdot\frac{\sqrt{3}}{2}=0.515$.

737. Replacing the increment of the function by the differential, calculate approximately:

a) $\cos 61°$; d) $\ln 0.9$;
b) $\tan 44°$; e) $\arctan 1.05$.
c) $e^{0.2}$;

738. What will be the approximate increase in the volume of a sphere if its radius $R=15$ cm increases by 2 mm?

739. Derive the approximate formula (for $|\Delta x|$ that are small compared to x)

$$\sqrt{x+\Delta x}\approx\sqrt{x}+\frac{\Delta x}{2\sqrt{x}}.$$

Using it, approximate $\sqrt{5}$, $\sqrt{17}$, $\sqrt{70}$, $\sqrt{640}$.

740. Derive the approximate formula

$$\sqrt[3]{x+\Delta x}\approx\sqrt[3]{x}+\frac{\Delta x}{3\sqrt[3]{x^2}}$$

and find approximate values for $\sqrt[3]{10}$, $\sqrt[3]{70}$, $\sqrt[3]{200}$.

741. Approximate the functions:

a) $y=x^3-4x^2+5x+3$ for $x=1.03$;
b) $f(x)=\sqrt{1+x}$ for $x=0.2$;
c) $f(x)=\sqrt[3]{\frac{1-x}{1+x}}$ for $x=0.1$;
d) $y=e^{1-x^2}$ for $x=1.05$.

742. Approximate tan 45°3′20″.

743. Find the approximate value of arc sin 0.54.

744. Approximate $\sqrt[4]{17}$.

745. Using Ohm's law, $I=\frac{E}{R}$, show that a small change in the current, due to a small change in the resistance, may be found approximately by the formula

$$\Delta I = -\frac{I}{R}\Delta R.$$

746. Show that, in determining the length of the radius, a relative error of 1% results in a relative error of approximately 2% in calculating the area of a circle and the surface of a sphere.

747. Compute d^2y, if $y=\cos 5x$.

Solution. $d^2y = y''(dx^2) = -25\cos 5x\,(dx)^2$.

748. $u=\sqrt{1-x^2}$, find d^2u.

749. $y=\arccos x$, find d^2y.

750. $y=\sin x \ln x$, find d^2y.

751. $z=\frac{\ln x}{x}$, find d^2z.

752. $z=x^2e^{-x}$, find d^3z.

753. $z=\frac{x^4}{2-x}$, find d^4z.

754. $u=3\sin(2x+5)$, find d^nu.

755. $y=e^{x\cos\alpha}\sin(x\sin\alpha)$, find d^ny.

Sec. 7. Mean-Value Theorems

1°. **Rolle's theorem.** If a function $f(x)$ is continuous on the interval $a \leqslant x \leqslant b$, has a derivative $f'(x)$ at every interior point of this interval, and

$$f(a)=f(b),$$

then the argument x has at least one value ξ, where $a<\xi<b$, such that

$$f'(\xi)=0.$$

2°. **Lagrange's theorem.** If a function $f(x)$ is continuous on the interval $a \leqslant x \leqslant b$ and has a derivative at every interior point of this interval, then

$$f(b)-f(a)=(b-a)f'(\xi),$$

where $a<\xi<b$.

3°. **Cauchy's theorem.** If the functions $f(x)$ and $F(x)$ are continuous on the interval $a \leqslant x \leqslant b$ and for $a<x<b$ have derivatives that do not vanish simultaneously, and $F(b) \neq F(a)$, then

$$\frac{f(b)-f(a)}{F(b)-F(a)}=\frac{f'(\xi)}{F'(\xi)}, \quad \text{where } a<\xi<b.$$

756. Show that the function $f(x)=x-x^3$ on the intervals $-1 \leqslant x \leqslant 0$ and $0 \leqslant x \leqslant 1$ satisfies the Rolle theorem. Find the appropriate values of ξ.

Solution. The function $f(x)$ is continuous and differentiable for all values of x, and $f(-1)=f(0)=f(1)=0$. Hence, the Rolle theorem is applicable on the intervals $-1 \leqslant x \leqslant 0$ and $0 \leqslant x < 1$. To find ξ we form the equation $f'(x)=1-3x^2=0$. Whence $\xi_1=-\sqrt{\frac{1}{3}}$; $\xi_2=\sqrt{\frac{1}{3}}$, where $-1<\xi_1<0$ and $0<\xi_2<1$.

757. The function $f(x)=\sqrt[3]{(x-2)^2}$ takes on equal values $f(0)=f(4)=\sqrt[3]{4}$ at the end-points of the interval [0.4]. Does the Rolle theorem hold for this function on [0.4]?

758. Does the Rolle theorem hold for the function

$$f(x)=\tan x$$

on the interval $[0, \pi]$?

759. Let

$$f(x)=x(x+1)(x+2)(x+3).$$

Show that the equation

$$f'(x)=0$$

has three real roots.

760. The equation

$$e^x=1+x$$

obviously has a root $x=0$. Show that this equation cannot have any other real root.

761. Test whether the Lagrange theorem holds for the function

$$f(x)=x-x^3$$

on the interval $[-2,1]$ and find the appropriate intermediate value of ξ.

Solution. The function $f(x)=x-x^3$ is continuous and differentiable for all values of x, and $f'(x)=1-3x^2$ Whence, by the Lagrange formula, we have $f(1)-f(-2)=0-6=[1-(-2)]f'(\xi)$, that is, $f'(\xi)=-2$ Hence, $1-3\xi^2=-2$ and $\xi=\pm 1$; the only suitable value is $\xi=-1$, for which the inequality $-2<\xi<1$ holds

762. Test the validity of the Lagrange theorem and find the appropriate intermediate point ξ for the function $f(x)=x^{4/3}$ on the interval $[-1,1]$.

763. Given a segment of the parabola $y=x^2$ lying between two points $A(1,1)$ and $B(3,9)$, find a point the tangent to which is parallel to the chord AB.

764. Using the Lagrange theorem, prove the formula

$$\sin(x+h)-\sin x=h\cos\xi,$$

where $x<\xi<x+h$.

765. a) For the functions $f(x)=x^2+2$ and $F(x)=x^3-1$ test whether the Cauchy theorem holds on the interval [1,2] and find ξ;

b) do the same with respect to $f(x)=\sin x$ and $F(x)=\cos x$ on the interval $\left[0, \frac{\pi}{2}\right]$.

Sec. 8. Taylor's Formula

If a function $f(x)$ is continuous and has continuous derivatives up to the $(n-1)$th order inclusive on the interval $a \leqslant x \leqslant b$ (or $b \leqslant x \leqslant a$), and there is a finite derivative $f^{(n)}(x)$ at each interior point of the interval, then *Taylor's formula*

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots$$

$$\ldots+\frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{(x-a)^n}{n!}f^{(n)}(\xi),$$

where $\xi=a+\theta(x-a)$ and $0<\theta<1$, holds true on the interval.

In particular, when $a=0$ we have (*Maclaurin's formula*)

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+\frac{x^n}{n!}f^{(n)}(\xi),$$

where $\xi=\theta x$, $0<\theta<1$.

766. Expand the polynomial $f(x)=x^3-2x^2+3x+5$ in positive integral powers of the binomial $x-2$.

Solution. $f'(x)=3x^2-4x+3$; $f''(x)=6x-4$; $f'''(x)=6$; $f^{(n)}(x)=0$ for $n \geqslant 4$. Whence

$$f(2)=11;\ f'(2)=7;\ f''(2)=8;\ f'''(2)=6.$$

Therefore,

$$x^3-2x^2+3x+5=11+(x-2)\cdot 7+\frac{(x-2)^2}{2!}\cdot 8+\frac{(x-2)^3}{3!}\cdot 6$$

or

$$x^3-2x^2+3x+5=11+7(x-2)+4(x-2)^2+(x-2)^3.$$

767. Expand the function $f(x)=e^x$ in powers of $x+1$ to the term containing $(x+1)^3$.

Solution. $f^{(n)}(x)=e^x$ for all n, $f^{(n)}(-1)=\frac{1}{e}$. Hence,

$$e^x=\frac{1}{e}+(x+1)\frac{1}{e}+\frac{(x+1)^2}{2!}\frac{1}{e}+\frac{(x+1)}{3!}\frac{1}{e}+\frac{(x+1)^4}{4!}e^{\xi},$$

where $\xi=-1+\theta(x+1)$; $0<\theta<1$.

768. Expand the function $f(x)=\ln x$ in powers of $x-1$ up to the term with $(x-1)^2$.

769. Expand $f(x)=\sin x$ in powers of x up to the term containing x^3 and to the term containing x^5.

770. Expand $f(x)=e^x$ in powers of x up to the term containing x^{n-1}.

771. Show that $\sin(a+h)$ differs from

$$\sin a+h\cos a$$

by not more than $1/2\,h^2$.

772. Determine the origin of the approximate formulas:

a) $\sqrt{1+x}\approx 1+\frac{1}{2}x-\frac{1}{8}x^2, \quad |x|<1,$

b) $\sqrt[3]{1+x}\approx 1+\frac{1}{3}x-\frac{1}{9}x^2, \quad |x|<1$

and evaluate their errors.

773. Evaluate the error in the formula

$$e\approx 2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}.$$

774. Due to its own weight, a heavy suspended thread lies in a catenary line $y=a\cosh\frac{x}{a}$. Show that for small $|x|$ the shape of the thread is approximately expressed by the parabola

$$y=a+\frac{x^2}{2a}.$$

775*. Show that for $|x|\ll a$, to within $\left(\frac{x}{a}\right)^2$, we have the approximate equality

$$e^{\frac{x}{a}}\approx\sqrt{\frac{a+x}{a-x}}.$$

Sec. 9. The L'Hospital-Bernoulli Rule for Evaluating Indeterminate Forms

1°. **Evaluating the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.** Let the single-valued functions $f(x)$ and $\varphi(x)$ be differentiable for $0<|x-a|<h$; the derivative of one of them does not vanish.

If $f(x)$ and $\varphi(x)$ are both infinitesimals or both infinites as $x\to a$; that is, if the quotient $\frac{f(x)}{\varphi(x)}$, at $x=a$, is one of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=\lim_{x\to a}\frac{f'(x)}{\varphi'(x)}$$

provided that the limit of the ratio of derivatives exists.

The rule is also applicable when $a=\infty$.

If the quotient $\frac{f'(x)}{\varphi'(x)}$ again yields an indeterminate form, at the point $x=a$, of one of the two above-mentioned types and $f'(x)$ and $\varphi'(x)$ satisfy all the requirements that have been stated for $f(x)$ and $\varphi(x)$, we can then pass to the ratio of second derivatives, etc.

However, it should be borne in mind that the limit of the ratio $\frac{f(x)}{\varphi(x)}$ may exist, whereas the ratios of the derivatives do not tend to any limit (see Example 809).

2°. Other indeterminate forms. To evaluate an indeterminate form like $0\cdot\infty$, transform the appropriate product $f_1(x)\cdot f_2(x)$, where $\lim\limits_{x\to a} f_1(x)=0$ and $\lim\limits_{x\to a} f_2(x)=\infty$, into the quotient $\frac{f_1(x)}{\frac{1}{f_2(x)}}$ (the form $\frac{0}{0}$ (or $\frac{f_2(x)}{\frac{1}{f_1(x)}}$ (the form $\frac{\infty}{\infty}$).

In the case of the indeterminate form $\infty-\infty$, one should transform the appropriate difference $f_1(x)-f_2(x)$ into the product $f_1(x)\left[1-\frac{f_2(x)}{f_1(x)}\right]$ and first evaluate the indeterminate form $\frac{f_2(x)}{f_1(x)}$; if $\lim\limits_{x\to a}\frac{f_2(x)}{f_1(x)}=1$, then we reduce the expression to the form

$$\frac{1-\frac{f_2(x)}{f_1(x)}}{\frac{1}{f_1(x)}} \quad \text{(the form } \frac{0}{0}\text{)}.$$

The indeterminate forms 1^∞, 0^0, ∞^0 are evaluated by first taking logarithms and then finding the limit of the logarithm of the power $[f_1(x)]^{f_2(x)}$ (which requires evaluating a form like $0\cdot\infty$).

In certain cases it is useful to combine the L'Hospital rule with the finding of limits by elementary techniques.

Example 1. Compute

$$\lim_{x\to 0}\frac{\ln x}{\cot x} \quad \text{(form } \frac{\infty}{\infty}\text{)}.$$

Solution. Applying the L'Hospital rule we have

$$\lim_{x\to 0}\frac{\ln x}{\cot x}=\lim_{x\to 0}\frac{(\ln x)'}{(\cot x)'}=-\lim_{x\to 0}\frac{\sin^2 x}{x}.$$

We get the indeterminate form $\frac{0}{0}$; however, we do not need to use the L'Hospital rule, since

$$\lim_{x\to 0}\frac{\sin^2 x}{x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\sin x=1\cdot 0=0.$$

We thus finally get

$$\lim_{x\to 0}\frac{\ln x}{\cot x}=0.$$

Example 2. Compute

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right) \text{ (form } \infty-\infty).$$

Reducing to a common denominator, we get

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x} \text{ (form } \frac{0}{0}).$$

Before applying the L'Hospital rule, we replace the denominator of the latter fraction by an equivalent infinitesimal (Ch. 1, Sec. 4) $x^2\sin^2 x \sim x^4$. We obtain

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^4} \text{ (form } \frac{0}{0}).$$

The L'Hospital rule gives

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=\lim_{x\to 0}\frac{2x-\sin 2x}{4x^3}=\lim_{x\to 0}\frac{2-2\cos 2x}{12x^2}.$$

Then, in elementary fashion, we find

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=\lim_{x\to 0}\frac{1-\cos 2x}{6x^2}=\lim_{x\to 0}\frac{2\sin^2 x}{6x^2}=\frac{1}{3}.$$

Example 3. Compute

$$\lim_{x\to 0}(\cos 2x)^{\frac{3}{x^2}} \text{ (form } 1^{\infty})$$

Taking logarithms and applying the L'Hospital rule, we get

$$\lim_{x\to 0}\ln(\cos 2x)^{\frac{3}{x^2}}=\lim_{x\to 0}\frac{3\ln\cos 2x}{x^2}=-6\lim_{x\to 0}\frac{\tan 2x}{2x}=-6.$$

Hence, $\lim_{x\to 0}(\cos 2x)^{\frac{3}{x^2}}=e^{-6}$.

Find the indicated limits of functions in the following examples.

776. $\lim_{x\to 1}\frac{x^3-2x^2-x+2}{x^3-7x+6}$.

Solution. $\lim_{x\to 1}\frac{x^3-2x^2-x+2}{x^3-7x+6}=\lim_{x\to 1}\frac{3x^2-4x-1}{3x^2-7}=\frac{1}{2}$.

777. $\lim_{x\to 0}\frac{x\cos x-\sin x}{x^3}$.

778. $\lim_{x\to 1}\frac{1-x}{1-\sin\frac{\pi x}{2}}$.

779. $\lim_{x\to 0}\frac{\cosh x-1}{1-\cos x}$.

780. $\lim_{x\to 0}\frac{\tan x-\sin x}{x-\sin x}$.

781. $\lim\limits_{x \to \frac{\pi}{4}} \frac{\sec^2 x - 2\tan x}{1+\cos 4x}$.

782. $\lim\limits_{x \to \frac{\pi}{2}} \frac{\tan x}{\tan 5x}$.

783. $\lim\limits_{x \to \infty} \frac{e^x}{x^5}$.

784. $\lim\limits_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}}$.

785. $\lim\limits_{x \to 0} \frac{\frac{\pi}{x}}{\cot \frac{\pi x}{2}}$.

786. $\lim\limits_{x \to 0} \frac{\ln(\sin mx)}{\ln \sin x}$.

787. $\lim\limits_{x \to 0} (1-\cos x)\cot x$.

Solution. $\lim\limits_{x \to 0} (1-\cos x)\cot x = \lim\limits_{x \to 0} \frac{(1-\cos x)\cos x}{\sin x} = \lim\limits_{x \to 0} \frac{(1-\cos x)\cdot 1}{\sin x} =$

$= \lim\limits_{x \to 0} \frac{\sin x}{\cos x} = 0$

788. $\lim\limits_{x \to 1} (1-x)\tan \frac{\pi x}{2}$.

789. $\lim\limits_{x \to 0} \arcsin x \cot x$.

790. $\lim\limits_{x \to 0} (x^n e^{-x})$, $n>0$.

791. $\lim\limits_{x \to \infty} x \sin \frac{a}{x}$.

792. $\lim\limits_{x \to \infty} x^n \sin \frac{a}{x}$, $n>0$.

793. $\lim\limits_{x \to 1} \ln x \ln (x-1)$.

794. $\lim\limits_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$.

Solution. $\lim\limits_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right) = \lim\limits_{x \to 1} \frac{x\ln x - x + 1}{(x-1)\ln x} =$

$= \lim\limits_{x \to 1} \frac{x \cdot \frac{1}{x} + \ln x - 1}{\ln x + \frac{1}{x}(x-1)} = \lim\limits_{x \to 1} \frac{\ln x}{\ln x - \frac{1}{x} + 1} = \lim\limits_{x \to 1} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{2}$.

795. $\lim\limits_{x \to 3} \left(\frac{1}{x-3} - \frac{5}{x^2-x-6} \right)$.

796. $\lim\limits_{x \to 1} \left[\frac{1}{2(1-\sqrt{x})} - \frac{1}{3(1-\sqrt[3]{x})} \right]$.

797. $\lim\limits_{x \to \frac{\pi}{2}} \left(\frac{x}{\cot x} - \frac{\pi}{2\cos x} \right)$.

798. $\lim\limits_{x \to 0} x^x$.

Solution. We have $x^x = y$; $\ln y = x \ln x$; $\lim\limits_{x \to 0} \ln y = \lim\limits_{x \to 0} x \ln x =$

$= \lim\limits_{x \to 0} \frac{\ln x}{\frac{1}{x}} = \lim\limits_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$, whence $\lim\limits_{x \to 0} y = 1$, that is, $\lim\limits_{x \to 0} x^x = 1$.

799. $\lim\limits_{x\to+\infty} x^{\frac{1}{x}}$.

800. $\lim\limits_{x\to 0} x^{\frac{3}{4+\ln x}}$.

801. $\lim\limits_{x\to 0} x^{\sin x}$.

802. $\lim\limits_{x\to 1} (1-x)^{\cos\frac{\pi x}{2}}$.

803. $\lim\limits_{x\to 0} (1+x^2)^{\frac{1}{x}}$.

804. $\lim\limits_{x\to 1} x^{\frac{1}{1-x}}$.

805. $\lim\limits_{x\to 1}\left(\tan\frac{\pi x}{4}\right)^{\tan\frac{\pi x}{2}}$.

806. $\lim\limits_{x\to 0} (\cot x)^{\frac{1}{\ln x}}$.

807. $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}$.

808. $\lim\limits_{x\to 0} (\cot x)^{\sin x}$.

809. Prove that the limits of

a) $\lim\limits_{x\to 0}\dfrac{x^2\sin\frac{1}{x}}{\sin x}=0$;

b) $\lim\limits_{x\to\infty}\dfrac{x-\sin x}{x+\sin x}=1$

cannot be found by the L'Hospital-Bernoulli rule. Find these limits directly.

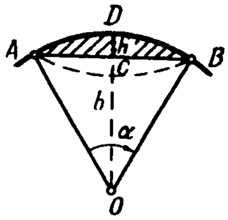

Fig. 20

810*. Show that the area of a circular segment with minor central angle α, which has a chord $AB=b$ and $CD=h$ (Fig. 20), is approximately

$$S\approx\frac{2}{3}bh$$

with an arbitrarily small relative error when $\alpha\to 0$.

Chapter III

THE EXTREMA OF A FUNCTION AND THE GEOMETRIC APPLICATIONS OF A DERIVATIVE

Sec. 1. The Extrema of a Function of One Argument

1°. Increase and decrease of functions. The function $y=f(x)$ is called *increasing (decreasing)* on some interval if, for any points x_1 and x_2 which belong to this interval, from the inequality $x_1<x_2$ we get the inequality $f(x_1)<f(x_2)$ (Fig. 21*a*) [$f(x_1)>f(x_2)$ (Fig. 21*b*)]. If $f(x)$ is continuous on the interval $[a, b]$ and $f'(x)>0$ [$f'(x)<0$] for $a<x<b$, then $f(x)$ increases (decreases) on the interval $[a, b]$.

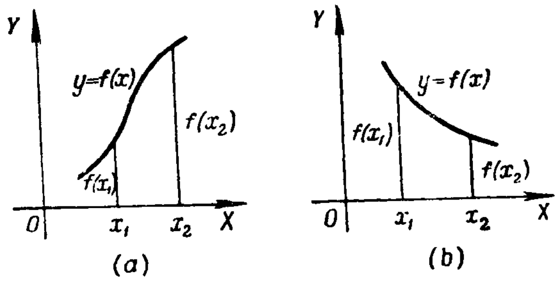

Fig. 21

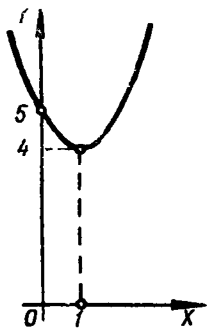

Fig. 22

In the simplest cases, the domain of definition of $f(x)$ may be subdivided into a finite number of intervals of increase and decrease of the function (*intervals of monotonicity*). These intervals are bounded by critical points x [where $f'(x)=0$ or $f'(x)$ does not exist].

Example 1. Test the following function for increase and decrease:

$$y=x^2-2x+5.$$

Solution. We find the derivative

$$y'=2x-2=2(x-1).$$

Whence $y'=0$ for $x=1$. On a number scale we get two intervals of monotonicity: $(-\infty, 1)$ and $(1, +\infty)$. From (1) we have: 1) if $-\infty<x<1$, then $y'<0$, and, hence, the function $f(x)$ decreases in the interval $(-\infty, 1)$; 2) if $1<x<+\infty$, then $y'>0$, and, hence, the function $f(x)$ increases in the interval $(1, +\infty)$ (Fig. 22).

Example 2. Determine the intervals of increase and decrease of the function

$$y=\frac{1}{x+2}.$$

Solution. Here, $x=-2$ is a discontinuity of the function and $y'=-\frac{1}{(x+2)^2}<0$ for $x\neq -2$. Hence, the function y decreases in the intervals $-\infty<x<-2$ and $-2<x<+\infty$.

Example 3. Test the following function for increase or decrease:

$$y=\frac{1}{5}x^5-\frac{1}{3}x^3.$$

Solution Here,

$$y'=x^4-x^2. \qquad (2)$$

Solving the equation $x^4-x^2=0$, we find the points $x_1=-1$, $x_2=0$, $x_3=1$ at which the derivative y' vanishes. Since y' can change sign only when passing through points at which it vanishes or becomes discontinuous (in the given case, y' has no discontinuities), the derivative in each of the intervals $(-\infty, -1)$, $(-1, 0)$, $(0,1)$ and $(1, +\infty)$ retains its sign; for this reason, the function under investigation is monotonic in each of these intervals. To determine in which of the indicated intervals the function increases and in which it decreases, one has to determine the sign of the derivative in each of the intervals. To determine what the sign of y' is in the interval $(-\infty, -1)$, it is sufficient to determine the sign of y' at some point of the interval; for example, taking $x=-2$, we get from (2) $y'=12>0$, hence, $y'>0$ in the interval $(-\infty, -1)$ and the function in this interval increases Similarly, we find that $y'<0$ in the interval $(-1, 0)$ (as a check, we can take $x=-\frac{1}{2}$), $y'<0$ in the interval $(0,1)$ (here, we can use $x=1/2$) and $y'>0$ in the interval $(1, +\infty)$.

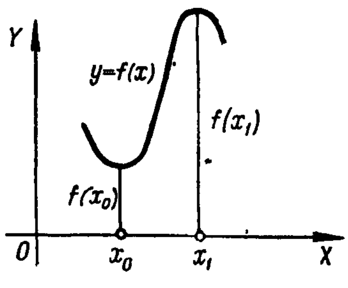

Fig 23

Thus, the function being tested increases in the interval $(-\infty, -1)$, decreases in the interval $(-1, 1)$ and again increases in the interval $(1, +\infty)$.

2°. **Extremum of a function.** If there exists a two-sided neighbourhood of a point x_0 such that for every point $x\neq x_0$ of this neighbourhood we have the inequality $f(x)>f(x_0)$, then the point x_0 is called the *minimum point* of the function $y=f(x)$, while the number $f(x_0)$ is called the *minimum* of the function $y=f(x)$. Similarly, if for any point $x\neq x_1$ of some neighbourhood of the point x_1, the inequality $f(x)<f(x_1)$ is fulfilled, then x_1 is called the *maximum point* of the function $f(x)$, and $f(x_1)$ is the *maximum* of the function (Fig. 23). The minimum point or maximum point of a function is its *extremal point* (bending point), and the minimum or maximum of a function is called the *extremum* of the function. If x_0 is an extremal point of the function $f(x)$, then $f'(x_0)=0$, or $f'(x_0)$ does not exist (necessary condition for the existence of an extremum). The converse is not true: points at which $f'(x)=0$, or $f'(x)$, does not exist (*critical points*) are not necessarily extremal points of the function $f(x)$.

The sufficient conditions for the existence and absence of an extremum of a continuous function $f(x)$ are given by the following rules:

1. If there exists a neighbourhood $(x_0-\delta,\ x_0+\delta)$ of a critical point x_0 such that $f'(x)>0$ for $x_0-\delta<x<x_0$ and $f'(x)<0$ for $x_0<x<x_0+\delta$, then x_0 is the maximum point of the function $f(x)$; and if $f'(x)<0$ for $x_0-\delta<x<x_0$ and $f'(x)>0$ for $x_0<x<x_0+\delta$, then x_0 is the minimum point of the function $f(x)$.

Finally, if there is some positive number δ such that $f'(x)$ retains its sign unchanged for $0<|x-x_0|<\delta$, then x_0 is not an extremal point of the function $f(x)$.

2. If $f'(x_0)=0$ and $f''(x_0)<0$, then x_0 is the maximum point; if $f'(x_0)=0$ and $f''(x_0)>0$, then x_0 is the minimum point; but if $f'(x_0)=0$, $f''(x_0)=0$, and $f'''(x_0)\neq 0$, then the point x_0 is not an extremal point.

More generally: let the first of the derivatives (not equal to zero at the point x_0) of the function $f(x)$ be of the order k. Then, if k is even, the point x_0 is an extremal point, namely, the maximum point, if $f^{(k)}(x_0)<0$; and it is the minimum point, if $f^{(k)}(x_0)>0$ But if k is odd, then x_0 is not an extremal point.

Example 4. Find the extrema of the function

$$y=2x+3\sqrt[3]{x^2}.$$

Solution. Find the derivative

$$y'=2+\frac{2}{\sqrt[3]{x}}=\frac{2}{\sqrt[3]{x}}(\sqrt[3]{x}+1). \tag{3}$$

Equating the derivative y' to zero, we get:

$$\sqrt[3]{x}+1=0.$$

Whence, we find the critical point $x_1=-1$. From formula (3) we have: if $x=-1-h$, where h is a sufficiently small positive number, then $y'>0$; but if $x=-1+h$, then $y'<0$*). Hence, $x_1=-1$ is the maximum point of the function y, and $y_{\max}=1$.

Equating the denominator of the expression of y' in (3) to zero, we get

$$\sqrt[3]{x}=0;$$

whence we find the second critical point of the function $x_2=0$, where there is no derivative y' For $x=-h$, we obviously have $y'<0$; for $x=h$ we have $y'>0$. Consequently, $x_2=0$ is the minimum point of the function y, and $y_{\min}=0$ (Fig. 24). It is also possible to test the behaviour of the function at the point $x=-1$ by means of the second derivative

$$y''=-\frac{2}{3x\sqrt[3]{x}}.$$

Here, $y''<0$ for $x_1=-1$ and, hence, $x_1=-1$ is the maximum point of the function.

3°. **Greatest and least values.** The least (greatest) value of a continuous function $f(x)$ on a given interval $[a, b]$ is attained either at the critical points of the function or at the end-points of the interval $[a, b]$.

*) If it is difficult to determine the sign of the derivative y', one can calculate arithmetically by taking for h a sufficiently small positive number.

Example 5. Find the greatest and least values of the function

$$y = x^3 - 3x + 3$$

on the interval $-1^1/_2 \leqslant x \leqslant 2^1/_2$.

Solution. Since

$$y' = 3x^2 - 3,$$

it follows that the critical points of the function y are $x_1 = -1$ and $x_2 = 1$.

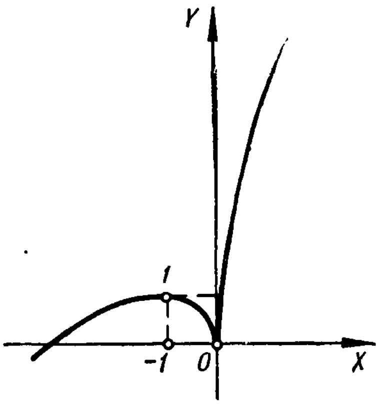

Fig. 24

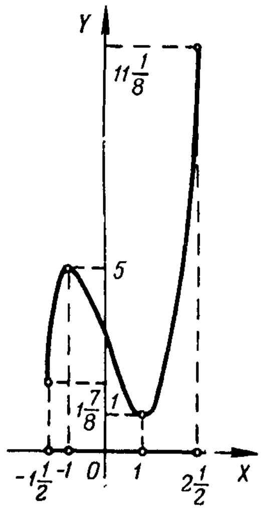

Fig. 25

Comparing the values of the function at these points and the values of the function at the end-points of the given interval

$$y(-1) = 5;\ y(1) = 1;\ y\left(-1\frac{1}{2}\right) = 4\frac{1}{8};\quad y\left(2\frac{1}{2}\right) = 11\frac{1}{8},$$

we conclude (Fig. 25) that the function attains its least value, $m = 1$, at the point $x = 1$ (at the minimum point), and the greatest value $M = 11\frac{1}{8}$ at the point $x = 2^1/_2$ (at the right-hand end-point of the interval).

Determine the intervals of decrease and increase of the functions:

811. $y = 1 - 4x - x^2$.

812. $y = (x-2)^2$.

813. $y = (x+4)^3$.

814. $y = x^2(x-3)$.

815. $y = \frac{x}{x-2}$.

816. $y = \frac{1}{(x-1)^2}$.

817. $y = \frac{x}{x^2 - 6x - 16}$.

818. $y = (x-3)\sqrt{x}$.

819. $y=\frac{x}{3}-\sqrt[3]{x}$.

820. $y=x+\sin x$.

821. $y=x\ln x$.

822. $y=\arcsin(1+x)$.

823. $y=2e^{x^2-4x}$.

824. $y=2^{\frac{1}{x-a}}$.

825. $y=\frac{e^x}{x}$.

Test the following functions for extrema:

826. $y=x^2+4x+6$.

Solution. We find the derivative of the given function, $y'=2x+4$. Equating y' to zero, we get the critical value of the argument $x=-2$. Since $y'<0$ when $x<-2$, and $y'>0$ when $x>-2$, it follows that $x=-2$ is the minimum point of the function, and $y_{min}=2$. We get the same result by utilizing the sign of the second derivative at the critical point $y''=2>0$.

827. $y=2+x-x^2$.

828. $y=x^3-3x^2+3x+2$.

829. $y=2x^3+3x^2-12x+5$.

Solution. We find the derivative

$$y'=6x^2+6x-12=6(x^2+x-2).$$

Equating the derivative y' to zero, we get the critical points $x_1=-2$ and $x_2=1$. To determine the nature of the extremum, we calculate the second derivative $y''=6(2x+1)$. Since $y''(-2)<0$, it follows that $x_1=-2$ is the maximum point of the function y, and $y_{max}=25$. Similarly, we have $y''(1)>0$; therefore, $x_2=1$ is the minimum point of the function y and $y_{min}=-2$.

830. $y=x^2(x-12)^2$.

831. $y=x(x-1)^2(x-2)^3$.

832. $y=\frac{x^3}{x^2+3}$.

833. $y=\frac{x^2-2x+2}{x-1}$.

834. $y=\frac{(x-2)(8-x)}{x^2}$.

835. $y=\frac{16}{x(4-x^2)}$.

836. $y=\frac{4}{\sqrt{x^2+8}}$.

837. $y=\frac{x}{\sqrt[3]{x^2-4}}$.

838. $y=\sqrt[3]{(x^2-1)^2}$.

839. $y=2\sin 2x+\sin 4x$.

840. $y=2\cos\frac{x}{2}+3\cos\frac{x}{3}$.

841. $y=x-\ln(1+x)$.

842. $y=x\ln x$.

843. $y=x\ln^2 x$.

844. $y=\cosh x$.

845. $y=xe^x$.

846. $y=x^2e^{-x}$.

847. $y=\frac{e^x}{x}$.

848. $y=x-\arctan x$.

Determine the least and greatest values of the functions on the indicated intervals (if the interval is not given, determine the greatest and least values of the function throughout the domain of definition).

849. $y=\frac{x}{1+x^2}$.

850. $y=\sqrt{x(10-x)}$.

851. $y=\sin^4 x+\cos^4 x$.

852. $y=\arccos x$.

853. $y=x^3$ on the interval $[-1,3]$.

854. $y=2x^3+3x^2-12x+1$
a) on the interval $[-1,5]$;
b) on the interval $[-10,12]$.

855. Show that for positive values of x we have the inequality

$$x+\frac{1}{x}\geqslant 2.$$

856. Determine the coefficients p and q of the quadratic trinomial $y=x^2+px+q$ so that this trinomial should have a minimum $y=3$ when $x=1$. Explain the result in geometrical terms.

857. Prove the inequality

$$e^x > 1+x \quad \text{when } x\neq 0.$$

Solution. Consider the function

$$f(x)=e^x-(1+x).$$

In the usual way we find that this function has a single minimum $f(0)=0$. Hence,

$$f(x)>f(0) \quad \text{when } x\neq 0,$$
$$\text{and so } e^x>1+x \quad \text{when } x\neq 0,$$

as we set out to prove.

Prove the inequalities:

858. $x-\frac{x^3}{6}<\sin x<x$ when $x>0$.

859. $\cos x>1-\frac{x^2}{2}$ when $x\neq 0$.

860. $x-\frac{x^2}{2}<\ln(1+x)<x$ when $x>0$.

861. Separate a given positive number a into two summands such that their product is the greatest possible.

862. Bend a piece of wire of length l into a rectangle so that the area of the latter is greatest.

863. What right triangle of given perimeter $2p$ has the greatest area?

864. It is required to build a rectangular playground so that it should have a wire net on three sides and a long stone wall on the fourth. What is the optimum (in the sense of area) shape of the playground if l metres of wire netting are available?

865. It is required to make an open rectangular box of greatest capacity out of a square sheet of cardboard with side a by cutting squares at each of the angles and bending up the ends of the resulting cross-like figure.

866. An open tank with a square base must have a capacity of v litres. What size will it be if the least amount of tin is used?

867. Which cylinder of a given volume has the least overall surface?

868. In a given sphere inscribe a cylinder with the greatest volume.

869. In a given sphere inscribe a cylinder having the greatest lateral surface.

870. In a given sphere inscribe a cone with the greatest volume.

871. Inscribe in a given sphere a right circular cone with the greatest lateral surface.

872. About a given cylinder circumscribe a right cone of least volume (the planes and centres of their circular bases coincide).

873. Which of the cones circumscribed about a given sphere has the least volume?

874. A sheet of tin of width a has to be bent into an open cylindrical channel (Fig. 26). What should the central angle φ be so that the channel will have maximum capacity?

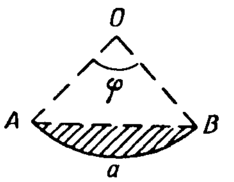

Fig. 26

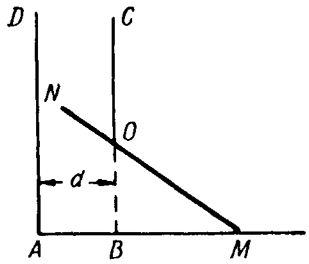

Fig. 27

875. Out of a circular sheet cut a sector such that when made into a funnel it will have the greatest possible capacity.

876. An open vessel consists of a cylinder with a hemisphere at the bottom; the walls are of constant thickness. What will the dimensions of the vessel be if a minimum of material is used for a given capacity?

877. Determine the least height $h = OB$ of the door of a vertical tower $ABCD$ so that this door can pass a rigid rod MN of length l, the end of which, M, slides along a horizontal straight line AB. The width of the tower is $d < l$ (Fig. 27).

878. A point $M_0(x_0, y_0)$ lies in the first quadrant of a coordinate plane. Draw a straight line through this point so that the triangle which it forms with the positive semi-axes is of least area.

879. Inscribe in a given ellipse a rectangle of largest area with sides parallel to the axes of the ellipse.

880. Inscribe a rectangle of maximum area in a segment of the parabola $y^2=2px$ cut off by the straight line $x=2a$.

881. On the curve $y=\frac{1}{1+x^2}$ find a point at which the tangent forms with the x-axis the greatest (in absolute value) angle.

882. A messenger leaving A on one side of a river has to get to B on the other side. Knowing that the velocity along the bank is k times that on the water, determine the angle at which the messenger has to cross the river so as to reach B in the shortest possible time. The width of the river is h and the distance between A and B along the bank is d.

883. On a straight line $AB=a$ connecting two sources of light A (of intensity p) and B (of intensity q), find the point M that receives least light (the intensity of illumination is inversely proportional to the square of the distance from the light source).

884. A lamp is suspended above the centre of a round table of radius r. At what distance should the lamp be above the table so that an object on the edge of the table will get the greatest illumination? (The intensity of illumination is directly proportional to the cosine of the angle of incidence of the light rays and is inversely proportional to the square of the distance from the light source.)

885. It is required to cut a beam of rectangular cross-section out of a round log of diameter d. What should the width x and the height y be of this cross-section so that the beam will offer maximum resistance a) to compression and b) to bending?

Note. The resistance of a beam to compression is proportional to the area of its cross-section, to bending—to the product of the width of the cross-section by the square of its height.

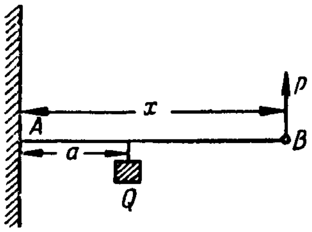

Fig. 2

886. A homogeneous rod AB, which can rotate about a point A (Fig. 28), is carrying a load Q kilograms at a distance of a cm from A and is held in equilibrium by a vertical force P applied to the free end B of the rod. A linear centimetre of the rod weighs q kilograms. Determine the length of the rod x so that the force P should be least, and find P_{min}.

887*. The centres of three elastic spheres A, B, C are situated on a single straight line. Sphere A of mass M moving with velocity v strikes B, which, having acquired a certain velocity, strikes C of mass m. What mass should B have so that C will have the greatest possible velocity?

888. N identical electric cells can be formed into a battery in different ways by combining n cells in series and then combining the resulting groups $\left(\text{the number of groups is } \frac{N}{n}\right)$ in parallel. The current supplied by this battery is given by the formula

$$I=\frac{Nn\mathcal{E}}{NR+n^2r},$$

where $\mathcal{E}$ is the electromotive force of one cell, r is its internal resistance, and R is its external resistance.

For what value of n will the battery produce the greatest current?

889. Determine the diameter y of a circular opening in the body of a dam for which the discharge of water per second Q will be greatest, if $Q=cy\sqrt{h-y}$, where h is the depth of the lowest point of the opening (h and the empirical coefficient c are constant).

890. If $x_1, x_2, \ldots, x_n$ are the results of measurements of equal precision of a quantity x, then its most probable value will be that for which the sum of the squares of the errors

$$\sigma=\sum_{i=1}^{n}(x-x_i)^2$$

is of least value (the principle of least squares).

Prove that the most probable value of x is the arithmetic mean of the measurements.

Sec. 2. The Direction of Concavity. Points of Inflection

1°. The concavity of the graph of a function. We say that the graph of a differentiable function $y=f(x)$ is *concave down* in the interval (a,b) [*concave up* in the interval (a_1,b_1)] if for $a<x<b$ the arc of the curve is below (or for $a_1<x<b_1$, above) the tangent drawn at any point of the interval (a,b) or of the interval (a_1,b_1)] (Fig. 29). A sufficient condition for the concavity downwards (upwards) of a graph $y=f(x)$ is that the following inequality be fulfilled in the appropriate interval:

$$f''(x)<0 \quad [f''(x)>0].$$

2°. Points of inflection. A point $[x_0, f(x_0)]$ at which the direction of concavity of the graph of some function changes is called a *point of inflection* (Fig. 29).

For the abscissa of the point of inflection x_0 of the graph of a function $y=f(x)$ there is no second derivative $f''(x_0)=0$ or $f''(x_0)$. Points at which $f''(x)=0$ or $f''(x)$ does not exist are called *critical points of the second kind*. The critical point of the second kind x_0 is the abscissa of the point of inflection if $f''(x)$ retains constant signs in the intervals $x_0-\delta<x<x_0$ and $x_0<x<x_0+\delta$, where δ is some positive number; provided these signs are opposite. And it is not a point of inflection if the signs of $f''(x)$ are the same in the above-indicated intervals.

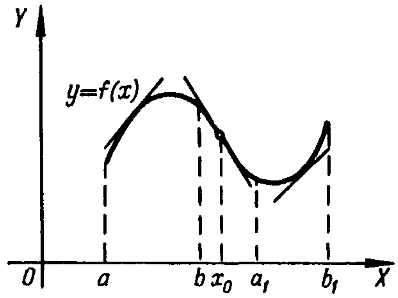

Fig. 29

Example 1. Determine the intervals of concavity and convexity and also the points of inflection of the Gaussian curve

$$y=e^{-x^2}.$$

Solution. We have

$$y'=-2xe^{-x}$$

and

$$y''=(4x^2-2)\,e^{-x^2}.$$

Equating the second derivative y'' to zero, we find the critical points of the second kind

$$x_1=-\frac{1}{\sqrt{2}} \quad \text{and} \quad x_2=\frac{1}{\sqrt{2}}.$$

These points divide the number scale $-\infty<x<+\infty$ into three intervals: I $(-\infty, x_1)$, II (x_1, x_2), and III $(x_2, +\infty)$. The signs of y'' will be, respec-

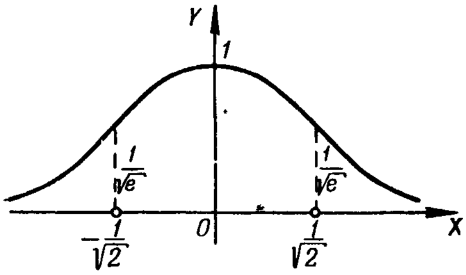

Fig. 30

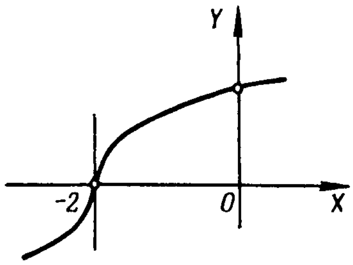

Fig. 31

tively, $+$, $-$, $+$ (this is obvious if, for example, we take one point in each of the intervals and substitute the corresponding values of x into y'') Therefore: 1) the curve is concave up when $-\infty<x<-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}<x<+\infty$; 2) the curve is concave down when $-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$. The points $\left(\frac{\pm 1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$ are points of inflection (Fig. 30).

It will be noted that due to the symmetry of the Gaussian curve about the y-axis, it would be sufficient to investigate the sign of the concavity of this curve on the semiaxis $0<x<+\infty$ alone.

Example 2. Find the points of inflection of the graph of the function

$$y=\sqrt[3]{x+2}.$$

Solution. We have:

$$y''=-\frac{2}{9}(x+2)^{-\frac{5}{3}}=\frac{-2}{9\sqrt[3]{(x+2)^5}}. \qquad (1)$$

It is obvious that y'' does not vanish anywhere.

Equating to zero the denominator of the fraction on the right of (1), we find that y'' does not exist for $x=-2$. Since $y''>0$ for $x<-2$ and $y''<0$ for $x>-2$, it follows that $(-2,0)$ is the point of inflection (Fig. 31). The tangent at this point is parallel to the axis of ordinates, since the first derivative y' is infinite at $x=-2$.

Find the intervals of concavity and the points of inflection of the graphs of the following functions:

891. $y=x^3-6x^2+12x+4$.

892. $y=(x+1)^4$.

893. $y=\frac{1}{x+3}$.

894. $y=\frac{x^3}{x^2+12}$.

895. $y=\sqrt[3]{4x^3-12x}$.

896. $y=\cos x$.

897. $y=x-\sin x$.

898. $y=x^2\ln x$.

899. $y=\arctan x-x$.

900. $y=(1+x^2)e^x$.

Sec. 3. Asymptotes

1°. **Definition.** If a point (x,y) is in continuous motion along a curve $y=f(x)$ in such a way that at least one of its coordinates approaches infinity (and at the same time the distance of the point from some straight line tends to zero), then this straight line is called an *asymptote* of the curve.

2°. **Vertical asymptotes.** If there is a number a such that

$$\lim_{x\to a} f(x)=\pm\infty,$$

then the straight line $x=a$ is an asymptote (*vertical asymptote).*

3° **Inclined asymptotes.** If there are limits

$$\lim_{x\to+\infty}\frac{f(x)}{x}=k_1$$

and

$$\lim_{x\to+\infty}[f(x)-k_1x]=b_1,$$

then the straight line $y=k_1x+b_1$ will be an asymptote (a *right inclined asymptote* or, when $k_1=0$, a *right horizontal asymptote*).

If there are limits

$$\lim_{x\to-\infty}\frac{f(x)}{x}=k_2$$

and

$$\lim_{x\to-\infty}[f(x)-k_2x]=b_2,$$

then the straight line $y=k_2x+b_2$ is an asymptote (a *left inclined asymptote* or, when $k_2=0$, a *left horizontal asymptote*). The graph of the function $y=f(x)$ (we assume the function is single-valued) cannot have more than one right (inclined or horizontal) and more than one left (inclined or horizontal) asymptote.

Example 1. Find the asymptotes of the curve

$$y=\frac{x^2}{\sqrt{x^2-1}}.$$

Solution. Equating the denominator to zero, we get two vertical asymptotes:

$$x=-1 \quad\text{and}\quad x=1.$$

We seek the inclined asymptotes. For $x\to+\infty$ we obtain

$$k_1=\lim_{x\to+\infty}\frac{y}{x}=\lim_{x\to+\infty}\frac{x^2}{x\sqrt{x^2-1}}=1,$$

$$b_1=\lim_{x\to+\infty}(y-x)=\lim_{x\to+\infty}\frac{x^2-x\sqrt{x^2-1}}{\sqrt{x^2-1}}=0,$$

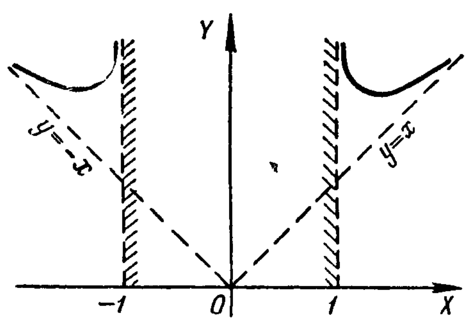

Fig. 32

hence, the straight line $y=x$ is the right asymptote. Similarly, when $x\to-\infty$, we have

$$k_2=\lim_{x\to-\infty}\frac{y}{x}=-1;$$

$$b_2=\lim_{x\to-\infty}(y+x)=0.$$

Thus, the left asymptote is $y=-x$ (Fig. 32). Testing a curve for asymptotes is simplified if we take into consideration the symmetry of the curve.

Example 2. Find the asymptotes of the curve

$$y=x+\ln x.$$

Solution. Since

$$\lim_{x\to+0} y=-\infty,$$

the straight line $x=0$ is a vertical asymptote (lower). Let us now test the curve only for the inclined right asymptote (since $x>0$).

We have:

$$k=\lim_{x\to+\infty}\frac{y}{x}=1,$$

$$b=\lim_{x\to+\infty}(y-x)=\lim_{x\to+\infty}\ln x=\infty.$$

Hence, there is no inclined asymptote.

If a curve is represented by the parametric equations $x=\varphi(t)$, $y=\psi(t)$, then we first test to find out whether there are any values of the parameter t for which one of the functions $\varphi(t)$ or $\psi(t)$ becomes infinite, while the other remains finite. When $\varphi(t_0)=\infty$ and $\psi(t_0)=c$, the curve has a horizontal asymptote $y=c$. When $\psi(t_0)=\infty$ and $\varphi(t_0)=c$, the curve has a vertical asymptote $x=c$.

If $\varphi(t_0)=\psi(t_0)=\infty$ and

$$\lim_{t\to t_0}\frac{\psi(t)}{\varphi(t)}=k;\ \lim_{t\to t_0}[\psi(t)-k\varphi(t)]=b,$$

then the curve has an inclined asymptote $y=kx+b$.

If the curve is represented by a polar equation $r=f(\varphi)$, then we can find its asymptotes by the preceding rule after transforming the equation of the curve to the parametric form by the formulas $x=r\cos\varphi=f(\varphi)\cos\varphi$; $y=r\sin\varphi=f(\varphi)\sin\varphi$.

Find the asymptotes of the following curves:

901. $y=\frac{1}{(x-2)^2}$.

902. $y=\frac{x}{x^2-4x+3}$.

903. $y=\frac{x^2}{x^2-4}$.

904. $y=\frac{x^3}{x^2+9}$.

905. $y=\sqrt{x^2-1}$.

906. $y=\frac{x}{\sqrt{x^2+3}}$.

907. $y=\frac{x^2+1}{\sqrt{x^2-1}}$.

908. $y=x-2+\frac{x^2}{\sqrt{x^2+9}}$.

909. $y=e^{-x^2}+2$.

910. $y=\frac{1}{1-e^x}$.

911. $y=e^{\frac{1}{x}}$.

912. $y=\frac{\sin x}{x}$.

913. $y=\ln(1+x)$.

914. $x=t;\ y=t+2\arctan t$.

915. Find the asymptote of the hyperbolic spiral $r=\frac{a}{\varphi}$.

Sec. 4. Graphing Functions by Characteristic Points

In constructing the graph of a function, first find its domain of definition and then determine the behaviour of the function on the boundary of this domain. It is also useful to note any peculiarities of the function (if there are any), such as symmetry, periodicity, constancy of sign, monotonicity, etc.

Then find any points of discontinuity, bending points, points of inflection, asymptotes, etc. These elements help to determine the general nature of the graph of the function and to obtain a mathematically correct outline of it.

Example 1. Construct the graph of the function

$$y=\frac{x}{\sqrt[3]{x^2-1}}.$$

Solution. a) The function exists everywhere except at the points $x=\pm 1$. The function is odd, and therefore the graph is symmetric about the point $O(0, 0)$. This simplifies construction of the graph

b) The discontinuities are $x=-1$ and $x=1$; and $\lim\limits_{x\to 1\pm 0} y=\pm\infty$ and $\lim\limits_{x\to -1\pm 0} y=\pm\infty$; hence, the straight lines $x=\pm 1$ are vertical asymptotes of the graph.

c) We seek inclined asymptotes, and find

$$k_1=\lim_{x\to+\infty}\frac{y}{x}=0,$$

$$b_1=\lim_{x\to+\infty} y=\infty,$$

thus, there is no right asymptote. From the symmetry of the curve it follows that there is no left-hand asymptote either.

d) We find the critical points of the first and second kinds, that is, points at which the first (or, respectively, the second) derivative of the given function vanishes or does not exist.

We have:

$$y'=\frac{x^2-3}{3\sqrt[3]{(x^2-1)^4}}, \tag{1}$$

$$y''=\frac{2x(9-x^2)}{9\sqrt[3]{(x^2-1)^7}}. \tag{2}$$

The derivatives y' and y'' are nonexistent only at $x=\pm 1$, that is, only at points where the function y itself does not exist; and so the critical points are only those at which y' and y'' vanish.

From (1) and (2) it follows that

$$y'=0 \quad \text{when } x=\pm\sqrt{3};$$
$$y''=0 \quad \text{when } x=0 \text{ and } x=\pm 3.$$

Thus, y' retains a constant sign in each of the intervals $(-\infty, -\sqrt{3})$, $(-\sqrt{3}, -1)$, $(-1, 1)$, $(1, \sqrt{3})$ and $(\sqrt{3}, +\infty)$, and y''—in each of the intervals $(-\infty, -3)$, $(-3, -1)$, $(-1, 0)$, $(0, 1)$, $(1, 3)$ and $(3, +\infty)$.

To determine the signs of y' (or, respectively, y'') in each of the indicated intervals, it is sufficient to determine the sign of y' (or y'') at some one point of each of these intervals.

It is convenient to tabulate the results of such an investigation (Table I), calculating also the ordinates of the characteristic points of the graph of the function. It will be noted that due to the oddness of the function y, it is enough to calculate only for $x \geqslant 0$; the left-hand half of the graph is constructed by the principle of odd symmetry.

Table I

x	0	(0, 1)	1	$(1, \sqrt{3})$	$\sqrt{3} \approx 1.73$	$(\sqrt{3}, 3)$	3	$(3, +\infty)$
y	0	−	$\pm\infty$	+	$\frac{\sqrt{3}}{\sqrt[3]{2}} \approx 1.37$	+	1.5	+
y'	−	−	non-exist	−	0	+	+	+
y''	0	−	non-exist	+	+	+	0	−
Conclusions	Point of inflection	Function decreases, graph is concave down	Discontinuit,	Function decreases, graph is concave up	Min. point	Function increases; graph is concave up	Point of inflection	Function increases; graph is concave down

e) Using the results of the investigation, we construct the graph of the function (Fig 33).

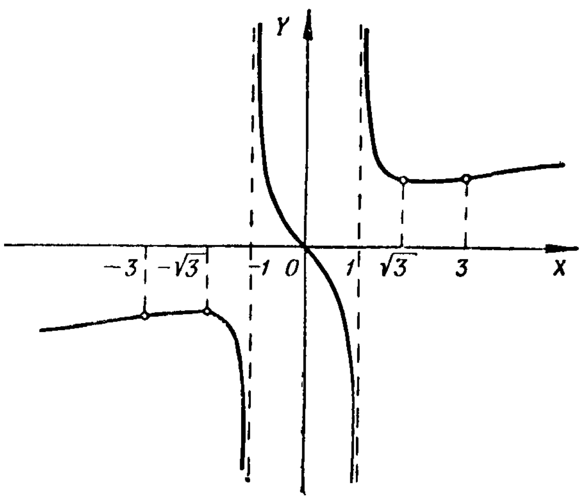

Fig. 33

Example 2. Graph the function

$$y=\frac{\ln x}{x}.$$

Solution. a) The domain of definition of the function is $0<x<+\infty$.

b) There are no discontinuities in the domain of definition, but as we approach the boundary point ($x=0$) of the domain of definition we have

$$\lim_{x\to 0} y=\lim_{x\to 0}\frac{\ln x}{x}=-\infty.$$

Hence, the straight line $x=0$ (ordinate axis) is a vertical asymptote.

c) We seek the right asymptote (there is no left asymptote, since x cannot tend to $-\infty$):

$$k=\lim_{x\to+\infty}\frac{y}{x}=0;$$

$$b=\lim_{x\to+\infty} y=0.$$

The right asymptote is the axis of abscissas: $y=0$.

d) We find the critical points; and have

$$y'=\frac{1-\ln x}{x^2},$$

$$y''=\frac{2\ln x-3}{x^3};$$

y' and y'' exist at all points of the domain of definition of the function and

$$y'=0 \quad \text{when } \ln x=1, \text{ that is, when } x=e;$$

$$y''=0 \quad \text{when } \ln x=\frac{3}{2}, \text{ that is, when } x=e^{3/2}.$$

We form a table, including the characteristic points (Table II). In addition to the characteristic points it is useful to find the points of intersection of

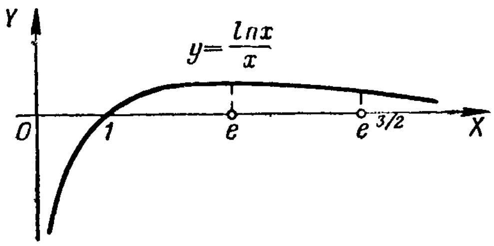

Fig. 34

the curve with the coordinate axes. Putting $y=0$, we find $x=1$ (the point of intersection of the curve with the axis of abscissas); the curve does not intersect the axis of ordinates

e) Utilizing the results of investigation, we construct the graph of the function (Fig. 34).

Table II

x	0	(0, 1)	1	(1, e)	$e \approx 2.72$	$\left(e, e^{\frac{3}{2}}\right)$	$e^{\frac{3}{2}} \approx 4.49$	$\left(e^{\frac{3}{2}}, +\infty\right)$
y	$-\infty$	$-$	0	+	$\frac{1}{e} \approx 0.37$	+	$\frac{3}{2\sqrt{e^3}} \approx 0.33$	+
y'	nonexist.	+	+	+	0	$-$	$-$	$-$
y''	nonexist	$-$	$-$	$-$	$-$	$-$	0	+
Conclusions	Boundary point of domain of def. of function Vertical asymptote	Funct increases graph is concave down	Funct incr., graph is concave down	Funct incr., graph is concave down	Max point of funct.	Funct decr., graph is concave down	Point of inflection	Function decreases; graph is concave up

Graph the following functions and determine for each function its domain of definition, discontinuities, extremal points, intervals of increase and decrease, points of inflection of its graph, the direction of concavity, and also the asymptotes.

916. $y=x^3-3x^2$.

917. $y=\frac{6x^2-x^4}{9}$.

918. $y=(x-1)^2(x+2)$.

919. $y=\frac{(x-2)^2(x+4)}{4}$.

920. $y=\frac{(x^2-5)^3}{125}$.

921. $y=\frac{x^2-2x+2}{x-1}$.

922. $y=\frac{x^4-3}{x}$.

923. $y=\frac{x^4+3}{x}$.

924. $y=x^2+\frac{2}{x}$.

925. $y=\frac{1}{x^2+3}$.

926. $y=\frac{8}{x^2-4}$.

927. $y=\frac{4x}{4+x^2}$.

928. $y=\frac{4x-12}{(x-2)^2}$.

929. $y=\frac{x}{x^2-4}$.

930. $y=\frac{16}{x^2(x-4)}$.

931. $y=\frac{3x^4+1}{x^3}$.

932. $y=\sqrt{x}+\sqrt{4-x}$.

933. $y=\sqrt{8+x}-\sqrt{8-x}$.

934. $y=x\sqrt{x+3}$.

935. $y=\sqrt{x^3-3x}$.

936. $y=\sqrt[3]{1-x^2}$.

937. $y=\sqrt[3]{1-x^3}$.

938. $y=2x+2-3\sqrt[3]{(x+1)^2}$.

939. $y=\sqrt[3]{x+1}-\sqrt[3]{x-1}$.

940. $y=\sqrt[3]{(x+4)^2}-\sqrt[3]{(x-4)^2}$.

941. $y=\sqrt[3]{(x-2)^2}+\sqrt[3]{(x-4)^2}$.

942. $y=\frac{4}{\sqrt{4-x^2}}$.

943. $y=\frac{8}{x\sqrt{x^2-4}}$.

944. $y=\frac{x}{\sqrt[3]{x^2-1}}$.

945. $y=\frac{x}{\sqrt[3]{(x-2)^2}}$.

946. $y=xe^{-x}$.

947. $y=\left(a+\frac{x^2}{a}\right)e^{\frac{x}{a}}$.

948. $y=e^{8x-x^2-14}$.

949. $y=(2+x^2)e^{-x^2}$.

950. $y=2|x|-x^2$.

951. $y=\frac{\ln x}{\sqrt{x}}$.

952. $y=\frac{x^2}{2}\ln\frac{x}{a}$.

953. $y=\frac{x}{\ln x}$.

954. $y=(x+1)\ln^2(x+1)$.

955. $y=\ln(x^2-1)+\frac{1}{x^2-1}$.

956. $y=\ln\frac{\sqrt{x^2+1}-1}{x}$.

957. $y=\ln(1+e^{-x})$.

958. $y=\ln\left(e+\frac{1}{x}\right)$.

959. $y=\sin x+\cos x$.

960. $y=\sin x+\frac{\sin 2x}{2}$.

961. $y=\cos x-\cos^2 x$.

962. $y=\sin^3 x+\cos^3 x$.

963. $y=\frac{1}{\sin x+\cos x}$.

964. $y=\frac{\sin x}{\sin\left(x+\frac{\pi}{4}\right)}$.

965. $y=\sin x\cdot\sin 2x$.

966. $y=\cos x\cdot\cos 2x$.

967. $y=x+\sin x$.

968. $y=\arcsin(1-\sqrt[3]{x^2})$.

969. $y=\frac{\arcsin x}{\sqrt{1-x^2}}$.

970. $y=2x-\tan x$.

971. $y=x\arctan x$.

972. $y=x\arctan\frac{1}{x}$ when $x\neq 0$ and $y=0$ when $x=0$.

973. $y=x-2\operatorname{arc\,cot} x$.

974. $y=\frac{x}{2}+\arctan x$.

975. $y=\ln\sin x$.

976. $y=\operatorname{arc\,cosh}\left(x+\frac{1}{x}\right)$.

977. $y=e^{\sin x}$.

978. $y=e^{\arcsin\sqrt{x}}$.

979. $y=e^{\arctan x}$.

980. $y=\ln\sin x$.

981. $y=\ln\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$.

982. $y=\ln x-\arctan x$.

983. $y=\cos x-\ln\cos x$.

984. $y=\arctan(\ln x)$.

985. $y=\arcsin\ln(x^2+1)$.

986. $y=x^x$.

987. $y=x^{\frac{1}{x}}$.

A good exercise is to graph the functions indicated in Examples 826-848.

Construct the graphs of the following functions represented parametrically.

988. $x=t^2-2t,\quad y=t^2+2t$.

989. $x=a\cos^3 t,\quad y=a\sin t\ (a>0)$.

990. $x=te^t,\quad y=te^{-t}$.

991. $x=t+e^{-t},\quad y=2t+e^{-2t}$.

992. $x=a(\sinh t-t),\quad y=a(\cosh t-1)\ (a>0)$.

Sec. 5. Differential of an Arc. Curvature

1°. Differential of an arc. The differential of an arc s of a plane curve represented by an equation in Cartesian coordinates x and y is expressed by the formula

$$ds=\sqrt{(dx)^2+(dy)^2};$$

here, if the equation of the curve is of the form

a) $y=f(x)$, then $ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx;$

b) $x=f_1(y)$, then $ds=\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy;$

c) $x=\varphi(t)$, $y=\psi(t)$, then $ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt;$

d) $F(x, y)=0$, then $ds=\frac{\sqrt{F_x'^2+F_y'^2}}{|F_y'|}\,dx=\frac{\sqrt{F_x'^2+F_y'^2}}{|F_x'|}\,dy.$

Denoting by α the angle formed by the tangent (in the direction of increasing arc of the curve s) with the positive x-direction, we get

$$\cos\alpha = \frac{dx}{ds},$$

$$\sin\alpha = \frac{dy}{ds}.$$

In polar coordinates,

$$ds = \sqrt{(dr)^2 + (r\,d\varphi)^2} = \sqrt{r^2 + \left(\frac{dr}{d\varphi}\right)^2}\,d\varphi$$

Denoting by β the angle between the radius vector of the point of the curve and the tangent to the curve at this point, we have

$$\cos\beta = \frac{dr}{ds},$$

$$\sin\beta = r\frac{d\varphi}{ds}.$$

2°. Curvature of a curve. The *curvature* K of a curve at one of its points M is the limit of the ratio of the angle between the positive directions of the tangents at the points M and N of the curve *(angle of contingence)* to the length of the arc $\smile{MN} = \Delta s$ when $N \to M$ (Fig. 35), that is,

$$K = \lim_{\Delta s \to 0} \frac{\Delta\alpha}{\Delta s} = \frac{d\alpha}{ds},$$

where α is the angle between the positive directions of the tangent at the point M and the x-axis.

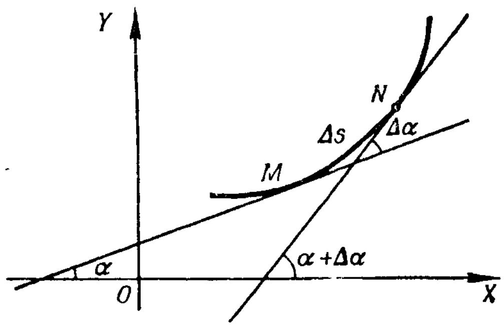

Fig. 35

The *radius of curvature* R is the reciprocal of the absolute value of the curvature, i. e.,

$$R = \frac{1}{|K|}.$$

The circle $\left(K = \frac{1}{a}\right.$, where a is the radius of the circle) and the straight line ($K = 0$) are lines of constant curvature.

We have the following formulas for computing the curvature in rectangular coordinates (accurate to within the sign):

1) if the curve is given by an equation explicitly, $y=f(x)$, then

$$K=\frac{y''}{(1+y'^2)^{3/2}};$$

2) if the curve is given by an equation implicitly, $F(x, y)=0$, then

$$K=\frac{\begin{vmatrix} F''_{xx} & F''_{xy} & F'_x \\ F''_{yx} & F''_{yy} & F'_y \\ F'_x & F'_y & 0 \end{vmatrix}}{(F'^2_x+F'^2_y)^{3/2}};$$

3) if the curve is represented by equations in parametric form, $x=\varphi(t)$, $y=\psi(t)$, then

$$K=\frac{\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}}{(x'^2+y'^2)^{3/2}},$$

where

$$x'=\frac{dx}{dt}, \quad y'=\frac{dy}{dt}, \quad x''=\frac{d^2x}{dt^2}, \quad y''=\frac{d^2y}{dt^2}.$$

In polar coordinates, when the curve is given by the equation $r=f(\varphi)$, we have

$$K=\frac{r^2+2r'^2-rr''}{(r^2+r'^2)^{3/2}},$$

where

$$r'=\frac{dr}{d\varphi} \quad \text{and} \quad r''=\frac{d^2r}{d\varphi^2}.$$

3°. Circle of curvature. The *circle of curvature* (or *osculating circle*) of a curve at the point M is the limiting position of a circle drawn through M and two other points of the curve, P and Q, as $P \to M$ and $Q \to M$.

The radius of the circle of curvature is equal to the radius of curvature, and the centre of the circle of curvature (the *centre of curvature*) lies on the normal to the curve drawn at the point M in the direction of concavity of the curve.

The coordinates X and Y of the centre of curvature of the curve are computed from the formulas

$$X=x-\frac{y'(1+y'^2)}{y''}, \quad Y=y+\frac{1+y'^2}{y''}.$$

The *evolute* of a curve is the locus of the centres of curvature of the curve.

If in the formulas for determining the coordinates of the centre of curvature we regard X and Y as the current coordinates of a point of the evolute, then these formulas yield parametric equations of the evolute with parameter x or y (or t, if the curve itself is represented by equations in parametric form)

Example 1. Find the equation of the evolute of the parabola $y=x^2$.

Solution. $X=-4x^3$, $Y=\frac{1+6x^2}{2}$. Eliminating the parameter x, we find the equation of the evolute in explicit form, $Y=\frac{1}{2}+3\left(\frac{X}{4}\right)^{2/3}$.

The *involute* of a curve is a curve for which the given curve is an evolute.

The normal MC of the involute Γ_2 is a tangent to the evolute Γ_1; the length of the arc $\breve{CC_1}$ of the evolute is equal to the corresponding increment in the radius of curvature $\breve{CC_1}=M_1C_1-MC$; that is why the involute Γ_2 is also called the *evolvent* of the curve Γ_1 obtained by unwinding a taut thread wound onto Γ_1 (Fig. 36). To each evolute there corresponds an infinitude of involutes, which are related to different initial lengths of thread.

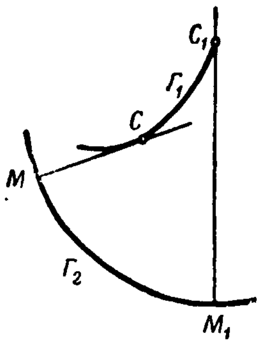

Fig. 36

4°. **Vertices of a curve.** The *vertex* of a curve is a point of the curve at which the curvature has a maximum or a minimum. To determine the vertices of a curve, we form the expression of the curvature K and find its extremal points. In place of the curvature K we can take the radius of curvature $R=\frac{1}{|K|}$ and seek its extremal points if the computations are simpler in this case.

Example 2. Find the vertex of the catenary $y=a\cosh\frac{x}{a}$ $(a>0)$.

Solution. Since $y'=\sinh\frac{x}{a}$ and $y''=\frac{1}{a}\cosh\frac{x}{a}$, it follows that $K=\frac{1}{a\cosh^2\frac{x}{a}}$ and, hence, $R=a\cosh^2\frac{x}{a}$. We have $\frac{dR}{dx}=\sinh 2\frac{x}{a}$. Equating the derivative $\frac{dR}{dx}$ to zero, we get $\sinh 2\frac{x}{a}=0$, whence we find the sole critical point $x=0$. Computing the second derivative $\frac{d^2R}{dx^2}$ and putting into it the value $x=0$, we get $\left.\frac{d^2R}{dx^2}\right|_{x=0}=\left.\frac{2}{a}\cosh 2\frac{x}{a}\right|_{x=0}=\frac{2}{a}>0$. Therefore, $x=0$ is the minimum point of the radius of curvature (or of the maximum of curvature) of the catenary. The vertex of the catenary $y=a\cosh\frac{x}{a}$ is, thus, the point $A\,(0,\ a)$.

Find the differential of the arc, and also the cosine and sine of the angle formed, with the positive x-direction, by the tangent to each of the following curves:

993. $x^2+y^2=a^2$ (circle).

994. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (ellipse).

995 $y^2=2px$ (parabola).

996. $x^{2/3}+y^{2/3}=a^{2/3}$ (astroid).

997. $y=a\cosh\frac{x}{a}$ (catenary).

998. $x=a(t-\sin t)$; $y=a(1-\cos t)$ (cycloid).

999. $x=a\cos^3 t$, $y=a\sin^3 t$ (astroid).

Find the differential of the arc, and also the cosine or sine of the angle formed by the radius vector and the tangent to each of the following curves:

1000. $r=a\varphi$ (spiral of Archimedes).

1001. $r=\frac{a}{\varphi}$ (hyperbolic spiral).

1002. $r=a\sec^2\frac{\varphi}{2}$ (parabola).

1003. $r=a\cos^2\frac{\varphi}{2}$ (cardioid).

1004. $r=a^{\varphi}$ (logarithmic spiral).

1005. $r^2=a^2\cos 2\varphi$ (lemniscate).

Compute the curvature of the given curves at the indicated points:

1006. $y=x^4-4x^3-18x^2$ at the coordinate origin.

1007. $x^2+xy+y^2=3$ at the point (1, 1).

1008. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the vertices $A(a, 0)$ and $B(0, b)$.

1009. $x=t^2$, $y=t^3$ at the point (1, 1).

1010. $r^2=2a^2\cos 2\varphi$ at the vertices $\varphi=0$ and $\varphi=\pi$.

1011. At what point of the parabola $y^2=8x$ is the curvature equal to 0.128?

1012. Find the vertex of the curve $y=e^x$.

Find the radii of curvature (at any point) of the given lines:

1013. $y=x^3$ (cubic parabola).

1014. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (ellipse).

1015. $x=\frac{y^2}{4}-\frac{\ln y}{2}$.

1016. $x=a\cos^3 t$; $y=a\sin^3 t$ (astroid).

1017. $x=a(\cos t+t\sin t)$; $y=a(\sin t-t\cos t)$ involute of a circle).

1018. $r=ae^{k\varphi}$ (logarithmic spiral).

1019. $r=a(1+\cos\varphi)$ (cardioid).

1020. Find the least value of the radius of curvature of the parabola $y^2=2px$.

1021. Prove that the radius of curvature of the catenary $y=a\cosh\frac{x}{a}$ is equal to a segment of the normal.

Compute the coordinates of the centre of curvature of the given curves at the indicated points:

1022. $xy=1$ at the point (1, 1).

1023. $ay^2=x^3$ at the point (a, a).

Write the equations of the circles of curvature of the given curves at the indicated points:

1024. $y=x^2-6x+10$ at the point (3, 1).

1025. $y=e^x$ at the point (0, 1).

Find the evolutes of the curves:

1026. $y^2=2px$ (parabola).

1027. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (ellipse).

1028. Prove that the evolute of the cycloid

$$x=a(t-\sin t), \quad y=a(1-\cos t)$$

is a displaced cycloid.

1029. Prove that the evolute of the logarithmic spiral

$$r=ae^{k\varphi}$$

is also a logarithmic spiral with the same pole.

1030. Show that the curve (the *involute of a circle*)

$$x=a(\cos t+t\sin t), \quad y=a(\sin t-t\cos t)$$

is the involute of the circle $x=a\cos t$; $y=a\sin t$.

Chapter IV

INDEFINITE INTEGRALS

Sec. 1. Direct Integration

1°. Basic rules of integration.

1) If $F'(x) = f(x)$, then

$$\int f(x)\,dx = F(x) + C,$$

where C is an arbitrary constant.

2) $\int Af(x)\,dx = A\int f(x)\,dx$, where A is a constant quantity.

3) $\int [f_1(x) \pm f_2(x)]\,dx = \int f_1(x)\,dx \pm \int f_2(x)\,dx$.

4) If $\int f(x)\,dx = F(x) + C$ and $u = \varphi(x)$, then

$$\int f(u)\,du = F(u) + C.$$

In particular,

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C \quad (a \neq 0).$$

2°. Table of standard integrals.

I. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1.$

II. $\int \frac{dx}{x} = \ln|x| + C.$

III. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\operatorname{arc\,tan}\frac{x}{a} + C = -\frac{1}{a}\operatorname{arc\,cot}\frac{x}{a} + C \quad (a \neq 0).$

IV $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C \quad (a \neq 0).$

$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C \quad (a \neq 0).$

V. $\int \frac{dx}{\sqrt{x^2+a}} = \ln\left|x+\sqrt{x^2+a}\right| + C \quad (a \neq 0).$

VI. $\int \frac{dx}{\sqrt{a^2-x^2}} = \operatorname{arc\,sin}\frac{x}{a} + C = -\operatorname{arc\,cos}\frac{x}{a} + C \quad (a > 0).$

VII. $\int a^x dx = \frac{a^x}{\ln a} + C \quad (a > 0); \quad \int e^x dx = e^x + C.$

VIII. $\int \sin x \, dx = -\cos x + C.$

IX. $\int \cos x \, dx = \sin x + C.$

X $\int \frac{dx}{\cos^2 x} = \tan x + C.$

XI. $\int \frac{dx}{\sin^2 x} = -\cot x + C.$

XII $\int \frac{dx}{\sin x} = \ln \left| \tan \frac{x}{2} \right| + C = \ln | \operatorname{cosec} x - \cot x | + C.$

XIII. $\int \frac{dx}{\cos x} = \ln \left| \tan \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| + C = \ln | \tan x + \sec x | + C.$

XIV. $\int \sinh x \, dx = \cosh x + C.$

XV. $\int \cosh x \, dx = \sinh x + C.$

XVI. $\int \frac{dx}{\cosh^2 x} = \tanh x + C.$

XVII. $\int \frac{dx}{\sinh^2 x} = -\coth x + C.$

Example 1.

$$\int (ax^2 + bx + c) \, dx = \int ax^2 dx + \int bx \, dx + \int c \, dx =$$
$$= a \int x^2 dx + b \int x \, dx + c \int dx = a \frac{x^3}{3} + b \frac{x^2}{2} + cx + C.$$

Applying the basic rules 1, 2, 3 and the formulas of integration, find the following integrals:

1031. $\int 5a^2 x^6 dx.$

1032. $\int (6x^2 + 8x + 3) \, dx.$

1033. $\int x (x + a)(x + b) \, dx.$

1034. $\int (a + bx^3)^2 \, dx.$

1035. $\int \sqrt{2px} \, dx.$

1036. $\int \frac{dx}{\sqrt[n]{x}}.$

1037. $\int (nx)^{\frac{1-n}{n}} dx.$

1038. $\int \left(a^{\frac{2}{3}} - x^{\frac{2}{3}} \right)^3 dx.$

1039. $\int (\sqrt{x} + 1)(x - \sqrt{x} + 1) dx.$

1040. $\int \frac{(x^2 + 1)(x^2 - 2)}{\sqrt[3]{x^2}} dx.$

1041. $\int \frac{(x^m - x^n)^2}{\sqrt{x}} dx.$

1042. $\int \frac{(\sqrt{a} - \sqrt{x})^4}{\sqrt{ax}} dx.$

1043. $\int \frac{dx}{x^2 + 7}.$

1044. $\int \frac{dx}{x^2 - 10}.$

1045. $\int \frac{dx}{\sqrt{4 + x^2}}.$

1046. $\int \frac{dx}{\sqrt{8 - x^2}}.$

1047. $\int \frac{\sqrt{2 + x^2} - \sqrt{2 - x^2}}{\sqrt{4 - x^4}} dx.$

1048*. a) $\int \tan^2 x\,dx$;
b) $\int \tanh^2 x\,dx$.

1049. a) $\int \cot^2 x\,dx$;
b) $\int \coth^2 x\,dx$.

1050. $\int 3^x e^x dx$.

3°. Integration under the sign of the differential. Rule 4 considerably expands the table of standard integrals: by virtue of this rule the table of integrals holds true irrespective of whether the variable of integration is an independent variable or a differentiable function.

Example 2.

$$\int \frac{dx}{\sqrt{5x-2}} = \frac{1}{5}\int (5x-2)^{-\frac{1}{2}}\,d(5x-2) =$$

$$= \frac{1}{5}\int u^{-\frac{1}{2}}\,du = \frac{1}{5}\cdot\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{5}\,\frac{(5x-2)^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{5}\sqrt{5x-2} + C,$$

where we put $u = 5x-2$. We took advantage of Rule 4 and tabular integral I.

Example 3. $\int \frac{x\,dx}{\sqrt{1+x^4}} = \frac{1}{2}\int \frac{d(x^2)}{\sqrt{1+(x^2)^2}} = \frac{1}{2}\ln(x^2+\sqrt{1+x^4}) + C.$

We implied $u = x^2$, and use was made of Rule 4 and tabular integral V.

Example 4. $\int x^2 e^{x^3} dx = \frac{1}{3}\int e^{x^3}\,d(x^3) = \frac{1}{3}e^{x^3} + C$ by virtue of Rule 4 and tabular integral VII.

In examples 2, 3, and 4 we reduced the given integral to the following form before making use of a tabular integral:

$$\int f(\varphi(x))\,\varphi'(x)\,dx = \int f(u)\,du, \quad \text{where } u = \varphi(x).$$

This type of transformation is called *integration under the differential sign.*

Some common transformations of differentials, which were used in Examples 2 and 3, are:

a) $dx = \frac{1}{a}\,d(ax+b)$ $(a \neq 0)$; b) $x\,dx = \frac{1}{2}\,d(x^2)$ and so on.

Using the basic rules and formulas of integration, find the following integrals:

1051.** $\int \frac{a\,dx}{a-x}$.

1052.** $\int \frac{2x+3}{2x+1}\,dx$.

1053. $\int \frac{1-3x}{3+2x}\,dx$.

1054. $\int \frac{x\,dx}{a+bx}$.

1055. $\int \frac{ax+b}{\alpha x+\beta}\,dx$.

1056. $\int \frac{x^2+1}{x-1}\,dx$.

1057. $\int \frac{x^2+5x+7}{x+3}\,dx$.

1058. $\int \frac{x^4+x^2+1}{x-1}\,dx$.

1059. $\int \left(a + \frac{b}{x-a}\right)^2 dx.$

1060*. $\int \frac{x}{(x+1)^2} dx.$

1061. $\int \frac{b\,dy}{\sqrt{1-y}}.$

1062. $\int \sqrt{a-bx}\,dx.$

1063*. $\int \frac{x}{\sqrt{x^2+1}} dx.$

1064. $\int \frac{\sqrt{x}+\ln x}{x} dx.$

1065. $\int \frac{dx}{3x^2+5}.$

1066. $\int \frac{dx}{7x^2-8}.$

1067. $\int \frac{dx}{(a+b)-(a-b)x^2}$ $(0<b<a).$

1068. $\int \frac{x^2}{x^2+2} dx.$

1069. $\int \frac{x^3}{a^2-x^2} dx.$

1070. $\int \frac{x^2-5x+6}{x^2+4} dx.$

1071. $\int \frac{dx}{\sqrt{7+8x^2}}.$

1072. $\int \frac{dx}{\sqrt{7-5x^2}}.$

1073. $\int \frac{2x-5}{3x^2-2} dx.$

1074. $\int \frac{3-2x}{5x^2+7} dx.$

1075. $\int \frac{3x+1}{\sqrt{5x^2+1}} dx.$

1076. $\int \frac{x+3}{\sqrt{x^2-4}} dx.$

1077. $\int \frac{x\,dx}{x^2-5}.$

1078. $\int \frac{x\,dx}{2x^2+3}.$

1079. $\int \frac{ax+b}{a^2x^2+b^2} dx.$

1080. $\int \frac{x\,dx}{\sqrt{a^4-x^4}}.$

1081. $\int \frac{x^2}{1+x^6} dx.$

1082. $\int \frac{x^2\,dx}{\sqrt{x^6-1}}.$

1083. $\int \sqrt{\frac{\arcsin x}{1-x^2}}\,dx.$

1084. $\int \frac{\arctan \frac{x}{2}}{4+x^2} dx.$

1085. $\int \frac{x-\sqrt{\arctan 2x}}{1+4x^2} dx.$

1086. $\int \frac{dx}{\sqrt{(1+x^2)\ln(x+\sqrt{1+x^2})}}.$

1087. $\int ae^{-mx}\,dx.$

1088. $\int 4^{2-3x}\,dx.$

1089. $\int (e^t-e^{-t})\,dt.$

1090. $\int \left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)^2 dx.$

1091. $\int \frac{(a^x-b^x)^2}{a^xb^x} dx.$

1092. $\int \frac{a^{2x}-1}{\sqrt{a^x}} dx.$

1093. $\int e^{-(x^2+1)}\,x\,dx.$

1094. $\int x\cdot 7^{x^2}\,dx.$

1095. $\int \frac{e^{\frac{1}{x}}}{x^2} dx.$

1096. $\int 5^{\sqrt{x}} \frac{dx}{\sqrt{x}}.$

1097. $\int \frac{e^x}{e^x-1} dx.$

1098. $\int e^x \sqrt{a-be^x}\,dx.$

1099. $\int \left(e^{\frac{x}{a}}+1\right)^{\frac{1}{3}} e^{\frac{x}{a}}\,dx.$

1100*. $\int \frac{dx}{2^x+3}.$

1101. $\int \frac{a^x\,dx}{1+a^{2x}}.$

1102. $\int \frac{e^{-bx}}{1-e^{-2bx}}\,dx.$

1103. $\int \frac{e^t\,dt}{\sqrt{1-e^{2t}}}.$

1104. $\int \sin(a+bx)\,dx.$

1105. $\int \cos\frac{x}{\sqrt{2}}\,dx.$

1106. $\int (\cos ax+\sin ax)^2\,dx.$

1107. $\int \cos\sqrt{x}\,\frac{dx}{\sqrt{x}}.$

1108. $\int \sin(\lg x)\,\frac{dx}{x}.$

1109*. $\int \sin^2 x\,dx.$

1110*. $\int \cos^2 x\,dx.$

1111. $\int \sec^2(ax+b)\,dx.$

1112. $\int \cot^2 ax\,dx.$

1113. $\int \frac{dx}{\sin\frac{x}{a}}.$

1114. $\int \frac{dx}{3\cos\left(5x-\frac{\pi}{4}\right)}.$

1115. $\int \frac{dx}{\sin(ax+b)}.$

1116. $\int \frac{x\,dx}{\cos^2 x^2}.$

1117. $\int x\sin(1-x^2)\,dx.$

1118. $\int \left(\frac{1}{\sin x\sqrt{2}}-1\right)^2 dx.$

1119. $\int \tan x\,dx.$

1120. $\int \cot x\,dx.$

1121. $\int \cot\frac{x}{a-b}\,dx.$

1122. $\int \frac{dx}{\tan\frac{x}{5}}.$

1123. $\int \tan\sqrt{x}\,\frac{dx}{\sqrt{x}}.$

1124. $\int x\cot(x^2+1)\,dx.$

1125. $\int \frac{dx}{\sin x\cos x}.$

1126. $\int \cos\frac{x}{a}\sin\frac{x}{a}\,dx.$

1127. $\int \sin^3 6x\cos 6x\,dx.$

1128. $\int \frac{\cos ax}{\sin^5 ax}\,dx.$

1129. $\int \frac{\sin 3x}{3+\cos 3x}\,dx.$

1130. $\int \frac{\sin x\cos x}{\sqrt{\cos^2 x-\sin^2 x}}\,dx.$

1131. $\int \sqrt{1+3\cos^2 x}\,\sin 2x\,dx.$

1132. $\int \tan^3\frac{x}{3}\sec^2\frac{x}{3}\,dx.$

1133. $\int \frac{\sqrt{\tan x}}{\cos^2 x}\,dx.$

1134. $\int \frac{\cot^{\frac{2}{3}} x}{\sin^2 x}\,dx.$

1135. $\int \frac{1+\sin 3x}{\cos^2 3x}\,dx.$

1136. $\int \frac{(\cos ax+\sin ax)^2}{\sin ax}\,dx.$

1137. $\int \frac{\operatorname{cosec}^2 3x}{b-a\cot 3x}\,dx.$

1138. $\int (2\sinh 5x-3\cosh 5x)\,dx.$

1139. $\int \sinh^2 x\,dx.$

1140. $\int \frac{dx}{\sinh x}$.

1141. $\int \frac{dx}{\cosh x}$.

1142. $\int \frac{dx}{\sinh x \cosh x}$.

1143. $\int \tanh x \, dx$.

1144. $\int \coth x \, dx$.

Find the indefinite integrals:

1145. $\int x \sqrt[5]{5-x^2}\, dx$.

1146. $\int \frac{x^3-1}{x^4-4x+1}\, dx$.

1147. $\int \frac{x^3}{x^8+5}\, dx$.

1148. $\int x e^{-x^2}\, dx$.

1149. $\int \frac{3-\sqrt{2+3x^2}}{2+3x^2}\, dx$.

1150. $\int \frac{x^3-1}{x-1}\, dx$.

1151. $\int \frac{dx}{\sqrt{e^x}}$.

1152. $\int \frac{1-\sin x}{x+\cos x}\, dx$.

1153. $\int \frac{\tan 3x - \cot 3x}{\sin 3x}\, dx$.

1154. $\int \frac{dx}{x \ln^2 x}$.

1155. $\int \frac{\sec^2 x}{\sqrt{\tan^2 x - 2}}\, dx$.

1156. $\int \left(2+\frac{x}{2x^2+1}\right)\frac{dx}{2x^2+1}$.

1157. $\int a^{\sin x} \cos x \, dx$.

1158. $\int \frac{x^2}{\sqrt[3]{x^3+1}}\, dx$.

1159. $\int \frac{x\,dx}{\sqrt{1-x^4}}$.

1160. $\int \tan^2 ax \, dx$.

1161. $\int \sin^2 \frac{x}{2}\, dx$.

1162. $\int \frac{\sec^2 x\, dx}{\sqrt{4-\tan^2 x}}$.

1163. $\int \frac{dx}{\cos \frac{x}{a}}$.

1164. $\int \frac{\sqrt[3]{1+\ln x}}{x}\, dx$.

1165. $\int \tan \sqrt{x-1}\, \frac{dx}{\sqrt{x-1}}$.

1166. $\int \frac{x\,dx}{\sin x^2}$.

1167. $\int \frac{e^{\arctan x} + x \ln(1+x^2) + 1}{1+x^2}\, dx$.

1168. $\int \frac{\sin x - \cos x}{\sin x + \cos x}\, dx$.

1169. $\int \frac{\left(1-\sin \frac{x}{\sqrt{2}}\right)^2}{\sin \frac{x}{\sqrt{2}}}\, dx$.

1170. $\int \frac{x^2}{x^2-2}\, dx$.

1171. $\int \frac{(1+x)^2}{x(1+x^2)}\, dx$.

1172. $\int e^{\sin^2 x} \sin 2x \, dx$.

1173. $\int \frac{5-3x}{\sqrt{4-3x^2}}\, dx$.

1174. $\int \frac{dx}{e^x+1}$.

1175. $\int \frac{dx}{(a+b)+(a-b)x^2}$ $(0 < b < a)$.

1176. $\int \frac{e^x}{\sqrt{e^{2x}-2}}\, dx$.

1177. $\int \frac{dx}{\sin ax \cos ax}$.

1178. $\int \sin\left(\frac{2\pi t}{T} + \varphi_0\right) dt.$

1179. $\int \frac{dx}{x(4-\ln^2 x)}.$

1180. $\int \frac{\arccos \frac{x}{2}}{\sqrt{4-x^2}} dx.$

1181. $\int e^{-\tan x} \sec^2 x\, dx.$

1182. $\int \frac{\sin x \cos x}{\sqrt{2-\sin^4 x}} dx.$

1183. $\int \frac{dx}{\sin^2 x \cos^2 x}.$

1184. $\int \frac{\arcsin x + x}{\sqrt{1-x^2}} dx.$

1185. $\int \frac{\sec x \tan x}{\sqrt{\sec^2 x + 1}} dx.$

1186. $\int \frac{\cos 2x}{4+\cos^2 2x} dx.$

1187. $\int \frac{dx}{1+\cos^2 x}.$

1188. $\int \sqrt{\frac{\ln(x+\sqrt{x^2+1})}{1+x^2}}\, dx.$

1189. $\int x^2 \cos(x^3+3)\, dx.$

1190. $\int \frac{3^{\tanh x}}{\cosh^2 x} dx.$

Sec. 2. Integration by Substitution

1°. Change of variable in an indefinite integral. Putting

$$x = \varphi(t),$$

where t is a new variable and φ is a continuously differentiable function, we will have:

$$\int f(x)\, dx = \int f[\varphi(t)]\, \varphi'(t)\, dt. \qquad (1)$$

The attempt is made to choose the function φ in such a way that the right side of (1) becomes more convenient for integration.

Example 1. Find

$$\int x\sqrt{x-1}\, dx.$$

Solution. It is natural to put $t=\sqrt{x-1}$, whence $x=t^2+1$ and $dx=2t\, dt$. Hence,

$$\int x\sqrt{x-1}\, dx = \int (t^2+1)\, t\cdot 2t\, dt = 2\int (t^4+t^2)\, dt =$$

$$= \frac{2}{5}t^5 + \frac{2}{3}t^3 + C = \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C.$$

Sometimes substitutions of the form

$$u = \varphi(x)$$

are used.

Suppose we succeeded in transforming the integrand $f(x)\, dx$ to the form

$$f(x)\, dx = g(u)\, du, \text{ where } u=\varphi(x).$$

If $\int g(u)\,du$ is known, that is,

$$\int g(u)\,du = F(u) + C,$$

then

$$\int f(x)\,dx = F[\varphi(x)] + C.$$

Actually, we have already made use of this method in Sec. 1,3°.
Examples 2, 3, 4 (Sec. 1) may be solved as follows:

Example 2. $u = 5x - 2$; $du = 5dx$; $dx = \frac{1}{5}du$.

$$\int \frac{dx}{\sqrt{5x-2}} = \frac{1}{5}\frac{du}{\sqrt{u}} = \frac{1}{5}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{5}\sqrt{5x-2} + C.$$

Example 3. $u = x^2$; $du = 2x\,dx$; $x\,dx = \frac{du}{2}$.

$$\int \frac{x\,dx}{\sqrt{1+x^4}} = \frac{1}{2}\int \frac{du}{\sqrt{1+u^2}} = \frac{1}{2}\ln(u + \sqrt{1+u^2}) + C = \frac{1}{2}\ln(x^2 + \sqrt{1+x^4}) + C.$$

Example 4. $u = x^3$; $du = 3x^2\,dx$; $x^2\,dx = \frac{du}{3}$.

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C.$$

2°. **Trigonometric substitutions.**

1) If an integral contains the radical $\sqrt{a^2-x^2}$, the usual thing is to put $x = a\sin t$; whence

$$\sqrt{a^2-x^2} = a\cos t.$$

2) If an integral contains the radical $\sqrt{x^2-a^2}$, we put $x = a\sec t$, whence

$$\sqrt{x^2-a^2} = a\tan t.$$

3) If an integral contains the radical $\sqrt{x^2+a^2}$, we put $x = a\tan t$; whence

$$\sqrt{x^2+a^2} = a\sec t.$$

It should be noted that trigonometric substitutions do not always turn out to be advantageous.

It is sometimes more convenient to make use of *hyperbolic substitutions*, which are similar to trigonometric substitutions (see Example 1209).

For more details about trigonometric and hyperbolic substitutions, see Sec. 9.

Example 5. Find

$$\int \frac{\sqrt{x^2+1}}{x^2}\,dx.$$

Solution. Put $x=\tan t$. Therefore, $dx=\frac{dt}{\cos^2 t}$.

$$\int \frac{\sqrt{x^2+1}}{x^2}\,dx=\int \frac{\sqrt{\tan^2 t+1}}{\tan^2 t}\,\frac{dt}{\cos^2 t}=\int \frac{\sec t \cos^2 t}{\sin^2 t}\,\frac{dt}{\cos^2 t}=$$

$$=\int \frac{dt}{\sin^2 t \cos t}=\int \frac{\sin^2 t+\cos^2 t}{\sin^2 t\cdot\cos t}\,dt=\int \frac{dt}{\cos t}+\int \frac{\cos t}{\sin^2 t}\,dt=$$

$$=\ln|\tan t+\sec t|-\frac{1}{\sin t}+C=\ln|\tan t+\sqrt{1+\tan^2 t}|-$$

$$-\frac{\sqrt{1+\tan^2 t}}{\tan t}+C=\ln|x+\sqrt{x^2+1}|-\frac{\sqrt{x^2+1}}{x}+C.$$

1191. Applying the indicated substitutions, find the following integrals:

a) $\int \frac{dx}{x\sqrt{x^2-2}}$, $\quad x=\frac{1}{t}$;

b) $\int \frac{dx}{e^x+1}$, $\quad x=-\ln t$,

c) $\int x(5x^2-3)^7\,dx$, $\quad 5x^2-3=t$;

d) $\int \frac{x\,dx}{\sqrt{x+1}}$, $\quad t=\sqrt{x+1}$;

e) $\int \frac{\cos x\,dx}{\sqrt{1+\sin^2 x}}$, $\quad t=\sin x$.

Applying suitable substitutions, find the following integrals:

1192. $\int x(2x+5)^{10}\,dx$.

1193. $\int \frac{1+x}{1+\sqrt{x}}\,dx$.

1194. $\int \frac{dx}{x\sqrt{2x+1}}$.

1195. $\int \frac{dx}{\sqrt{e^x-1}}$.

1196. $\int \frac{\ln 2x\,dx}{\ln 4x\,x}$.

1197. $\int \frac{(\arcsin x)^2}{\sqrt{1-x^2}}\,dx$.

1198. $\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx$.

1199. $\int \frac{\sin^3 x}{\sqrt{\cos x}}\,dx$.

1200*. $\int \frac{dx}{x\sqrt{1+x^2}}$.

Applying trigonometric substitutions, find the following integrals:

1201. $\int \frac{x^2\,dx}{\sqrt{1-x^2}}$.

1202. $\int \frac{x^3\,dx}{\sqrt{2-x^2}}$.

1203. $\int \frac{\sqrt{x^2-a^2}}{x}\,dx$.

1204*. $\int \frac{dx}{x\sqrt{x^2-1}}$.

1205. $\int \frac{\sqrt{x^2+1}}{x}\,dx.$ 1206*. $\int \frac{dx}{x^2\sqrt{4-x^2}}.$

1207. $\int \sqrt{1-x^2}\,dx.$

1208. Evaluate the integral

$$\int \frac{dx}{\sqrt{x(1-x)}}$$

by means of the substitution $x=\sin^2 t$.

1209. Find

$$\int \sqrt{a^2+x^2}\,dx,$$

by applying the hyperbolic substitution $x=a\sinh t$.

Solution. We have: $\sqrt{a^2+x^2}=\sqrt{a^2+a^2\sinh^2 t}=a\cosh t$ and $dx=a\cosh t\,dt$. Whence

$$\int \sqrt{a^2+x^2}\,dx=\int a\cosh t\cdot a\cosh t\,dt=$$
$$=a^2\int \cosh^2 t\,dt=a^2\int \frac{\cosh 2t+1}{2}\,dt=\frac{a^2}{2}\left(\frac{1}{2}\sinh 2t+t\right)+C=$$
$$=\frac{a^2}{2}(\sinh t\cosh t+t)+C.$$

Since

$$\sinh t=\frac{x}{a},\quad \cosh t=\frac{\sqrt{a^2+x^2}}{a}$$

and

$$e^t=\cosh t+\sinh t=\frac{x+\sqrt{a^2+x^2}}{a}$$

we finally get

$$\int \sqrt{a^2+x^2}\,dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+C_1,$$

where $C_1=C-\frac{a^2}{2}\ln a$ is a new arbitrary constant.

1210. Find

$$\int \frac{x^2\,dx}{\sqrt{x^2-a^2}},$$

putting $x=a\cosh t$.

Sec. 3. Integration by Parts

A formula for integration by parts. If $u=\varphi(x)$ and $v=\psi(x)$ are differentiable functions, then

$$\int u\,dv=uv-\int v\,du.$$

Example 1. Find

$$\int x \ln x \, dx.$$

Putting $u=\ln x$, $dv=x\,dx$, we have $du=\frac{dx}{x}$, $v=\frac{x^2}{2}$ Whence

$$\int x \ln x \, dx=\frac{x^2}{2}\ln x-\int \frac{x^2}{2}\frac{dx}{x}=\frac{x^2}{2}\ln x-\frac{x^2}{4}+C.$$

Sometimes, to reduce a given integral to tabular form, one has to apply the formula of integration by parts several times. In certain cases, integration by parts yields an equation from which the desired integral is determined.

Example 2. Find

$$\int e^x \cos x \, dx.$$

We have

$$\int e^x \cos x\,dx=\int e^x d(\sin x)=e^x \sin x-\int e^x \sin x\,dx=e^x \sin x+$$
$$+\int e^x d(\cos x)=e^x \sin x+e^x \cos x-\int e^x \cos x\,dx.$$

Hence,

$$\int e^x \cos x\,dx=e^x \sin x+e^x \cos x-\int e^x \cos x\,dx,$$

whence

$$\int e^x \cos x\,dx=\frac{e^x}{2}(\sin x+\cos x)+C.$$

Applying the formula of integration by parts, find the following integrals:

1211. $\int \ln x\,dx.$

1212. $\int \arctan x\,dx.$

1213. $\int \arcsin x\,dx.$

1214. $\int x \sin x\,dx.$

1215. $\int x \cos 3x\,dx.$

1216. $\int \frac{x}{e^x}\,dx.$

1217. $\int x\cdot 2^{-x}\,dx.$

1218.** $\int x^2 e^{3x}\,dx.$

1219*. $\int (x^2-2x+5)\,e^{-x}\,dx.$

1220*. $\int x^3 e^{-\frac{x}{3}}\,dx.$

1221. $\int x \sin x \cos x\,dx$

1222* $\int (x^2+5x+6)\cos 2x\,dx.$

1223. $\int x^2 \ln x\,dx.$

1224. $\int \ln^2 x\,dx.$

1225. $\int \frac{\ln x}{x^3}\,dx.$

1226. $\int \frac{\ln x}{\sqrt{x}}\,dx.$

1227. $\int x \arctan x\,dx.$

1228. $\int x \arcsin x\,dx.$

1229. $\int \ln(x+\sqrt{1+x^2})\,dx.$

1230. $\int \frac{x\,dx}{\sin^2 x}.$

1231. $\int \frac{x\cos x}{\sin^2 x}\,dx$.

1232. $\int e^x \sin x\,dx$.

1233. $\int 3^x \cos x\,dx$.

1234. $\int e^{ax} \sin bx\,dx$.

1235. $\int \sin(\ln x)\,dx$.

Applying various methods, find the following integrals:

1236. $\int x^3 e^{-x^2}\,dx$.

1237. $\int e^{\sqrt{x}}\,dx$.

1238. $\int (x^2-2x+3)\ln x\,dx$.

1239. $\int x\ln\frac{1-x}{1+x}\,dx$.

1240. $\int \frac{\ln^2 x}{x^2}\,dx$.

1241. $\int \frac{\ln(\ln x)}{x}\,dx$.

1242. $\int x^2 \arctan 3x\,dx$.

1243. $\int x(\arctan x)^2\,dx$.

1244. $\int (\arcsin x)^2\,dx$.

1245. $\int \frac{\arcsin x}{x^2}\,dx$.

1246. $\int \frac{\arcsin\sqrt{x}}{\sqrt{1-x}}\,dx$.

1247. $\int x\tan^2 2x\,dx$.

1248. $\int \frac{\sin^2 x}{e^x}\,dx$.

1249. $\int \cos^2(\ln x)\,dx$.

1250**. $\int \frac{x^2}{(x^2+1)^2}\,dx$.

1251*. $\int \frac{dx}{(x^2+a^2)^2}$.

1252*. $\int \sqrt{a^2-x^2}\,dx$.

1253*. $\int \sqrt{A+x^2}\,dx$.

1254*. $\int \frac{x^2\,dx}{\sqrt{9-x^2}}$.

Sec. 4. Standard Integrals Containing a Quadratic Trinomial

1°. Integrals of the form

$$\int \frac{mx+n}{ax^2+bx+c}\,dx.$$

The principal calculation procedure is to reduce the quadratic trinomial to the form

$$ax^2+bx+c=a(x+k)^2+l, \qquad (1)$$

where k and l are constants. To perform the transformations in (1), it is best to take the perfect square out of the quadratic trinomial. The following substitution may also be used:

$$2ax+b=t.$$

If $m=0$, then, reducing the quadratic trinomial to the form (1), we get the tabular integrals III or IV (see Table).

Example 1.

$$\int \frac{dx}{2x^2-5x+7}=\frac{1}{2}\int \frac{dx}{\left(x^2-2\cdot\frac{5}{4}x+\frac{25}{16}\right)+\left(\frac{7}{2}-\frac{25}{16}\right)}=$$

$$=\frac{1}{2}\int \frac{d\left(x-\frac{5}{4}\right)}{\left(x-\frac{5}{4}\right)^2+\frac{31}{16}}=\frac{1}{2}\,\frac{1}{\frac{\sqrt{31}}{4}}\arctan\frac{x-\frac{5}{4}}{\frac{\sqrt{31}}{4}}+C=$$

$$=\frac{2}{\sqrt{31}}\arctan\frac{4x-5}{\sqrt{31}}+C.$$

If $m\neq 0$, then from the numerator we can take the derivative $2ax+b$ out of the quadratic trinomial

$$\int \frac{mx+n}{ax^2+bx+c}\,dx=\int \frac{\frac{m}{2a}(2ax+b)+\left(n-\frac{mb}{2a}\right)}{ax^2+bx+c}\,dx=$$

$$=\frac{m}{2a}\ln|ax^2+bx+c|+\left(n-\frac{mb}{2a}\right)\int\frac{dx}{ax^2+bx+c},$$

and thus we arrive at the integral discussed above.

Example 2.

$$\int \frac{x-1}{x^2-x-1}\,dx=\int \frac{\frac{1}{2}(2x-1)-\frac{1}{2}}{x^2-x-1}\,dx=\frac{1}{2}\ln|x^2-x-1|-$$

$$-\frac{1}{2}\int \frac{d\left(x-\frac{1}{2}\right)}{\left(x-\frac{1}{2}\right)^2-\frac{5}{4}}=\frac{1}{2}\ln|x^2-x-1|-\frac{1}{2\sqrt{5}}\ln\left|\frac{2x-1-\sqrt{5}}{2x-1+\sqrt{5}}\right|+C.$$

2°. **Integrals of the form** $\int \frac{mx+n}{\sqrt{ax^2+bx+c}}\,dx$. The methods of calculation are similar to those analyzed above. The integral is finally reduced to tabular integral V, if $a>0$, and VI, if $a<0$.

Example 3.

$$\int \frac{dx}{\sqrt{2+3x-2x^2}}=\frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\frac{25}{16}-\left(x-\frac{3}{4}\right)^2}}=\frac{1}{\sqrt{2}}\arcsin\frac{4x-3}{5}+C.$$

Example 4.

$$\int \frac{x+3}{\sqrt{x^2+2x+2}}\,dx=\frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+2}}\,dx+2\int \frac{dx}{\sqrt{(x+1)^2+1}}=$$

$$=\sqrt{x^2+2x+2}+2\ln\left(x+1+\sqrt{x^2+2x+2}\right)+C.$$

3°. Integrals of the form $\int \frac{dx}{(mx+n)\sqrt{ax^2+bx+c}}$. By means of the inverse substitution

$$\frac{1}{mx+n}=t$$

these integrals are reduced to integrals of the form 2°.

Example 5. Find

$$\int \frac{dx}{(x+1)\sqrt{x^2+1}}.$$

Solution. We put

$$x+1=\frac{1}{t},$$

whence

$$dx=-\frac{dt}{t^2}.$$

We have:

$$\int \frac{dx}{(x+1)\sqrt{x^2+1}}=\int \frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2+1}}=-\int \frac{dt}{\sqrt{1-2t+2t^2}}=$$

$$=-\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}}=-\frac{1}{\sqrt{2}}\ln\left|t-\frac{1}{2}+\sqrt{t^2-t+\frac{1}{2}}\right|+$$

$$+C=-\frac{1}{\sqrt{2}}\ln\left|\frac{1-x+\sqrt{2(x^2+1)}}{x+1}\right|+C.$$

4°. Integrals of the form $\int \sqrt{ax^2+bx+c}\,dx$. By taking the perfect square out of the quadratic trinomial, the given integral is reduced to one of the following two basic integrals (see examples 1252 and 1253):

1) $\int \sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C;$

$(a>0);$

2) $\int \sqrt{x^2+A}\,dx=\frac{x}{2}\sqrt{x^2+A}+\frac{A}{2}\ln|x+\sqrt{x^2+A}|+C.$

Example 6.

$$\int \sqrt{1-2x-x^2}\,dx=\int \sqrt{2-(1+x)^2}\,d(1+x)=$$

$$=\frac{1+x}{2}\sqrt{1-2x-x^2}+\arcsin\frac{1+x}{\sqrt{2}}+C.$$

Find the following integrals:

1255. $\int \frac{dx}{x^2+2x+5}.$

1256. $\int \frac{dx}{x^2+2x}.$

1257. $\int \frac{dx}{3x^2-x+1}.$

1258. $\int \frac{x\,dx}{x^2-7x+13}.$

1259. $\int \frac{3x-2}{x^2-4x+5}\,dx$.

1260. $\int \frac{(x-1)^2}{x^2+3x+4}\,dx$.

1261. $\int \frac{x^2\,dx}{x^2-6x+10}$

1262. $\int \frac{dx}{\sqrt{2+3x-2x^2}}$.

1263. $\int \frac{dx}{\sqrt{x-x^2}}$

1264. $\int \frac{dx}{\sqrt{x^2+px+q}}$.

1265. $\int \frac{3x-6}{\sqrt{x^2-4x+5}}\,dx$.

1266. $\int \frac{2x-8}{\sqrt{1-x-x^2}}\,dx$.

1267. $\int \frac{x}{\sqrt{5x^2-2x+1}}\,dx$.

1268. $\int \frac{dx}{x\sqrt{1-x^2}}$.

1269. $\int \frac{dx}{x\sqrt{x^2+x-1}}$.

1270. $\int \frac{dx}{(x-1)\sqrt{x^2-2}}$.

1271. $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$.

1272. $\int \sqrt{x^2+2x+5}\,dx$.

1273. $\int \sqrt{x-x^2}\,dx$

1274. $\int \sqrt{2-x-x^2}\,dx$.

1275. $\int \frac{x\,dx}{x^4-4x^2+3}$.

1276. $\int \frac{\cos x}{\sin^2 x-6\sin x+12}\,dx$.

1277. $\int \frac{e^x dx}{\sqrt{1+e^x+e^{2x}}}$

1278. $\int \frac{\sin x\,dx}{\sqrt{\cos^2 x+4\cos x+1}}$.

1279. $\int \frac{\ln x\,dx}{x\sqrt{1-4\ln x-\ln^2 x}}$.

Sec. 5. Integration of Rational Functions

1°. The method of undetermined coefficients. Integration of a rational function, after taking out the whole part, reduces to integration of the *proper rational fraction*

$$\frac{P(x)}{Q(x)}, \tag{1}$$

where $P(x)$ and $Q(x)$ are integral polynomials, and the degree of the numerator $P(x)$ is lower than that of the denominator $Q(x)$.

If

$$Q(x)=(x-a)^{\alpha}\ldots(x-l)^{\lambda},$$

where $a, \ldots, l$ are real distinct roots of the polynomial $Q(x)$, and $\alpha, \ldots, \lambda$ are natural numbers (root multiplicities), then decomposition of (1) into partial fractions is justified:

$$\frac{P(x)}{Q(x)}=\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\ldots+\frac{A_\alpha}{(x-a)^\alpha}+\ldots$$
$$\ldots+\frac{L_1}{x-l}+\frac{L_2}{(x-l)^2}+\ldots+\frac{L_\lambda}{(x-l)^\lambda}. \tag{2}$$

To calculate the undetermined coefficients $A_1, A_2, \ldots$, both sides of the identity (2) are reduced to an integral form, and then the coefficients of like powers of the variable x are equated (**first method**). These coefficients may likewise be determined by putting [in equation (2) or in an equivalent equation] x equal to suitably chosen numbers (**second method**).

Example 1. Find

$$\int \frac{x\,dx}{(x-1)(x+1)^2}=I.$$

Solution. We have:

$$\frac{x}{(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B_1}{x+1}+\frac{B_2}{(x+1)^2}.$$

Whence

$$x\equiv A(x+1)^2+B_1(x-1)(x+1)+B_2(x-1). \qquad (3)$$

a) *First method of determining the coefficients.* We rewrite identity (3) in the form $x\equiv(A+B_1)x^2+(2A+B_2)x+(A-B_1-B_2)$ Equating the coefficients of identical powers of x, we get:

$$0=A+B_1;\quad 1=2A+B_2;\quad 0=A-B_1-B_2.$$

Whence

$$A=\frac{1}{4};\quad B_1=-\frac{1}{4};\quad B_2=\frac{1}{2}.$$

b) *Second method of determining the coefficients.* Putting $x=1$ in identity (3), we will have:

$$1=A\cdot A,\quad \text{i. e.,}\quad A={}^1/_4.$$

Putting $x=-1$, we get:

$$-1=-B_2\cdot 2,\quad \text{i. e.,}\quad B_2={}^1/_2.$$

Further, putting $x=0$, we will have:

$$0=A-B_1-B_2,$$

or $B_1=A-B_2=-{}^1/_4$.
Hence,

$$I=\frac{1}{4}\int\frac{dx}{x-1}-\frac{1}{4}\int\frac{dx}{x+1}+\frac{1}{2}\int\frac{dx}{(x+1)^2}=$$

$$=\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|-\frac{1}{2(x+1)}+C=$$

$$=-\frac{1}{2(x+1)}+\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|+C.$$

Example 2. Find

$$\int\frac{dx}{x^3-2x^2+x}=I.$$

Solution. We have:

$$\frac{1}{x^3-2x^2+x}=\frac{1}{x(x-1)^2}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

and

$$1=A(x-1)^2+Bx(x-1)+Cx. \qquad (4)$$

When solving this example it is advisable to combine the two methods of determining coefficients. Applying the second method, we put $x=0$ in identity (4). We get $1=A$. Then, putting $x=1$, we get $1=C$. Further, applying the first method, we equate the coefficients of x^2 in identity (4), and get·

$$0=A+B,\quad \text{i. e.,}\quad B=-1.$$

Hence,

$$A=1,\quad B=-1,\quad \text{and}\quad C=1.$$

Consequently,

$$I=\int \frac{dx}{x}-\int \frac{dx}{x-1}+\int \frac{dx}{(x-1)^2}=\ln|x|-\ln|x-1|-\frac{1}{x-1}+C.$$

If the polynomial $Q(x)$ has complex roots $a \pm ib$ of multiplicity k, then partial fractions of the form

$$\frac{A_1x+B_1}{x^2+px+q}+\ldots+\frac{A_kx+B_k}{(x^2+px+q)^k} \tag{5}$$

will enter into the expansion (2). Here,

$$x^2+px+q=[x-(a+ib)]\,[x-(a-ib)]$$

and $A_1, B_1, \ldots, A_k, B_k$ are undetermined coefficients which are determined by the methods given above. For $k=1$, the fraction (5) is integrated directly; for $k>1$, use is made of the *reduction method*; here, it is first advisable to represent the quadratic trinomial x^2+px+q in the form $\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)$ and make the substitution $x+\frac{p}{2}=z$.

Example 3. Find

$$\int \frac{x+1}{(x^2+4x+5)^2}\,dx=I.$$

Solution. Since

$$x^2+4x+5=(x+2)^2+1,$$

then, putting $x+2=z$, we get

$$I=\int \frac{z-1}{(z^2+1)^2}\,dz=\int \frac{z\,dz}{(z^2+1)^2}-\int \frac{(1+z^2)-z^2}{(z^2+1)^2}\,dz=$$

$$=-\frac{1}{2(z^2+1)}-\int \frac{dz}{z^2+1}+\int z\,d\left[-\frac{1}{2(z^2+1)}\right]=-\frac{1}{2(z^2+1)}-$$

$$-\arctan z-\frac{z}{2(z^2+1)}+\frac{1}{2}\arctan z=-\frac{z+1}{2(z^2+1)}-$$

$$-\frac{1}{2}\arctan z+C=-\frac{x+3}{2(x^2+4x+5)}-\frac{1}{2}\arctan(x+2)+C.$$

2°. The Ostrogradsky method. If $Q(x)$ has multiple roots, then

$$\int \frac{P(x)}{Q(x)}\,dx=\frac{X(x)}{Q_1(x)}+\int \frac{Y(x)}{Q_2(x)}\,dx, \tag{6}$$

where $Q_1(x)$ is the greatest common divisor of the polynomial $Q(x)$ and its derivative $Q'(x)$;

$$Q_2(x)=Q(x):Q_1(x);$$

$X(x)$ and $Y(x)$ are polynomials with undetermined coefficients, whose degrees are, respectively, less by unity than those of $Q_1(x)$ and $Q_2(x)$.

The undetermined coefficients of the polynomials $X(x)$ and $Y(x)$ are computed by differentiating the identity (6).

Example 4. Find

$$\int \frac{dx}{(x^3-1)^2}.$$

Solution.

$$\int \frac{dx}{(x^3-1)^2}=\frac{Ax^2+Bx+C}{x^3-1}+\int \frac{Dx^2+Ex+F}{x^3-1}\,dx$$

Differentiating this identity, we get

$$\frac{1}{(x^3-1)^2}=\frac{(2Ax+B)(x^3-1)-3x^2(Ax^2+Bx+C)}{(x^3-1)^2}+\frac{Dx^2+Ex+F}{x^3-1}$$

or

$$1=(2Ax+B)(x^3-1)-3x^2(Ax^2+Bx+C)+(Dx^2+Ex+F)(x^3-1).$$

Equating the coefficients of the respective degrees of x, we will have:

$$D=0;\quad E-A=0;\quad F-2B=0;\quad D+3C=0;\quad E+2A=0;\quad B+F=-1;$$

whence

$$A=0;\quad B=-\frac{1}{3};\quad C=0;\quad D=0;\quad E=0;\quad F=-\frac{2}{3}$$

and, consequently,

$$\int \frac{dx}{(x^3-1)^2}=-\frac{1}{3}\frac{x}{x^3-1}-\frac{2}{3}\int \frac{dx}{x^3-1} \qquad (7)$$

To compute the integral on the right of (7), we decompose the fraction $\frac{1}{x^3-1}$ into partial fractions:

$$\frac{1}{x^3-1}=\frac{L}{x-1}+\frac{Mx+N}{x^2+x+1},$$

that is,

$$1=L(x^2+x+1)+Mx(x-1)+N(x-1). \qquad (8)$$

Putting $x=1$, we get $L=\frac{1}{3}$.

Equating the coefficients of identical degrees of x on the right and left of (8), we find

$$L+M=0;\quad L-N=1,$$

or

$$M=-\frac{1}{3};\quad N=-\frac{2}{3}.$$

Therefore,

$$\int \frac{dx}{x^3-1}=\frac{1}{3}\int \frac{dx}{x-1}-\frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx=$$

$$=\frac{1}{3}\ln|x-1|-\frac{1}{6}\ln(x^2+x+1)-\frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C$$

and

$$\int \frac{dx}{(x^3-1)^2}=-\frac{x}{3(x^3-1)}+\frac{1}{9}\ln\frac{x^2+x+1}{(x-1)^2}+\frac{2}{3\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C.$$

Find the following integrals:

1280. $\int \frac{dx}{(x+a)(x+b)}$.

1281. $\int \frac{x^2-5x+9}{x^2-5x+6}\,dx$.

1282. $\int \frac{dx}{(x+1)(x+2)(x+3)}$.

1283. $\int \frac{2x^2+41x-91}{(x-1)(x+3)(x-4)}\,dx$.

1284. $\int \frac{5x^3+2}{x^3-5x^2+4x}\,dx.$

1285. $\int \frac{dx}{x(x+1)^2}.$

1286. $\int \frac{x^3-1}{4x^3-x}\,dx.$

1287. $\int \frac{x^4-6x^3+12x^2+6}{x^3-6x^2+12x-8}\,dx.$

1288. $\int \frac{5x^2+6x+9}{(x-3)^2(x+1)^2}\,dx.$

1289. $\int \frac{x^2-8x+7}{(x^2-3x-10)^2}\,dx.$

1290. $\int \frac{2x-3}{(x^2-3x+2)^3}\,dx.$

1291. $\int \frac{x^3+x+1}{x(x^2+1)}\,dx.$

1292. $\int \frac{x^4}{x^4-1}\,dx.$

1293. $\int \frac{dx}{(x^2-4x+3)(x^2+4x+5)}.$

1294. $\int \frac{dx}{x^3+1}.$

1295. $\int \frac{dx}{x^4+1}.$

1296. $\int \frac{dx}{x^4+x^2+1}.$

1297. $\int \frac{dx}{(1+x^2)^2}.$

1298. $\int \frac{3x+5}{(x^2+2x+2)^2}\,dx.$

1299. $\int \frac{dx}{(x+1)(x^2+x+1)^2}.$

1300. $\int \frac{x^3+1}{(x^2-4x+5)^2}\,dx.$

Applying Ostrogradsky's method, find the following integrals:

1301. $\int \frac{dx}{(x+1)^2(x^2+1)^2}.$

1302. $\int \frac{dx}{(x^4-1)^2}.$

1303. $\int \frac{dx}{(x^2+1)^4}.$

1304. $\int \frac{x^4-2x^2+2}{(x^2-2x+2)^2}\,dx.$

Applying different procedures, find the integrals:

1305. $\int \frac{x^5}{(x^3+1)(x^3+8)}\,dx.$

1306. $\int \frac{x^7+x^3}{x^{12}-2x^4+1}\,dx.$

1307. $\int \frac{x^2-x+14}{(x-4)^3(x-2)}\,dx.$

1308. $\int \frac{dx}{x^4(x^3+1)^2}$

1309. $\int \frac{dx}{x^3-4x^2+5x-2}.$

1310*. $\int \frac{dx}{x(x^7+1)}.$

1311. $\int \frac{dx}{x(x^5+1)^2}.$

1312. $\int \frac{dx}{(x^2+2x+2)(x^2+2x+5)}.$

1313. $\int \frac{x^2dx}{(x-1)^{10}}.$

1314. $\int \frac{dx}{x^8+x^6}.$

Sec. 6. Integrating Certain Irrational Functions

1°. Integrals of the form

$$\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_1}{q_1}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_2}{q_2}}, \ldots\right]dx, \qquad (1)$$

where R is a rational function and p_1, q_1, p_2, q_2 are whole numbers.

Integrals of form (1) are found by the substitution

$$\frac{ax+b}{cx+d}=z^n,$$

where n is the least common multiple of the numbers $q_1, q_2, \ldots$

Example 1. Find

$$\int \frac{dx}{\sqrt{2x-1}-\sqrt[4]{2x-1}}.$$

Solution. The substitution $2x-1=z^4$ leads to an integral of the form

$$\int \frac{dx}{\sqrt{2x-1}-\sqrt[4]{2x-1}}=\int \frac{2z^3dz}{z^2-z}=2\int \frac{z^2dz}{z-1}=$$

$$=2\int \left(z+1+\frac{1}{z-1}\right)dz=(z+1)^2+2\ln|z-1|+C=$$

$$=(1+\sqrt[4]{2x-1})^2+\ln(\sqrt[4]{2x-1}-1)^2+C.$$

Find the integrals:

1315. $\int \frac{x^3}{\sqrt{x-1}}dx.$

1316. $\int \frac{x\,dx}{\sqrt[3]{ax+b}}.$

1317. $\int \frac{dx}{\sqrt{x+1}+\sqrt{(x+1)^3}}.$

1318. $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}.$

1319. $\int \frac{\sqrt{x}-1}{\sqrt[3]{x}+1}dx.$

1320. $\int \frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}}dx.$

1321. $\int \frac{\sqrt{x}}{x+2}dx.$

1322. $\int \frac{dx}{(2-x)\sqrt{1-x}}.$

1323. $\int x\sqrt{\frac{x-1}{x+1}}dx.$

1324. $\int \sqrt[3]{\frac{x+1}{x-1}}dx.$

1325. $\int \frac{x+3}{x^2\sqrt{2x+3}}dx.$

2°. Integrals of the form

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx, \qquad (2)$$

where $P_n(x)$ is a polynomial of degree n

Put

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx=Q_{n-1}(x)\sqrt{ax^2+bx+c}+\lambda\int \frac{dx}{\sqrt{ax^2+bx+c}}, \qquad (3)$$

where $Q_{n-1}(x)$ is a polynomial of degree $(n-1)$ with undetermined coefficients and λ is a number.

The coefficients of the polynomial $Q_{n-1}(x)$ and the number λ are found by differentiating identity (3).

Example 2.

$$\int x^2 \sqrt{x^2+4}\, dx = \int \frac{x^4+4x^2}{\sqrt{x^2+4}}\, dx =$$

$$= (Ax^3+Bx^2+Cx+D)\sqrt{x^2+4} + \lambda \int \frac{dx}{\sqrt{x^2+4}}.$$

Whence

$$\frac{x^4+4x^2}{\sqrt{x^2+4}} = (3Ax^2+2Bx+C)\sqrt{x^2+4} + \frac{(Ax^3+Bx^2+Cx+D)\,x}{\sqrt{x^2+4}} + \frac{\lambda}{\sqrt{x^2+4}}.$$

Multiplying by $\sqrt{x^2+4}$ and equating the coefficients of identical degrees of x, we obtain

$$A=\frac{1}{4};\quad B=0;\quad C=\frac{1}{2};\quad D=0;\quad \lambda=-2$$

Hence,

$$\int x^2\sqrt{x^2+4}\, dx = \frac{x^3+2x}{4}\sqrt{x^2+4} - 2\ln\left(x+\sqrt{x^2+4}\right) + C.$$

3°. Integrals of the form

$$\int \frac{dx}{(x-\alpha)^n \sqrt{ax^2+bx+c}}. \qquad (4)$$

They are reduced to integrals of the form (2) by the substitution:

$$\frac{1}{x-\alpha} = t.$$

Find the integrals:

1326. $\int \frac{x^2\,dx}{\sqrt{x^2-x+1}}$.

1327. $\int \frac{x^5}{\sqrt{1-x^2}}\, dx$.

1328. $\int \frac{x^6}{\sqrt{1+x^2}}\, dx$.

1329. $\int \frac{dx}{x^5\sqrt{x^2-1}}$.

1330. $\int \frac{dx}{(x+1)^3\sqrt{x^2+2x}}$.

1331. $\int \frac{x^2+x+1}{x\sqrt{x^2-x+1}}\, dx$.

4°. Integrals of the binomial differentials

$$\int x^m (a+bx^n)^p\, dx, \qquad (5)$$

where m, n and p are rational numbers.

Chebyshev's conditions. The integral (5) can be expressed in terms of a finite combination of elementary functions only in the following three cases:

1) if p is a whole number;

2) if $\frac{m+1}{n}$ is a whole number. Here, we make the substitution $a+bx^n = z^s$, where s is the denominator of the fraction p;

3) if $\frac{m+1}{n}+p$ is a whole number. Here, use is made of the substitution $ax^{-n}+b=z^s$.

Example 3. Find

$$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx = I.$$

Solution. Here, $m=-\frac{1}{2}$; $n=\frac{1}{4}$; $p=\frac{1}{3}$; $\frac{m+1}{n}=\frac{-\frac{1}{2}+1}{\frac{1}{4}}=2$. Hence, we have here Case 2 integrability.

The substitution

$$1+x^{\frac{1}{4}}=z^3$$

yields $x=(z^3-1)^4$; $dx=12z^2(z^3-1)^3\,dz$ Therefore,

$$I=\int x^{-\frac{1}{2}}\left(1+x^{\frac{1}{4}}\right)^{\frac{1}{3}}dx=12\int \frac{z^3(z^3-1)^3}{(z^3-1)^2}\,dz=$$

$$=12\int (z^6-z^3)\,dz=\frac{12}{7}z^7-3z^4+C,$$

where $z=\sqrt[3]{1+\sqrt[4]{x}}$.

Find the integrals:

1332. $\int x^3(1+2x^2)^{-\frac{3}{2}}\,dx.$

1333. $\int \frac{dx}{\sqrt[4]{1+x^4}}.$

1334. $\int \frac{dx}{x^4\sqrt{1+x^2}}.$

1335. $\int \frac{dx}{x\sqrt[3]{1+x^5}}.$

1336. $\int \frac{dx}{x^2(2+x^3)^{\frac{5}{3}}}.$

1337. $\int \frac{dx}{\sqrt{x^3}\sqrt[3]{1+\sqrt[4]{x^3}}}.$

Sec. 7. Integrating Trigonometric Functions

1°. Integrals of the form

$$\int \sin^m x\cos^n x\,dx=I_{m,n}, \quad (1)$$

where m and n are integers.

1) If $m=2k+1$ is an odd positive number, then we put

$$I_{m,n}=-\int \sin^{2k}x\cos^n x\,d(\cos x)=-\int (1-\cos^2 x)^k\cos^n x\,d(\cos x).$$

We do the same if n is an odd positive number.

Example 1.

$$\int \sin^{10}x\cos^3 x\,dx=\int \sin^{10}x(1-\sin^2 x)\,d(\sin x)=$$

$$=\frac{\sin^{11}x}{11}-\frac{\sin^{13}x}{13}+C.$$

2) If m and n are even positive numbers, then the integrand (1) is transformed by means of the formulas

$$\sin^2 x = \frac{1}{2}(1-\cos 2x), \quad \cos^2 x = \frac{1}{2}(1+\cos 2x),$$
$$\sin x \cos x = \frac{1}{2}\sin 2x.$$

Example 2. $\int \cos^2 3x \sin^4 3x \, dx = \int (\cos 3x \sin 3x)^2 \sin^2 3x \, dx =$

$$= \int \frac{\sin^2 6x}{4}\,\frac{1-\cos 6x}{2}\,dx = \frac{1}{8}\int (\sin^2 6x - \sin^2 6x \cos 6x)\,dx =$$
$$= \frac{1}{8}\int \left(\frac{1-\cos 12x}{2} = \sin^2 6x \cos 6x\right) dx =$$
$$= \frac{1}{8}\left(\frac{x}{2} - \frac{\sin 12x}{24} - \frac{1}{18}\sin^3 6x\right) + C.$$

3) If $m = -\mu$ and $n = -\nu$ are integral negative numbers of identical parity, then

$$I_{m,\,n} = \int \frac{dx}{\sin^{\mu} x \cos^{\nu} x} = \int \operatorname{cosec}^{\mu} x \sec^{\nu-2} x d(\tan x) =$$
$$= \int \left(1 + \frac{1}{\tan^2 x}\right)^{\frac{\mu}{2}} (1+\tan^2 x)^{\frac{\nu-2}{2}} d(\tan x) = \int \frac{(1+\tan^2 x)^{\frac{\mu+\nu}{2}-1}}{\tan^{\mu} x}\, d(\tan x).$$

In particular, the following integrals reduce to this case:

$$\int \frac{dx}{\sin^{\mu} x} = \frac{1}{2^{\mu-1}} \int \frac{d\left(\frac{x}{2}\right)}{\sin^{\mu}\frac{x}{2}\cos^{\mu}\frac{x}{2}} \quad \text{and} \quad \int \frac{dx}{\cos^{\nu} x} = \int \frac{d\left(x+\frac{\pi}{2}\right)}{\sin^{\nu}\left(x+\frac{\pi}{2}\right)}.$$

Example 3. $\int \frac{dx}{\cos^4 x} = \int \sec^2 x d(\tan x) = \int (1+\tan^2 x)\, d(\tan x) =$

$$= \tan x + \frac{1}{3}\tan^3 x + C.$$

Example 4. $\int \frac{dx}{\sin^3 x} = \frac{1}{2^3}\int \frac{dx}{\sin^3\frac{x}{2}\cos^3\frac{x}{2}} = \frac{1}{8}\int \tan^{-3}\frac{x}{2}\sec^6\frac{x}{2}\,dx =$

$$= \frac{1}{8}\int \frac{\left(1+\tan^2\frac{x}{2}\right)^2}{\tan^3\frac{x}{2}}\sec^2\frac{x}{2}\,dx = \frac{2}{8}\int \left[\tan^{-3}\frac{x}{2} + \frac{2}{\tan\frac{x}{2}} + \right.$$
$$\left. + \tan\frac{x}{2}\right] d\left(\tan\frac{x}{2}\right) = \frac{1}{4}\left[-\frac{1}{2\tan^2\frac{x}{2}} + 2\ln\left|\tan\frac{x}{2}\right| + \frac{\tan^2\frac{x}{2}}{2}\right] + C.$$

4) Integrals of the form $\int \tan^m x\,dx$ (or $\int \cot^m x\,dx$), where m is an integral positive number, are evaluated by the formula

$$\tan^2 x = \sec^2 x - 1$$

(or, respectively, $\cot^2 x = \operatorname{cosec}^2 x - 1$).

Example 5. $\int \tan^4 x\,dx = \int \tan^2 x(\sec^2 x - 1)\,dx = \frac{\tan^3 x}{3} - \int \tan^2 x\,dx = \frac{\tan^3 x}{3} - \int (\sec^2 x - 1)\,dx = \frac{\tan^3 x}{3} - \tan x + x + C.$

5) In the general case, integrals $I_{m,n}$ of the form (1) are evaluated by means of *reduction formulas* that are usually derived by integration by parts.

Example 6. $\int \frac{dx}{\cos^3 x} = \int \frac{\sin^2 x + \cos^2 x}{\cos^3 x}\,dx =$

$= \int \sin x \cdot \frac{\sin x}{\cos^3 x}\,dx + \int \frac{dx}{\cos x} = \sin x \cdot \frac{1}{2\cos^2 x} - \frac{1}{2}\int \frac{\cos x}{\cos^2 x}\,dx + \int \frac{dx}{\cos x} =$

$= \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\ln|\tan x + \sec x| + C.$

Find the integrals:

1338. $\int \cos^2 x\,dx.$

1339. $\int \sin^5 x\,dx.$

1340. $\int \sin^2 x \cos^3 x\,dx.$

1341. $\int \sin^3 \frac{x}{2} \cos^5 \frac{x}{2}\,dx.$

1342. $\int \frac{\cos^5 x}{\sin^3 x}\,dx.$

1343. $\int \sin^4 x\,dx.$

1344. $\int \sin^2 x \cos^2 x\,dx.$

1345. $\int \sin^2 x \cos^4 x\,dx.$

1346. $\int \cos^6 3x\,dx.$

1347. $\int \frac{dx}{\sin^4 x}.$

1348. $\int \frac{dx}{\cos^6 x}.$

1349. $\int \frac{\cos^2 x}{\sin^6 x}\,dx.$

1350. $\int \frac{dx}{\sin^2 x \cos^4 x}.$

1351. $\int \frac{dx}{\sin^5 x \cos^3 x}.$

1352. $\int \frac{dx}{\sin \frac{x}{2} \cos^3 \frac{x}{2}}.$

1353. $\int \frac{\sin\left(x + \frac{\pi}{4}\right)}{\sin x \cos x}\,dx.$

1354. $\int \frac{dx}{\sin^5 x}.$

1355. $\int \sec^5 4x\,dx.$

1356. $\int \tan^2 5x\,dx.$

1357. $\int \cot^3 x\,dx.$

1358. $\int \cot^4 x\,dx.$

1359. $\int \left(\tan^3 \frac{x}{3} + \tan^4 \frac{x}{4}\right) dx.$

1360. $\int x \sin^2 x^2\,dx.$

1361. $\int \frac{\cos^2 x}{\sin^4 x}\,dx.$

1362. $\int \sin^3 x \sqrt[3]{\cos x}\,dx.$

1363. $\int \frac{dx}{\sqrt{\sin x \cos^3 x}}.$

1364. $\int \frac{dx}{\sqrt{\tan x}}.$

2°. Integrals of the form $\int \sin mx \cos nx\,dx$, $\int \sin mx \sin nx\,dx$ and $\int \cos mx \cos nx\,dx$. In these cases the following formulas are used:

1) $\sin mx \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x]$;

2) $\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$;

3) $\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$.

Example 7. $\int \sin 9x \sin x\,dx = \int \frac{1}{2}[\cos 8x - \cos 10x]\,dx =$
$= \frac{1}{16}\sin 8x - \frac{1}{20}\sin 10x + C$.

Find the integrals:

1365. $\int \sin 3x \cos 5x\,dx$.

1366. $\int \sin 10x \sin 15x\,dx$.

1367. $\int \cos\frac{x}{2}\cos\frac{x}{3}\,dx$.

1368. $\int \sin\frac{x}{3}\sin\frac{2x}{3}\,dx$.

1369. $\int \cos(ax+b)\cos(ax-b)\,dx$.

1370. $\int \sin\omega t \sin(\omega t + \varphi)\,dt$.

1371. $\int \cos x \cos^2 3x\,dx$.

1372. $\int \sin x \sin 2x \sin 3x\,dx$.

3°. Integrals of the form

$$\int R(\sin x, \cos x)\,dx, \qquad (2)$$

where R is a rational function.

1) By means of substitution

$$\tan\frac{x}{2} = t,$$

whence

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2dt}{1+t^2},$$

integrals of form (2) are reduced to integrals of rational functions by the new variable t.

Example 8. Find

$$\int \frac{dx}{1+\sin x+\cos x} = I.$$

Solution. Putting $\tan\frac{x}{2} = t$, we will have

$$I = \int \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} = \int \frac{dt}{1+t} = \ln|1+t| + C = \ln\left|1+\tan\frac{x}{2}\right| + C.$$

2) If we have the identity

$$R(-\sin x, \ -\cos x) \equiv R(\sin x, \ \cos x),$$

then we can use the substitution $\tan x = t$ to reduce the integral (2) to a rational form.

Here,

$$\sin x = \frac{t}{\sqrt{1+t^2}}, \ \cos x = \frac{1}{\sqrt{1+t^2}}$$

and

$$x = \arctan t, \ dx = \frac{dt}{1+t^2}.$$

Example 9. Find

$$\int \frac{dx}{1+\sin^2 x} = I. \tag{3}$$

Solution. Putting

$$\tan x = t, \quad \sin^2 x = \frac{t^2}{1+t^2}, \quad dx = \frac{dt}{1+t^2},$$

we will have

$$I = \int \frac{dt}{(1+t^2)\left(1+\frac{t^2}{1+t^2}\right)} = \int \frac{dt}{1+2t^2} = \frac{1}{\sqrt{2}} \int \frac{d(t\sqrt{2})}{1+(t\sqrt{2})^2} =$$

$$= \frac{1}{\sqrt{2}} \arctan(t\sqrt{2}) + C = \frac{1}{\sqrt{2}} \arctan(\sqrt{2} \tan x) + C.$$

We note that the integral (3) is evaluated faster if the numerator and denominator of the fraction are first divided by $\cos^2 x$.

In individual cases, it is useful to apply artificial procedures (see, for example, 1379).

Find the integrals:

1373. $\int \frac{dx}{3+5\cos x}$.

1374. $\int \frac{dx}{\sin x + \cos x}$.

1375. $\int \frac{\cos x}{1+\cos x}\, dx$.

1376. $\int \frac{\sin x}{1-\sin x}\, dx$.

1377. $\int \frac{dx}{8-4\sin x+7\cos x}$.

1378. $\int \frac{dx}{\cos x+2\sin x+3}$.

1379.** $\int \frac{3\sin x+2\cos x}{2\sin x+3\cos x}\, dx$.

1380. $\int \frac{1+\tan x}{1-\tan x}\, dx$.

1381*. $\int \frac{dx}{1+3\cos^2 x}$.

1382*. $\int \frac{dx}{3\sin^2 x+5\cos^2 x}$.

1383*. $\int \frac{dx}{\sin^2 x+3\sin x\cos x-\cos^2 x}$.

1384*. $\int \frac{dx}{\sin^2 x-5\sin x\cos x}$.

1385. $\int \frac{\sin x}{(1-\cos x)^3}\, dx$.

1386. $\int \frac{\sin 2x}{1+\sin^2 x}\, dx$.

1387. $\int \frac{\cos 2x}{\cos^4 x+\sin^4 x}\, dx$.

1388. $\int \frac{\cos x}{\sin^2 x-6\sin x+5}\, dx$.

1389*. $\int \frac{dx}{(2-\sin x)(3-\sin x)}$.

1390*. $\int \frac{1-\sin x+\cos x}{1+\sin x-\cos x}\, dx$.

Sec. 8. Integration of Hyperbolic Functions

Integration of hyperbolic functions is completely analogous to the integration of trigonometric functions.

The following basic formulas should be remembered:

1) $\cosh^2 x - \sinh^2 x = 1$;

2) $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$;

3) $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$;

4) $\sinh x \cosh x = \frac{1}{2}\sinh 2x$.

Example 1. Find

$$\int \cosh^2 x\,dx.$$

Solution. We have

$$\int \cosh^2 x\,dx = \int \frac{1}{2}(\cosh 2x + 1)\,dx = \frac{1}{4}\sinh 2x + \frac{1}{2}x + C.$$

Example 2. Find

$$\int \cosh^3 x\,dx.$$

Solution. We have

$$\int \cosh^3 x\,dx = \int \cosh^2 x\,d(\sinh x) = \int (1 + \sinh^2 x)\,d(\sinh x) = \sinh x + \frac{\sinh^3 x}{3} + C.$$

Find the integrals:

1391. $\int \sinh^3 x\,dx$.

1392. $\int \cosh^4 x\,dx$.

1393. $\int \sinh^3 x \cosh x\,dx$.

1394. $\int \sinh^2 x \cosh^2 x\,dx$.

1395. $\int \frac{dx}{\sinh x \cosh^2 x}$.

1396. $\int \frac{dx}{\sinh^2 x \cosh^2 x}$.

1397. $\int \tanh^3 x\,dx$.

1398. $\int \coth^4 x\,dx$.

1399. $\int \frac{dx}{\sinh^2 x + \cosh^2 x}$.

1400. $\int \frac{dx}{2\sinh x + 3\cosh x}$.

1401*. $\int \frac{dx}{\tanh x - 1}$.

1402. $\int \frac{\sinh x\,dx}{\sqrt{\cosh 2x}}$.

Sec. 9. Using Trigonometric and Hyperbolic Substitutions for Finding Integrals of the Form

$$\int R(x, \sqrt{ax^2 + bx + c})\,dx, \qquad (1)$$

where R is a rational function.

Transforming the quadratic trinomial ax^2+bx+c into a sum or difference of squares, the integral (1) becomes reducible to one of the following types of integrals:

1) $\int R(z, \sqrt{m^2-z^2})\,dz$;

2) $\int R(z, \sqrt{m^2+z^2})\,dz$;

3) $\int R(z, \sqrt{z^2-m^2})\,dz$.

The latter integrals are, respectively, taken by means of substitutions:

1) $z=m\sin t$ or $z=m\tanh t$,
2) $z=m\tan t$ or $z=m\sinh t$,
3) $z=m\sec t$ or $z=m\cosh t$.

Example 1. Find

$$\int \frac{dx}{(x+1)^2\sqrt{x^2+2x+2}}=I.$$

Solution. We have

$$x^2+2x+2=(x+1)^2+1.$$

Putting $x+1=\tan z$, we then have $dx=\sec^2 z\,dz$ and

$$I=\int\frac{dx}{(x+1)^2\sqrt{(x+1)^2+1}}=\int\frac{\sec^2 z\,dz}{\tan^2 z\sec z}=\int\frac{\cos z}{\sin^2 z}\,dz=$$

$$=-\frac{1}{\sin z}+C=\frac{\sqrt{x^2+2x+2}}{x+1}+C.$$

Example 2. Find

$$\int x\sqrt{x^2+x+1}\,dx=I.$$

Solution. We have

$$x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}.$$

Putting

$$x+\frac{1}{2}=\frac{\sqrt{3}}{2}\sinh t \quad\text{and}\quad dx=\frac{\sqrt{3}}{2}\cosh t\,dt,$$

we get

$$I=\int\left(\frac{\sqrt{3}}{2}\sinh t-\frac{1}{2}\right)\frac{\sqrt{3}}{2}\cosh t\cdot\frac{\sqrt{3}}{2}\cosh t\,dt=$$

$$=\frac{3\sqrt{3}}{8}\int\sinh t\cosh^2 t\,dt-\frac{3}{8}\int\cosh^2 t\,dt=$$

$$=\frac{3\sqrt{3}}{8}\,\frac{\cosh^3 t}{3}-\frac{3}{8}\left(\frac{1}{2}\sinh t\cosh t+\frac{1}{2}t\right)+C.$$

Since

$$\sinh t=\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right),\quad \cosh t=\frac{2}{\sqrt{3}}\sqrt{x^2+x+1}$$

and

$$t=\ln\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)+\ln\frac{2}{\sqrt{3}},$$

we finally have

$$I=\frac{1}{3}(x^2+x+1)^{\frac{3}{2}}-\frac{1}{4}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1}-$$
$$-\frac{3}{16}\ln\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)+C.$$

Find the integrals:

1403. $\int\sqrt{3-2x-x^2}\,dx.$

1404. $\int\sqrt{2+x^2}\,dx.$

1405. $\int\frac{x^2}{\sqrt{9+x^2}}\,dx.$

1406. $\int\sqrt{x^2-2x+2}\,dx.$

1407. $\int\sqrt{x^2-4}\,dx.$

1408. $\int\sqrt{x^2+x}\,dx.$

1409. $\int\sqrt{x^2-6x-7}\,dx.$

1410. $\int(x^2+x+1)^{\frac{3}{2}}\,dx.$

1411. $\int\frac{dx}{(x-1)\sqrt{x^2-3x+2}}.$

1412. $\int\frac{dx}{(x^2-2x+5)^{\frac{3}{2}}}.$

1413. $\int\frac{dx}{(1+x^2)\sqrt{1-x^2}}.$

1414. $\int\frac{dx}{(1-x^2)\sqrt{1+x^2}}.$

Sec. 10. Integration of Various Transcendental Functions

Find the integrals:

1415. $\int(x^2+1)^2e^{2x}\,dx.$

1416. $\int x^2\cos^2 3x\,dx.$

1417. $\int x\sin x\cos 2x\,dx.$

1418. $\int e^{2x}\sin^2 x\,dx.$

1419. $\int e^x\sin x\sin 3x\,dx.$

1420. $\int xe^x\cos x\,dx.$

1421. $\int\frac{dx}{e^{2x}+e^x-2}.$

1422. $\int\frac{dx}{\sqrt{e^{2x}+e^x+1}}.$

1423. $\int x^2\ln\frac{1+x}{1-x}\,dx.$

1424. $\int\ln^2(x+\sqrt{1+x^2})\,dx.$

1425. $\int x\arccos(5x-2)\,dx.$

1426. $\int\sin x\sinh x\,dx.$

Sec. 11. Using Reduction Formulas

Derive the reduction formulas for the following integrals:

1427. $I_n=\int\frac{dx}{(x^2+a^2)^n}$; find I_2 and I_3.

1428. $I_n=\int\sin^n x\,dx$; find I_4 and I_5.

1429. $I_n=\int \frac{dx}{\cos^n x}$; find I_3 and I_4.

1430. $I_n=\int x^n e^{-x}\,dx$; find I_{10}.

Sec. 12. Miscellaneous Examples on Integration

1431. $\int \frac{dx}{2x^2-4x+9}$.

1432. $\int \frac{x-5}{x^2-2x+2}\,dx$.

1433. $\int \frac{x^3}{x^2+x+\frac{1}{2}}\,dx$.

1434. $\int \frac{dx}{x(x^2+5)}$.

1435. $\int \frac{dx}{(x+2)^2(x+3)^2}$.

1436. $\int \frac{dx}{(x+1)^2(x^2+1)}$.

1437. $\int \frac{dx}{(x^2+2)^2}$.

1438. $\int \frac{dx}{x^4-2x^2+1}$.

1439. $\int \frac{x\,dx}{(x^2-x+1)^3}$.

1440. $\int \frac{3-4x}{(1-2\sqrt{x})^2}\,dx$.

1441. $\int \frac{(\sqrt{x}+1)^2}{x^3}\,dx$.

1442. $\int \frac{dx}{\sqrt{x^2+x+1}}$.

1443. $\int \frac{1-\sqrt[3]{2x}}{\sqrt{2x}}\,dx$.

1444. $\int \frac{dx}{(\sqrt[3]{x^2}+\sqrt[3]{x})^2}$.

1445. $\int \frac{2x+1}{\sqrt{(4x^2-2x+1)^3}}\,dx$.

1446. $\int \frac{dx}{\sqrt[4]{5-x}+\sqrt{5-x}}$.

1447. $\int \frac{x^2}{\sqrt{(x^2-1)^3}}\,dx$.

1448. $\int \frac{x\,dx}{(1+x^2)\sqrt{1-x^4}}$.

1449. $\int \frac{x\,dx}{\sqrt{1-2x^2-x^4}}$.

1450. $\int \frac{x+1}{(x^2+1)^{\frac{3}{2}}}\,dx$.

1451*. $\int \frac{dx}{(x^2+4x)\sqrt{4-x^2}}$.

1452. $\int \sqrt{x^2-9}\,dx$.

1453. $\int \sqrt{x-4x^2}\,dx$.

1454. $\int \frac{dx}{x\sqrt{x^2+x+1}}$.

1455. $\int x\sqrt{x^2+2x+2}\,dx$.

1456. $\int \frac{dx}{x^4\sqrt{x^2-1}}$.

1457. $\int \frac{dx}{x\sqrt{1-x^3}}$.

1458. $\int \frac{dx}{\sqrt[3]{1+x^3}}$.

1459. $\int \frac{5x}{\sqrt{1+x^4}}\,dx$.

1460. $\int \cos^4 x\,dx$.

1461. $\int \frac{dx}{\cos x\sin^5 x}$.

1462. $\int \frac{1+\sqrt{\cot x}}{\sin^2 x}\,dx$.

1463. $\int \frac{\sin^3 x}{\sqrt[5]{\cos^3 x}}\,dx$.

1464. $\int \operatorname{cosec}^5 5x\,dx$.

1465. $\int \frac{\sin^2 x}{\cos^6 x}\,dx$.

1466. $\int \sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}+x\right) dx.$

1467. $\int \tan^2\left(\frac{x}{2}+\frac{\pi}{4}\right) dx.$

1468. $\int \frac{dx}{2\sin x+3\cos x-5}.$

1469. $\int \frac{dx}{2+3\cos^2 x}.$

1470. $\int \frac{dx}{\cos^2 x+2\sin x\cos x+2\sin^2 x}.$

1471. $\int \frac{dx}{\sin x \sin 2x}.$

1472. $\int \frac{dx}{(2+\cos x)(3+\cos x)}.$

1473. $\int \frac{\sec^2 x}{\sqrt{\tan^2 x+4\tan x+1}} dx.$

1474. $\int \frac{\cos ax}{\sqrt{a^2+\sin^2 ax}} dx.$

1475. $\int \frac{x\,dx}{\cos^2 3x}.$

1476. $\int x\sin^2 x\, dx.$

1477. $\int x^2 e^{x^3} dx.$

1478. $\int xe^{2x} dx.$

1479. $\int x^2 \ln\sqrt{1-x}\, dx.$

1480. $\int \frac{x \arctan x}{\sqrt{1+x^2}} dx.$

1481. $\int \sin^2\frac{x}{2}\cos\frac{3x}{2} dx.$

1482. $\int \frac{dx}{(\sin x+\cos x)^2}.$

1483. $\int \frac{dx}{(\tan x+1)\sin^2 x}.$

1484. $\int \sinh x \cosh x\, dx.$

1485. $\int \frac{\sinh\sqrt{1-x}}{\sqrt{1-x}} dx.$

1486. $\int \frac{\sinh x\cosh x}{\sinh^2 x+\cosh^2 x} dx.$

1487. $\int \frac{x}{\sinh^2 x} dx.$

1488. $\int \frac{dx}{e^{2x}-2e^x}.$

1489. $\int \frac{e^x}{e^{2x}-6e^x+13} dx.$

1490. $\int \frac{e^{2x}}{(e^x+1)^{\frac{1}{4}}} dx.$

1491. $\int \frac{2^x}{1-4^x} dx.$

1492. $\int (x^2-1)\,10^{-2x} dx.$

1493. $\int \sqrt{e^x+1}\, dx.$

1494. $\int \frac{\arctan x}{x^2} dx.$

1495. $\int x^3 \arcsin\frac{1}{x} dx.$

1496. $\int \cos(\ln x)\, dx.$

1497. $\int (x^2-3x)\sin 5x\, dx.$

1498. $\int x \arctan(2x+3)\, dx.$

1499. $\int \arcsin\sqrt{x}\, dx.$

1500. $\int |x|\, dx.$

Chapter V

DEFINITE INTEGRALS

Sec. 1. The Definite Integral as the Limit of a Sum

1°. Integral sum. Let a function $f(x)$ be defined on an interval $a \leqslant x \leqslant b$, and $a = x_0 < x_1 < \ldots < x_n = b$ is an arbitrary partition of this interval into n subintervals (Fig. 37). A sum of the form

$$S_n = \sum_{i=0}^{n-1} f(\xi_i)\,\Delta x_i, \tag{1}$$

where

$$x_i \leqslant \xi_i \leqslant x_{i+1}; \quad \Delta x_i = x_{i+1} - x_i;$$
$$i = 0, 1, 2, \ldots (n-1),$$

is called the *integral sum* of the function $f(x)$ on $[a, b]$. Geometrically, S_n is the algebraic area of a step-like figure (see Fig. 37).

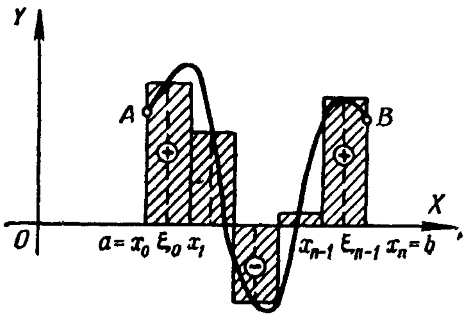

Fig. 37

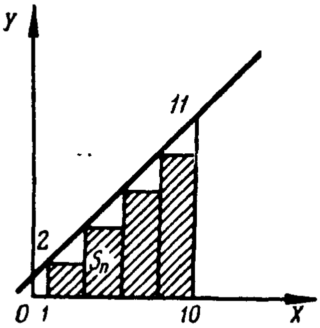

Fig. 38

2°. The definite integral. The limit of the sum S_n, provided that the number of subdivisions n tends to infinity, and the largest of them, Δx_i, to zero, is called the *definite integral* of the function $f(x)$ within the limits from $x = a$ to $x = b$; that is,

$$\lim_{\max \Delta x_i \to 0} \sum_{i=0}^{n-1} f(\xi_i)\,\Delta x_i = \int_a^b f(x)\,dx. \tag{2}$$

If the function $f(x)$ is continuous on $[a, b]$, it is integrable on $[a, b]$; i.e., the limit of (2) exists and is independent of the mode of partition of the interval of integration $[a, b]$ into subintervals and is independent of the choice of points ξ_i in these subintervals. Geometrically, the definite integral (2) is the algebraic sum of the areas of the figures that make up the curvilinear trapezoid $aABb$, in which the areas of the parts located above the x-axis are plus, those below the x-axis, minus (Fig. 37).

The definitions of integral sum and definite integral are naturally generalized to the case of an interval $[a, b]$, where $a > b$.

Example 1. Form the integral sum S_n for the function

$$f(x) = 1 + x$$

on the interval [1,10] by dividing the interval into n equal parts and choosing points ξ_i that coincide with the left end-points of the subintervals $[x_i, x_{i+1}]$. What is the $\lim\limits_{n \to \infty} S_n$ equal to?

Solution. Here, $\Delta x_i = \frac{10-1}{n} = \frac{9}{n}$ and $\xi_i = x_i = x_0 + i\Delta x_i = 1 + \frac{9i}{n}$. Whence $(\xi_i) = 1 + 1 + \frac{9i}{n} = 2 + \frac{9i}{n}$. Hence (Fig. 38),

$$S_n = \sum_{i=0}^{n-1} f(\xi_i)\,\Delta x_i = \sum_{i=0}^{n-1}\left(2 + \frac{9i}{n}\right)\frac{9}{n} = \frac{18}{n}n + \frac{81}{n^2}(0 + 1 + \ldots + n - 1) =$$

$$= 18 + \frac{81}{n^2}\,\frac{n(n-1)}{2} = 18 + \frac{81}{2}\left(1 - \frac{1}{n}\right) = 58\frac{1}{2} - \frac{81}{2n},$$

$$\lim_{n \to \infty} S_n = 58\frac{1}{2}.$$

Example 2. Find the area bounded by an arc of the parabola $y = x^2$, the x-axis, and the ordinates $x = 0$, and $x = a$ $(a > 0)$.

Solution. Partition the base a into n equal parts $\Delta x = \frac{a}{n}$. Choosing the value of the function at the beginning of each subinterval, we will have

$$y_1 = 0;\; y_2 = \left(\frac{a}{n}\right)^2;\; y_3 = \left[2\left(\frac{a}{n}\right)^2\right];\; \ldots;$$

$$y_n = \left[(n-1)\frac{a}{n}\right]^2.$$

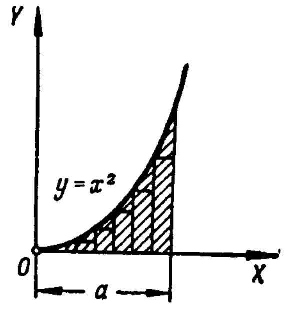

Fig. 39

The areas of the rectangles are obtained by multiplying each y_k by the base $\Delta x = \frac{a}{n}$ (Fig. 39). Summing, we get the area of the step-like figure

$$S_n = \frac{a}{n}\left(\frac{a}{n}\right)^2 [1 + 2^2 + 3^2 + \ldots + (n-1)^2].$$

Using the formula for the sum of the squares of integers,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$

we find

$$S_n = \frac{a^3 n(n-1)(2n-1)}{6n^3},$$

and, passing to the limit, we obtain

$$S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{a^3(n-1)n(2n-1)}{6n^3} = \frac{a^3}{3}.$$

Evaluate the following definite integrals, regarding them as the limits of appropriate integral sums:

1501. $\int_a^b dx.$

1502. $\int_0^T (v_0 + gt)\, dt,$

v_0 and g are constant.

1503. $\int_{-2}^{1} x^2 dx.$

1504. $\int_0^{10} 2^x dx.$

1505*. $\int_1^5 x^3 dx.$

1506*. Find the area of a curvilinear trapezoid bounded by the hyperbola

$$y = \frac{1}{x},$$

by two ordinates: $x = a$ and $x = b$ $(0 < a < b)$, and the x-axis.

1507*. Find

$$f(x) = \int_0^x \sin t\, dt.$$

Sec. 2. Evaluating Definite Integrals by Means of Indefinite Integrals

1°. A definite integral with variable upper limit. If a function $f(t)$ is continuous on an interval $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\, dt$$

is the antiderivative of the function $f(x)$; that is,

$$F'(x) = f(x) \quad \text{for} \quad a \leqslant x \leqslant b.$$

2°. The Newton-Leibniz formula. If $F'(x) = f(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

The antiderivative $F(x)$ is computed by finding the indefinite integral

$$\int f(x)\,dx = F(x) + C.$$

Example 1. Find the integral

$$\int_{-1}^{3} x^4\,dx.$$

Solution. $\int_{-1}^{3} x^4\,dx = \frac{x^5}{5}\Big|_{-1}^{3} = \frac{3^5}{5} - \frac{(-1)^5}{5} = 48\frac{4}{5}.$

1508. Let

$$I = \int_a^b \frac{dx}{\ln x} \quad (b > a > 1).$$

Find

$$1)\ \frac{dI}{da};\quad 2)\ \frac{dI}{db}.$$

Find the derivatives of the following functions:

1509. $F(x) = \int_1^x \ln t\,dt \quad (x > 0)$.

1510. $F(x) = \int_x^0 \sqrt{1+t^4}\,dt$.

1511. $F(x) = \int_x^{x^2} e^{-t^2}\,dt$.

1512. $I = \int_{\frac{1}{x}}^{\sqrt{x}} \cos(t^2)\,dt \quad (x > 0)$.

1513. Find the points of the extremum of the function

$$y = \int_0^x \frac{\sin t}{t}\,dt \text{ in the region } x > 0.$$

Applying the Newton-Leibniz formula, find the integrals:

1514. $\int_0^1 \frac{dx}{1+x}$.

1515. $\int_{-2}^{-1} \frac{dx}{x^3}$.

1516. $\int_{-x}^{x} e^t\,dt$.

1517. $\int_0^x \cos t\,dt$.

Using definite integrals, find the limits of the sums:

1518**. $\lim\limits_{n\to\infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \ldots + \frac{n-1}{n^2}\right)$.

1519**. $\lim\limits_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}\right)$.

1520. $\lim\limits_{n\to\infty} \frac{1^p + 2^p + \ldots + n^p}{n^{p+1}} \quad (p > 0)$.

Evaluate the integrals:

1521. $\int_{1}^{2} (x^2-2x+3)\,dx.$

1522. $\int_{0}^{8} (\sqrt{2x}+\sqrt[3]{x})\,dx.$

1523. $\int_{1}^{4} \frac{1+\sqrt{y}}{y^2}\,dy.$

1524. $\int_{2}^{6} \sqrt{x-2}\,dx.$

1525. $\int_{0}^{-3} \frac{dx}{\sqrt{25+3x}}.$

1526. $\int_{-2}^{-3} \frac{dx}{x^2-1}.$

1527. $\int_{0}^{1} \frac{x\,dx}{x^2+3x+2}.$

1528. $\int_{-1}^{1} \frac{y^5\,dy}{y+2}.$

1529. $\int_{0}^{1} \frac{dx}{x^2+4x+5}.$

1530. $\int_{3}^{4} \frac{dx}{x^2-3x+2}.$

1531. $\int_{0}^{1} \frac{z^3}{z^8+1}\,dz.$

1532. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2\alpha\,d\alpha.$

1533. $\int_{0}^{\frac{\sqrt{2}}{2}} \frac{dx}{\sqrt{1-x^2}}.$

1534. $\int_{2}^{5} \frac{dx}{\sqrt{5+4x-x^2}}.$

1535. $\int_{0}^{1} \frac{y^2\,dy}{\sqrt{y^6+4}}.$

1536. $\int_{0}^{\frac{\pi}{4}} \cos^2\alpha\,d\alpha.$

1537. $\int_{0}^{\frac{\pi}{2}} \sin^3\varphi\,d\varphi.$

1538. $\int_{e}^{e^2} \frac{dx}{x\ln x}.$

1539. $\int_{1}^{e} \frac{\sin(\ln x)}{x}\,dx.$

1540. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan x\,dx.$

1541. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cot^4\varphi\,d\varphi.$

1542. $\int_{0}^{1} \frac{e^x}{1+e^{2x}}\,dx.$

1543. $\int_{0}^{1} \cosh x\,dx.$

1544. $\int_{\ln 2}^{\ln 3} \frac{dx}{\cosh^2 x}.$

1545. $\int_{0}^{\pi} \sinh^2 x\,dx.$

Sec. 3. Improper Integrals

1°. Integrals of unbounded functions. If a function $f(x)$ is not bounded in any neighbourhood of a point c of an interval $[a, b]$ and is continuous for $a \leqslant x < c$ and $c < x \leqslant b$, then by definition we put

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)\,dx + \lim_{\varepsilon \to 0} \int_{c+\varepsilon}^b f(x)\,dx. \qquad (1)$$

If the limits on the right side of (1) exist and are finite, the improper integral is called *convergent*, otherwise it is *divergent*. When $c = a$ or $c = b$, the definition is correspondingly simplified.

If there is a continuous function $F(x)$ on $[a, b]$ such that $F'(x) = f(x)$ when $x \neq c$ *(generalized antiderivative)*, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \qquad (2)$$

If $|f(x)| \leqslant F(x)$ when $a \leqslant x \leqslant b$ and $\int_a^b F(x)\,dx$ converges, then the integral (1) also converges *(comparison test)*.

If $f(x) \geqslant 0$ and $\lim_{x \to c} f(x)\,|c - x|^m = A \neq \infty$, $A \neq 0$, i. e., $f(x) \sim \frac{A}{|c-x|^m}$ when $x \to c$, then 1) for $m < 1$ the integral (1) converges, 2) for $m \geqslant 1$ the integral (1) diverges.

2°. Integrals with infinite limits. If the function $f(x)$ is continuous when $a \leqslant x < \infty$, then we assume

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx \qquad (3)$$

and depending on whether there is a finite limit or not on the right of (3), the respective integral is called *convergent* or *divergent*.

Similarly,

$$\int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{a \to -\infty \\ b \to +\infty}} \int_a^b f(x)\,dx.$$

If $|f(x)| \leqslant F(x)$ and the integral $\int_a^\infty F(x)\,dx$ converges, then the integral (3) converges as well.

If $f(x) \geqslant 0$ and $\lim_{x \to \infty} f(x)\,x^m = A \neq \infty$, $A \neq 0$, i. e., $f(x) \sim \frac{A}{x^m}$ when $x \to \infty$, then 1) for $m > 1$ the integral (3) converges, 2) for $m \leqslant 1$ the integral (3) diverges.

Example 1.

$$\int_{-1}^{1} \frac{dx}{x^2} = \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} \frac{dx}{x^2} + \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} \frac{dx}{x^2} = \lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon} - 1\right) + \lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon} - 1\right) = \infty$$

and the integral diverges.

Example 2.

$$\int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{b \to \infty} \int_{0}^{b} \frac{dx}{1+x^2} = \lim_{b \to \infty} (\arctan b - \arctan 0) = \frac{\pi}{2}.$$

Example 3. Test the convergence of the *probability integral*

$$\int_{0}^{\infty} e^{-x^2}\, dx. \tag{4}$$

Solution. We put

$$\int_{0}^{\infty} e^{-x^2}\, dx = \int_{0}^{1} e^{-x^2}\, dx + \int_{1}^{\infty} e^{-x^2}\, dx.$$

The first of the two integrals on the right is not an improper integral, while the second one converges, since $e^{-x^2} \leqslant e^{-x}$ when $x \geqslant 1$ and

$$\int_{1}^{\infty} e^{-x}\, dx = \lim_{b \to \infty} \int_{1}^{b} e^{-x}\, dx = \lim_{b \to \infty} (-e^{-b} + e^{-1}) = e^{-1};$$

hence, the integral (4) converges.

Example 4. Test the following integral for convergence:

$$\int_{1}^{\infty} \frac{dx}{\sqrt{x^3+1}}. \tag{5}$$

Solution. When $x \to +\infty$, we have

$$\frac{1}{\sqrt{x^3+1}} = \frac{1}{\sqrt{x^3\left(1+\frac{1}{x^3}\right)}} = \frac{1}{x^{\frac{3}{2}}} \frac{1}{\sqrt{1+\frac{1}{x^3}}} \sim \frac{1}{x^{\frac{3}{2}}}.$$

Since the integral

$$\int_{1}^{\infty} \frac{dx}{x^{\frac{3}{2}}}$$

converges, our integral (5) likewise converges.

Example 5. Test for convergence the elliptic integral

$$\int_{0}^{1} \frac{dx}{\sqrt{1-x^4}}. \tag{6}$$

Solution. The point of discontinuity of the integrand is $x=1$. Applying the Lagrange formula we get

$$\frac{1}{\sqrt{1-x^4}}=\frac{1}{\sqrt{(1-x)\cdot 4x_1^3}}=\frac{1}{(1-x)^{\frac{1}{4}}}\cdot\frac{1}{2x_1^{\frac{3}{2}}},$$

where $x<x_1<1$. Hence, for $x\to 1$ we have

$$\frac{1}{\sqrt{1-x^4}}\sim\frac{1}{2}\left(\frac{1}{1-x}\right)^{\frac{1}{4}}.$$

Since the integral

$$\int_0^1\left(\frac{1}{1-x}\right)^{\frac{1}{4}}dx$$

converges, the given integral (6) converges as well.

Evaluate the improper integrals (or establish their divergence):

1546. $\int_0^1\frac{dx}{\sqrt{x}}$.

1547. $\int_{-1}^2\frac{dx}{x}$.

1548. $\int_0^1\frac{dx}{x^p}$.

1549. $\int_0^3\frac{dx}{(x-1)^2}$.

1550. $\int_0^1\frac{dx}{\sqrt{1-x^2}}$.

1551. $\int_1^\infty\frac{dx}{x}$.

1552. $\int_1^\infty\frac{dx}{x^2}$.

1553. $\int_1^\infty\frac{dx}{x^p}$.

1554. $\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$.

1555. $\int_{-\infty}^{\infty}\frac{dx}{x^2+4x+9}$.

1556. $\int_0^\infty\sin x\,dx$.

1557. $\int_0^{\frac{1}{2}}\frac{dx}{x\ln x}$.

1558. $\int_0^{\frac{1}{2}}\frac{dx}{x\ln^2 x}$.

1559. $\int_a^\infty\frac{dx}{x\ln x}$ $(a>1)$.

1560. $\int_a^\infty\frac{dx}{x\ln^2 x}$ $(a>1)$.

1561. $\int_0^{\frac{\pi}{2}}\cot x\,dx$.

1562. $\int\limits_0^\infty e^{-kx}\,dx \quad (k>0).$

1563. $\int\limits_0^\infty \frac{\operatorname{arc\,tan} x}{x^2+1}\,dx.$

1564. $\int\limits_2^\infty \frac{dx}{(x^2-1)^2}.$

1565. $\int\limits_0^\infty \frac{dx}{x^3+1}.$

1566. $\int\limits_0^1 \frac{dx}{x^3-5x^2}.$

Test the convergence of the following integrals:

1567. $\int\limits_0^{100} \frac{dx}{\sqrt[3]{x}+2\sqrt[4]{x}+x^3}.$

1568. $\int\limits_1^{+\infty} \frac{dx}{2x+\sqrt[3]{x^2+1}+5}.$

1569. $\int\limits_{-1}^\infty \frac{dx}{x^2+\sqrt[3]{x^4+1}}.$

1570. $\int\limits_0^\infty \frac{x\,dx}{\sqrt{x^5+1}}.$

1571. $\int\limits_0^1 \frac{dx}{\sqrt[3]{1-x^4}}.$

1572. $\int\limits_1^2 \frac{dx}{\ln x}.$

1573. $\int\limits_{\frac{\pi}{2}}^\infty \frac{\sin x}{x^2}\,dx.$

1574*. Prove that the Euler integral of the first kind *(beta-function)*

$$B(p,\ q)=\int\limits_0^1 x^{p-1}(1-x)^{q-1}\,dx$$

converges when $p>0$ and $q>0$.

1575*. Prove that the Euler integral of the second kind *(gamma-function)*

$$\Gamma(p)=\int\limits_0^\infty x^{p-1}e^{-x}\,dx$$

converges for $p>0$.

Sec. 4. Change of Variable in a Definite Integral

If a function $f(x)$ is continuous over $a \leqslant x \leqslant b$ and $x=\varphi(t)$ is a function continuous together with its derivative $\varphi'(t)$ over $\alpha \leqslant t \leqslant \beta$, where $a=\varphi(\alpha)$ and $b=\varphi(\beta)$, and $f[\varphi(t)]$ is defined and continuous on the interval $\alpha \leqslant t \leqslant \beta$, then

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f[\varphi(t)]\,\varphi'(t)\,dt.$$

Example 1. Find

$$\int_0^a x^2 \sqrt{a^2-x^2}\,dx \qquad (a>0).$$

Solution. We put

$$x = a\sin t;$$
$$dx = a\cos t\,dt.$$

Then $t=\arc\sin\frac{x}{a}$ and, consequently, we can take $\alpha=\arcsin 0=0$, $\beta=\arcsin 1=\frac{\pi}{2}$. Therefore, we shall have

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx = \int_0^{\frac{\pi}{2}} a^2\sin^2 t\,\sqrt{a^2-a^2\sin^2 t}\; a\cos t\,dt =$$

$$= a^4\int_0^{\frac{\pi}{2}} \sin^2 t\cos^2 t\,dt = \frac{a^4}{4}\int_0^{\frac{\pi}{2}}\sin^2 2t\,dt = \frac{a^4}{8}\int_0^{\frac{\pi}{2}}(1-\cos 4t)\,dt =$$

$$= \frac{a^4}{8}\left(t-\frac{1}{4}\sin 4t\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{\pi a^4}{16}.$$

1576. Can the substitution $x=\cos t$ be made in the integral

$$\int_0^2 \sqrt[3]{1-x^2}\,dx?$$

Transform the following definite integrals by means of the indicated substitutions:

1577. $\int_1^3 \sqrt{x+1}\,dx$, $x=2t-1$.

1578. $\int_{\frac{1}{2}}^{1} \frac{dx}{\sqrt{1-x^4}}$, $x=\sin t$.

1579. $\int_{\frac{3}{4}}^{\frac{4}{3}} \frac{dx}{\sqrt{x^2+1}}$, $x=\sinh t$.

1580. $\int_0^{\frac{\pi}{2}} f(x)\,dx$, $x=\arctan t$.

1581. For the integral

$$\int_a^b f(x)\,dx \quad (b>a)$$

indicate an integral linear substitution

$$x = \alpha t + \beta,$$

as a result of which the limits of integration would be 0 and 1, respectively.

Applying the indicated substitutions, evaluate the following integrals:

1582. $\int\limits_0^4 \frac{dx}{1+\sqrt{x}}$, $\qquad x = t^2$.

1583. $\int\limits_3^{29} \frac{(x-2)^{2/3}}{(x-2)^{2/3}+3}\, dx$, $\qquad x-2 = z^3$.

1584. $\int\limits_0^{\ln 2} \sqrt{e^x-1}\, dx$, $\qquad e^x - 1 = z^2$.

1585. $\int\limits_0^{\pi} \frac{dt}{3+2\cos t}$, $\qquad \tan\frac{t}{2} = z$.

1586. $\int\limits_0^{\frac{\pi}{2}} \frac{dx}{1+a^2\sin^2 x}$, $\qquad \tan x = t$.

Evaluate the following integrals by means of appropriate substitutions:

1587. $\int\limits_{\frac{\sqrt{2}}{2}}^{1} \frac{\sqrt{1-x^2}}{x^2}\, dx$.

1588. $\int\limits_1^2 \frac{\sqrt{x^2-1}}{x}\, dx$.

1589. $\int\limits_0^{\ln 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}\, dx$.

1590. $\int\limits_0^5 \frac{dx}{2x+\sqrt{3x+1}}$.

Evaluate the integrals:

1591. $\int\limits_1^3 \frac{dx}{x\sqrt{x^2+5x+1}}$.

1592. $\int\limits_{-1}^{1} \frac{dx}{(1+x^2)^2}$.

1593. $\int\limits_0^a \sqrt{ax-x^2}\, dx$.

1594. $\int\limits_0^{2\pi} \frac{dx}{5-3\cos x}$.

1595. Prove that if $f(x)$ is an even function, then

$$\int\limits_{-a}^{a} f(x)\, dx = 2\int\limits_0^a f(x)\, dx.$$

But if $f(x)$ is an odd function, then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

1596. Show that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_{0}^{\infty} e^{-x^2}dx = \int_{0}^{\infty} \frac{e^{-x}}{\sqrt{x}}\,dx.$$

1597. Show that

$$\int_{0}^{1} \frac{dx}{\arccos x} = \int_{0}^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx.$$

1598. Show that

$$\int_{0}^{\frac{\pi}{2}} f(\sin x)\,dx = \int_{0}^{\frac{\pi}{2}} f(\cos x)\,dx.$$

Sec. 5. Integration by Parts

If the functions $u(x)$ and $v(x)$ are continuously differentiable on the interval $[a, b]$, then

$$\int_{a}^{b} u(x)\,v'(x)\,dx = u(x)\,v(x)\Big|_{a}^{b} - \int_{a}^{b} v(x)\,u'(x)\,dx. \tag{1}$$

Applying the formula for integration by parts, evaluate the following integrals:

1599. $\int_{0}^{\frac{\pi}{2}} x\cos x\,dx.$

1600. $\int_{1}^{e} \ln x\,dx.$

1601. $\int_{0}^{1} x^3 e^{2x}dx.$

1602. $\int_{0}^{\pi} e^{x}\sin x\,dx.$

1603. $\int_{0}^{\infty} xe^{-x}dx.$

1604. $\int_{0}^{\infty} e^{-ax}\cos bx\,dx \quad (a>0).$

1605. $\int_{0}^{\infty} e^{-ax}\sin bx\,dx \quad (a>0).$

1606.** Show that for the gamma-function (see Example 1575) the following reduction formula holds true:

$$\Gamma(p+1) = p\Gamma(p) \quad (p>0).$$

From this derive that $\Gamma(n+1) = n!$, if n is a natural number.

1607. Show that for the integral

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$$

the *reduction formula*

$$I_n = \frac{n-1}{n} I_{n-2}$$

holds true.

Find I_n, if n is a natural number. Using the formula obtained, evaluate I_9 and I_{10}.

1608. Applying repeated integration by parts, evaluate the integral (see Example 1574)

$$\mathrm{B}(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\, dx,$$

where p and q are positive integers.

1609*. Express the following integral in terms of B (beta-function):

$$I_{n,m} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\, dx,$$

if m and n are nonnegative integers.

Sec. 6. Mean-Value Theorem

1°. Evaluation of integrals. If $f(x) \leqslant F(x)$ for $a \leqslant x \leqslant b$, then

$$\int_a^b f(x)\, dx \leqslant \int_a^b F(x)\, dx. \tag{1}$$

If $f(x)$ and $\varphi(x)$ are continuous for $a \leqslant x \leqslant b$ and, besides, $\varphi(x) \geqslant 0$, then

$$m\int_a^b \varphi(x)\, dx \leqslant \int_a^b f(x)\varphi(x)\, dx \leqslant M\int_a^b \varphi(x)\, dx, \tag{2}$$

where m is the smallest and M is the largest value of the function $f(x)$ on the interval $[a, b]$.

In particular, if $\varphi(x) \equiv 1$, then

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a). \qquad (3)$$

The inequalities (2) and (3) may be replaced, respectively, by their equivalent equalities:

$$\int_a^b f(x)\varphi(x)\,dx = f(c)\int_a^b \varphi(x)\,dx$$

and

$$\int_a^b f(x)\,dx = f(\xi)(b-a),$$

where c and ξ are certain numbers lying between a and b.

Example 1. Evaluate the integral

$$I = \int_0^{\frac{\pi}{2}} \sqrt{1+\frac{1}{2}\sin^2 x}\,dx.$$

Solution. Since $0 \leqslant \sin^2 x \leqslant 1$, we have

$$\frac{\pi}{2} < I < \frac{\pi}{2}\sqrt{\frac{3}{2}},$$

that is,

$$1.57 < I < 1.91.$$

2°. The mean value of a function. The number

$$\mu = \frac{1}{b-a}\int_a^b f(x)\,dx$$

is called the *mean value* of the function $f(x)$ on the interval $a \leqslant x \leqslant b$.

1610*. Determine the signs of the integrals without evaluating them:

a) $\int_{-1}^{2} x^3\,dx$;

b) $\int_0^{\pi} x\cos x\,dx$;

c) $\int_0^{2\pi} \frac{\sin x}{x}\,dx$.

1611. Determine (without evaluating) which of the following integrals is greater:

a) $\int_0^1 \sqrt{1+x^2}\,dx$ or $\int_0^1 dx$;

b) $\int_0^1 x^2 \sin^2 x\,dx$ or $\int_0^1 x \sin^2 x\,dx$;

c) $\int_1^2 e^{x^2} dx$ or $\int_1^2 e^x dx$.

Find the mean values of the functions on the indicated intervals:

1612. $f(x)=x^2$, $0 \leqslant x \leqslant 1$.

1613. $f(x)=a+b\cos x$, $-\pi \leqslant x \leqslant \pi$.

1614. $f(x)=\sin^2 x$, $0 \leqslant x \leqslant \pi$.

1615. $f(x)=\sin^4 x$, $0 \leqslant x \leqslant \pi$.

1616. Prove that $\int_0^1 \frac{dx}{\sqrt{2+x-x^2}}$ lies between $\frac{2}{3} \approx 0.67$ and $\frac{1}{\sqrt{2}} \approx$ ≈ 0.70. Find the exact value of this integral.

Evaluate the integrals:

1617. $\int_0^1 \sqrt{4+x^2}\,dx$.

1618. $\int_{-1}^{+1} \frac{dx}{8+x^3}$.

1619. $\int_0^{2\pi} \frac{dx}{10+3\cos x}$.

1620*. $\int_0^{\frac{\pi}{4}} x\sqrt{\tan x}$.

1621. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx$.

1622. Integrating by parts, prove that

$$0 < \int_{100\pi}^{200\pi} \frac{\cos x}{x}\,dx < \frac{1}{100\pi}.$$

Sec. 7. The Areas of Plane Figures

1°. Area in rectangular coordinates. If a continuous curve is defined in rectangular coordinates by the equation $y=f(x)$ $[f(x)\geqslant 0]$, the area of the curvilinear trapezoid bounded by this curve, by two vertical lines at the

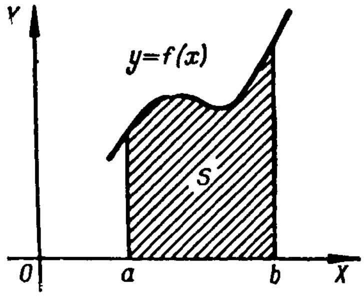

Fig. 40

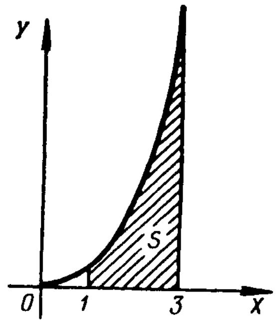

Fig. 41

points $x=a$ and $x=b$ and by a segment of the x-axis $a\leqslant x\leqslant b$ (Fig. 40), is given by the formula

$$S=\int_a^b f(x)\,dx. \qquad (1)$$

Example 1. Compute the area bounded by the parabola $y=\frac{x^2}{2}$, the straight lines $x=1$ and $x=3$, and the x-axis (Fig. 41).

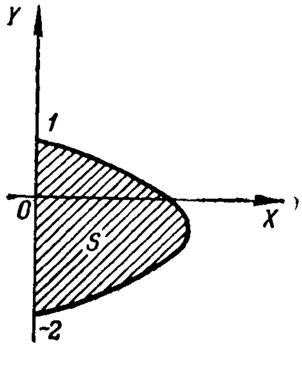

Fig. 42

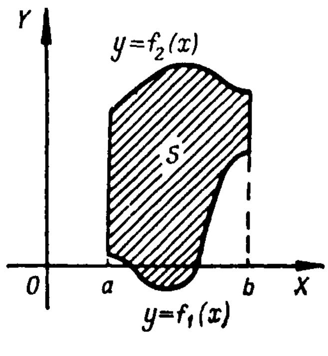

Fig. 43

Solution. The sought-for area is expressed by the integral

$$S=\int_1^3 \frac{x^2}{2}\,dx=4\frac{1}{3}.$$

Example 2. Evaluate the area bounded by the curve $x=2-y-y^2$ and the y-axis (Fig. 42).

Solution. Here, the roles of the coordinate axes are changed and so the sought-for area is expressed by the integral

$$S=\int_{-2}^{1}(2-y-y^2)\,dy=4\frac{1}{2},$$

where the limits of integration $y_1=-2$ and $y_2=1$ are found as the ordinates of the points of intersection of the curve with the y-axis.

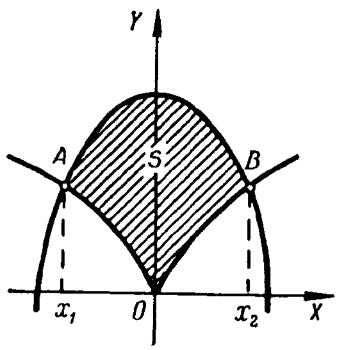

Fig. 44

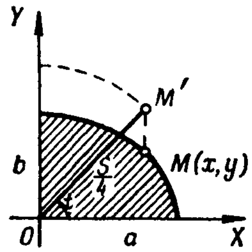

Fig. 45

In the more general case, if the area S is bounded by two continuous curves $y=f_1(x)$ and $y=f_2(x)$ and by two vertical lines $x=a$ and $x=b$, where $f_1(x)\leqslant f_2(x)$ when $a\leqslant x\leqslant b$ (Fig. 43), we will then have:

$$S=\int_{a}^{b}[f_2(x)-f_1(x)]\,dx. \tag{2}$$

Example 3. Evaluate the area S contained between the curves

$$y=2-x^2 \text{ and } y^3=x^2 \tag{3}$$

(Fig. 44).

Solution. Solving the set of equations (3) simultaneously, we find the limits of integration: $x_1=-1$ and $x_2=1$. By virtue of formula (2), we obtain

$$S=\int_{-1}^{1}(2-x^2-x^{2/3})\,dx=\left(2x-\frac{x^3}{3}-\frac{3}{5}x^{\frac{5}{3}}\right)\Big|_{-1}^{1}=2\frac{2}{15}.$$

If the curve is defined by equations in parametric form $x=\varphi(t)$, $y=\psi(t)$, then the area of the curvilinear trapezoid bounded by this curve, by two vertical lines ($x=a$ and $x=b$), and by a segment of the x-axis is expressed by the integral

$$S=\int_{t_1}^{t_2} \psi(t)\,\varphi'(t)\,dt,$$

where t_1 and t_2 are determined from the equations

$a=\varphi(t_1)$ and $b=\varphi(t_2)$ [$\psi(t)\geqslant 0$ on the interval $[t_1, t_2]$].

Example 4. Find the area of the ellipse (Fig. 45) by using its parametric equations

$$\begin{cases} x=a\cos t, \\ y=b\sin t. \end{cases}$$

Solution. Due to the symmetry, it is sufficient to compute the area of a quadrant and then multiply the result by four. If in the equation $x=a\cos t$ we first put $x=0$ and then $x=a$, we get the limits of integration $t_1=\frac{\pi}{2}$ and $t_2=0$. Therefore,

$$\frac{1}{4}S=\int_{\frac{\pi}{2}}^{0} b\sin a\,(-\sin t)\,dt=ab\int_{0}^{\frac{\pi}{2}}\sin^2 t\,dt=\frac{\pi ab}{4}$$

and, hence, $S=\pi ab$.

2°. The area in polar coordinates. If a curve is defined in polar coordinates by the equation $r=f(\varphi)$, then the area of the sector AOB (Fig. 46), bounded by an arc of the curve, and by two radius vectors OA and OB,

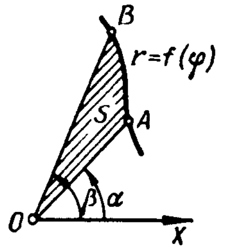

Fig. 46

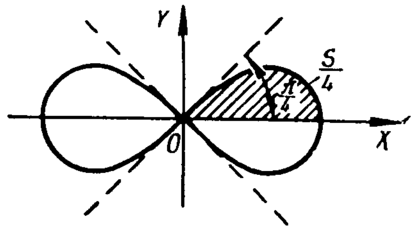

Fig. 47

which correspond to the values $\varphi_1=\alpha$ and $\varphi_2=\beta$, is expressed by the integral

$$S=\frac{1}{2}\int_{\alpha}^{\beta}[f(\varphi)]^2\,d\varphi.$$

Example 5. Find the area contained inside Bernoulli's lemniscate $r^2=a^2\cos 2\varphi$ (Fig. 47).

Solution. By virtue of the symmetry of the curve we determine first one quadrant of the sought-for area:

$$\frac{1}{4}S=\frac{1}{2}\int_0^{\frac{\pi}{4}} a^2\cos 2\varphi\, d\varphi=\frac{a^2}{2}\left[\frac{1}{2}\sin 2\varphi\right]_0^{\frac{\pi}{4}}=\frac{a^2}{4}.$$

Whence $S=a^2$.

1623. Compute the area bounded by the parabola $y=4x-x^2$ and the x-axis.

1624. Compute the area bounded by the curve $y=\ln x$, the x-axis and the straight line $x=e$.

1625*. Find the area bounded by the curve $y=x(x-1)(x-2)$ and the x-axis.

1626. Find the area bounded by the curve $y^3=x$, the straight line $y=1$ and the vertical line $x=8$.

1627. Compute the area bounded by a single half-wave of the sinusoidal curve $y=\sin x$ and the x-axis.

1628. Compute the area contained between the curve $y=\tan x$, the x-axis and the straight line $x=\frac{\pi}{3}$.

1629. Find the area contained between the hyperbola $xy=m^2$, the vertical lines $x=a$ and $x=3a$ $(a>0)$ and the x-axis.

1630. Find the area contained between the witch of Agnesi $y=\frac{a^3}{x^2+a^2}$ and the x-axis.

1631. Compute the area of the figure bounded by the curve $y=x^3$, the straight line $y=8$ and the y-axis.

1632. Find the area bounded by the parabolas $y^2=2px$ and $x^2=2py$.

1633. Evaluate the area bounded by the parabola $y=2x-x^2$ and the straight line $y=-x$.

1634. Compute the area of a segment cut off by the straight line $y=3-2x$ from the parabola $y=x^2$.

1635. Compute the area contained between the parabolas $y=x^2$, $y=\frac{x^2}{2}$ and the straight line $y=2x$.

1636. Compute the area contained between the parabolas $y=\frac{x^2}{3}$ and $y=4-\frac{2}{3}x^2$.

1637. Compute the area contained between the witch of Agnesi $y=\frac{1}{1+x^2}$ and the parabola $y=\frac{x^2}{2}$.

1638. Compute the area bounded by the curves $y=e^x$, $y=e^{-x}$ and the straight line $x=1$.

1639. Find the area of the figure bounded by the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and the straight line $x=2a$.

1640*. Find the entire area bounded by the astroid

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}.$$

1641. Find the area between the catenary

$$y=a\cosh\frac{x}{a},$$

the y-axis and the straight line $y=\frac{a}{2e}(e^2+1)$.

1642. Find the area bounded by the curve $a^2y^2=x^2(a^2-x^2)$.

1643. Compute the area contained within the curve

$$\left(\frac{x}{5}\right)^2+\left(\frac{y}{4}\right)^{\frac{2}{3}}=1.$$

1644. Find the area between the equilateral hyperbola $x^2-y^2=9$, the x-axis and the diameter passing through the point (5,4).

1645. Find the area between the curve $y=\frac{1}{x^2}$, the x-axis, and the ordinate $x=1$ $(x>1)$.

1646*. Find the area bounded by the cissoid $y^2=\frac{x^3}{2a-x}$ and its asymptote $x=2a$ $(a>0)$.

1647*. Find the area between the strophoid $y^2=\frac{x(x-a)^2}{2a-x}$ and its asymptote $(a>0)$.

1648. Compute the area of the two parts into which the circle $x^2+y^2=8$ is divided by the parabola $y^2=2x$.

1649. Compute the area contained between the circle $x^2+y^2=16$ and the parabola $x^2=12(y-1)$.

1650. Find the area contained within the astroid

$$x=a\cos^3 t;\quad y=b\sin^3 t.$$

1651. Find the area bounded by the x-axis and one arc of the cycloid

$$\begin{cases} x=a(t-\sin t), \\ y=a(1-\cos t). \end{cases}$$

1652. Find the area bounded by one branch of the trochoid

$$\begin{cases} x=at-b\sin t, \\ y=a-b\cos t \end{cases} \quad (0<b\leqslant a)$$

and a tangent to it at its lower points.

1653. Find the area bounded by the cardioid

$$\begin{cases} x = a(2\cos t - \cos 2t), \\ y = a(2\sin t - \sin 2t). \end{cases}$$

1654*. Find the area of the loop of the folium of Descartes

$$x = \frac{3at}{1+t^3}; \quad y = \frac{3at^2}{1+t^3}.$$

1655*. Find the entire area of the cardioid $r = a(1 + \cos\varphi)$.

1656*. Find the area contained between the first and second turns of Archimedes' spiral, $r = a\varphi$ (Fig. 48).

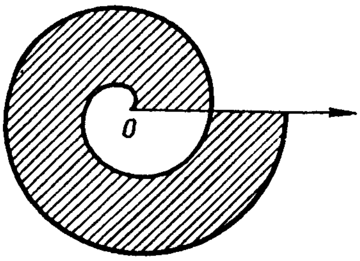

Fig. 48

1657. Find the area of one of the leaves of the curve $r = a\cos 2\varphi$.

1658. Find the entire area bounded by the curve $r^2 = a^2 \sin 4\varphi$.

1659*. Find the area bounded by the curve $r = a\sin 3\varphi$.

1660. Find the area bounded by Pascal's limaçon

$$r = 2 + \cos\varphi.$$

1661. Find the area bounded by the parabola $r = a\sec^2\frac{\varphi}{2}$ and the two half-lines $\varphi = \frac{\pi}{4}$ and $\varphi = \frac{\pi}{2}$.

1662. Find the area of the ellipse $r = \frac{p}{1+\varepsilon\cos\varphi}$ $(\varepsilon < 1)$.

1663. Find the area bounded by the curve $r = 2a\cos 3\varphi$ and lying outside the circle $r = a$.

1664*. Find the area bounded by the curve $x^4 + y^4 = x^2 + y^2$.

Sec. 8. The Arc Length of a Curve

1°. The arc length in rectangular coordinates. The arc length s of a curve $y = f(x)$ contained between two points with abscissas $x = a$ and $x = b$ is

$$s = \int_a^b \sqrt{1 + y'^2}\,dx.$$

Example 1. Find the length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ (Fig. 49).

Solution. Differentiating the equation of the astroid, we get

$$y' = -\frac{y^{1/3}}{x^{1/3}}.$$

For this reason, we have for the arc length of a quarter of the astroid:

$$\frac{1}{4}s=\int_0^a \sqrt{1+\frac{y^{2/3}}{x^{2/3}}}\,dx=\int_0^a \frac{a^{1/3}}{x^{1/3}}\,dx=\frac{3}{2}a.$$

Whence $s=6a$.

2°. The arc length of a curve represented parametrically. If a curve is represented by equations in parametric form, $x=\varphi(t)$ and $y=\psi(t)$, then the arc length s of the curve is

$$s=\int_{t_1}^{t_2}\sqrt{x'^2+y'^2}\,dt,$$

where t_1 and t_2 are values of the parameter that correspond to the extremities of the arc.

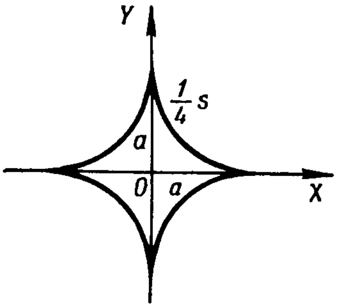

Fig 49

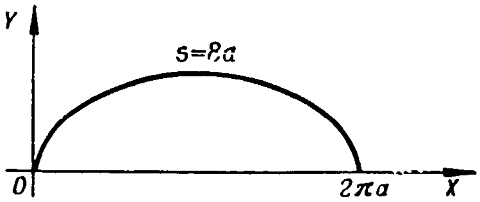

Fig. 50

Example 2. Find the length of one arc of the cycloid (Fig. 50)

$$\begin{cases} x=a(t-\sin t), \\ y=a(1-\cos t). \end{cases}$$

Solution. We have $\frac{dx}{dt}=a(1-\cos t)$ and $\frac{dy}{dt}=a\sin t$. Therefore,

$$s=\int_0^{2\pi}\sqrt{a^2(1-\cos t)^2+a^2\sin^2 t}\,dt=2a\int_0^{2\pi}\sin\frac{t}{2}\,dt=8a.$$

The limits of integration $t_1=0$ and $t_2=2\pi$ correspond to the extreme points of the arc of the cycloid.

If a curve is defined by the equation $r=f(\varphi)$ in polar coordinates, then the arc length s is

$$s=\int_\alpha^\beta \sqrt{r^2+r'^2}\,d\varphi,$$

where α and β are the values of the polar angle at the extreme points of the arc.

Example 3. Find the length of the entire curve $r = a\sin^3\frac{\varphi}{3}$ (Fig. 51). The entire curve is described by a point as φ ranges from 0 to 3π.

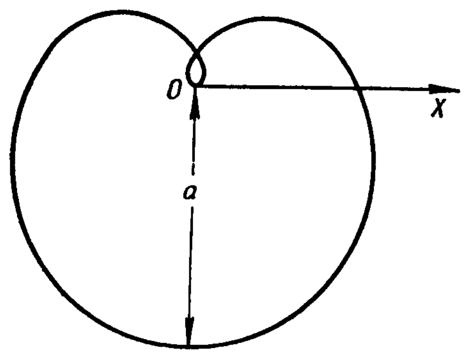

Fig. 51

Solution. We have $r' = a\sin^2\frac{\varphi}{3}\cos\frac{\varphi}{3}$, therefore the entire arc length of the curve is

$$s = \int_0^{3\pi}\sqrt{a^2\sin^6\frac{\varphi}{3} + a^2\sin^4\frac{\varphi}{3}\cos^2\frac{\varphi}{3}}\,d\varphi = a\int_0^{3\pi}\sin^2\frac{\varphi}{3}\,d\varphi = \frac{3\pi a}{2}.$$

1665. Compute the arc length of the semicubical parabola $y^2 = x^3$ from the coordinate origin to the point $x = 4$.

1666*. Find the length of the catenary $y = a\cosh\frac{x}{a}$ from the vertex $A(0,a)$ to the point $B(b,h)$.

1667. Compute the arc length of the parabola $y = 2\sqrt{x}$ from $x = 0$ to $x = 1$.

1668. Find the arc length of the curve $y = e^x$ lying between the points (0,1) and $(1,e)$.

1669. Find the arc length of the curve $y = \ln x$ from $x = \sqrt{3}$ to $x = \sqrt{8}$.

1670. Find the arc length of the curve $y = \arcsin(e^{-x})$ from $x = 0$ to $x = 1$.

1671. Compute the arc length of the curve $x = \ln\sec y$, lying between $y = 0$ and $y = \frac{\pi}{3}$.

1672. Find the arc length of the curve $x = \frac{1}{4}y^2 - \frac{1}{2}\ln y$ from $y = 1$ to $y = e$.

1673. Find the length of the right branch of the tractrix

$$x=\sqrt{a^2-y^2}+a\ln\left|\frac{a+\sqrt{a^2-y^2}}{y}\right| \text{ from } y=a \text{ to } y=b\ (0<b<a).$$

1674. Find the length of the closed part of the curve $9ay^2 = x(x-3a)^2$.

1675. Find the length of the curve $y=\ln\left(\coth\frac{x}{a}\right)$ from $x=a$ to $x=b$ $(0<a<b)$.

1676*. Find the arc length of the involute of the circle

$$\left.\begin{aligned} x&=a(\cos t+t\sin t),\\ y&=a(\sin t-t\cos t)\end{aligned}\right\} \text{ from } t=0 \text{ to } t=T.$$

1677. Find the length of the evolute of the ellipse

$$x=\frac{c^2}{a}\cos^3 t;\quad y=\frac{c^2}{b}\sin^3 t\quad (c^2=a^2-b^2).$$

1678. Find the length of the curve

$$\left.\begin{aligned} x&=a(2\cos t-\cos 2t),\\ y&=a(2\sin t-\sin 2t).\end{aligned}\right\}$$

1679. Find the length of the first turn of Archimedes' spiral $r=a\varphi$.

1680. Find the entire length of the cardioid $r=a(1+\cos\varphi)$.

1681. Find the arc length of that part of the parabola $r=a\sec^2\frac{\varphi}{2}$ which is cut off by a vertical line passing through the pole.

1682. Find the length of the hyperbolic spiral $r\varphi=1$ from the point $(2, 1/2)$ to the point $(1/2, 2)$.

1683. Find the arc length of the logarithmic spiral $r=ae^{m\varphi}$, lying inside the circle $r=a$.

1684. Find the arc length of the curve $\varphi=\frac{1}{2}\left(r+\frac{1}{r}\right)$ from $r=1$ to $r=3$.

Sec. 9. Volumes of Solids

1°. The volume of a solid of revolution. The volumes of solids formed by the revolution of a curvilinear trapezoid [bounded by the curve $y=f(x)$, the x-axis and two vertical lines $x=a$ and $x=b$] about the x- and y-axes are

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expressed, respectively, by the formulas:

$$1)\ V_X=\pi\int_a^b y^2\,dx;\quad 2)\ V_Y=2\pi\int_a^b xy\,dx\ ^{*}).$$

Example 1. Compute the volumes of solids formed by the revolution of a figure bounded by a single lobe of the sinusoidal curve $y=\sin x$ and by the segment $0\leqslant x\leqslant\pi$ of the x-axis about: a) the x-axis and b) the y-axis.

Solution.

$$\text{a)}\ V_X=\pi\int_0^{\pi}\sin^2 x\,dx=\frac{\pi^2}{2};$$

$$\text{b)}\ V_Y=2\pi\int_0^{\pi}x\sin x\,dx=2\pi\,(-x\cos x+\sin x)_0^{\pi}=2\pi^2.$$

The volume of a solid formed by revolution about the y-axis of a figure bounded by the curve $x=g(y)$, the y-axis and by two parallel lines $y=c$ and $y=d$, may be determined from the formula

$$V_Y=\pi\int_c^d x^2\,dy,$$

obtained from formula (1), given above, by interchanging the coordinates x and y.

If the curve is defined in a different form (parametrically, in polar coordinates, etc.), then in the foregoing formulas we must change the variable of integration in appropriate fashion.

In the more general case, the volumes of solids formed by the revolution about the x- and y-axes of a figure bounded by the curves $y_1=f_1(x)$ and $y_2=f_2(x)$ [where $f_1(x)\leqslant f_2(x)$], and the straight lines $x=a$ and $x=b$ are, respectively, equal to

$$V_X=\pi\int_a^b (y_2^2-y_1^2)\,dx$$

and

$$V_Y=2\pi\int_a^b x\,(y_2-y_1)\,dx.$$

Example 2. Find the volume of a torus formed by the rotation of the circle $x^2+(y-b)^2=a^2$ $(b\geqslant a)$ about the x-axis (Fig. 52).

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*) The solid is formed by the revolution, about the y-axis, of a curvilinear trapezoid bounded by the curve $y=f(x)$ and the straight lines $x=a$, $x=b$, and $y=0$. For a volume element we take the volume of that part of the solid formed by revolving about the y-axis a rectangle with sides y and dx at a distance x from the y-axis. Then the volume element $dV_Y=2\pi xy\,dx$, whence $V_Y=2\pi\int_a^b xy\,dx$.

Solution. We have

$$y_1 = b - \sqrt{a^2 - x^2} \text{ and } y_2 = b + \sqrt{a^2 - x^2}.$$

Therefore,

$$V_X = \pi \int_{-a}^{a} [(b + \sqrt{a^2 - x^2})^2 - (b - \sqrt{a^2 - x^2})^2]\, dx =$$

$$= 4\pi b \int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = 2\pi^2 a^2 b$$

(the latter integral is taken by the substitution $x = a \sin t$).

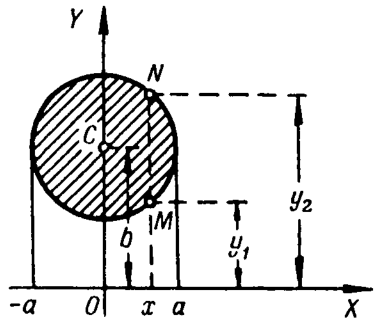

Fig 52

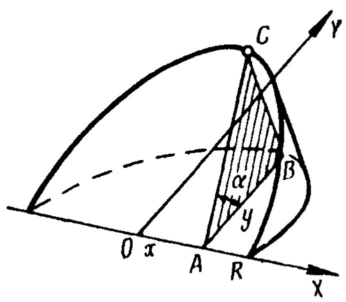

Fig. 53

The volume of a solid obtained by the rotation, about the polar axis, of a sector formed by an arc of the curve $r = F(\varphi)$ and by two radius vectors $\varphi = \alpha$, $\varphi = \beta$ may be computed from the formula

$$V_P = \frac{2}{3} \pi \int_{\alpha}^{\beta} r^3 \sin \varphi\, d\varphi.$$

This same formula is conveniently used when seeking the volume obtained by the rotation, about the polar axis, of some closed curve defined in polar coordinates.

Example 3. Determine the volume formed by the rotation of the curve $r = a \sin 2\varphi$ about the polar axis.

Solution.

$$V_P = 2 \cdot \frac{2}{3} \pi \int_0^{\frac{\pi}{2}} r^3 \sin \varphi\, d\varphi = \frac{4}{3} \pi a^3 \int_0^{\frac{\pi}{2}} \sin^3 2\varphi \sin \varphi\, d\varphi =$$

$$= \frac{32}{3} \pi a^3 \int_0^{\frac{\pi}{2}} \sin^4 \varphi \cos^3 \varphi\, d\varphi = \frac{64}{105} \pi a^3.$$

2°. Computing the volumes of solids from known cross-sections. If $S=S(x)$ is the cross-sectional area cut off by a plane perpendicular to some straight line (which we take to be the x-axis) at a point with abscissa x, then the volume of the solid is

$$V=\int_{x_1}^{x_2} S(x)\,dx,$$

where x_1 and x_2 are the abscissas of the extreme cross-sections of the solid.

Example 4. Determine the volume of a wedge cut off a circular cylinder by a plane passing through the diameter of the base and inclined to the base at an angle α. The radius of the base is R (Fig. 53).

Solution. For the x-axis we take the diameter of the base along which the cutting plane intersects the base, and for the y-axis we take the diameter of the base perpendicular to it. The equation of the circumference of the base is $x^2+y^2=R^2$.

The area of the section ABC at a distance x from the origin O is $S(x)=\text{area}\ \triangle ABC=\frac{1}{2}AB\cdot BC=\frac{1}{2}yy\tan\alpha=\frac{y^2}{2}\tan\alpha$. Therefore, the sought-for volume of the wedge is

$$V=2\cdot\frac{1}{2}\int_0^R y^2\tan\alpha\,dx=\tan\alpha\int_0^R (R^2-x^2)\,dx=\frac{2}{3}\tan\alpha\,R^3.$$

1685. Find the volume of a solid formed by rotation, about the x-axis, of an area bounded by the x-axis and the parabola $y=ax-x^2$ $(a>0)$.

1686. Find the volume of an ellipsoid formed by the rotation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the x-axis.

1687. Find the volume of a solid formed by the rotation, about the x-axis, of an area bounded by the catenary $y=a\cosh\frac{x}{a}$, the x-axis, and the straight lines $x=\pm a$.

1688. Find the volume of a solid formed by the rotation, about the x-axis, of the curve $y=\sin^2 x$ in the interval between $x=0$ and $x=\pi$.

1689. Find the volume of a solid formed by the rotation, about the x-axis, of an area bounded by the semicubical parabola $y^2=x^3$, the x-axis, and the straight line $x=1$.

1690. Find the volume of a solid formed by the rotation of the same area (as in Problem 1689) about the y-axis.

1691. Find the volumes of the solids formed by the rotation of an area bounded by the lines $y=e^x$, $x=0$, $y=0$ about: a) the x-axis and b) the y-axis.

1692. Find the volume of a solid formed by the rotation, about the y-axis, of that part of the parabola $y^2=4ax$ which is cut off by the straight line $x=a$.

1693. Find the volume of a solid formed by the rotation, about the straight line $x=a$, of that part of the parabola $y^2=4ax$ which is cut off by this line.

1694. Find the volume of a solid formed by the rotation, about the straight line $y=-p$, of a figure bounded by the parabola $y^2=2px$ and the straight line $x=\frac{p}{2}$.

1695. Find the volume of a solid formed by the rotation, about the x-axis, of the area contained between the parabolas $y=x^2$ and $y=\sqrt{x}$.

1696. Find the volume of a solid formed by the rotation, about the x-axis, of a loop of the curve $(x-4a)y^2=ax(x-3a)$ $(a>0)$.

1697. Find the volume of a solid generated by the rotation of the cyssoid $y^2=\frac{x^3}{2a-x}$ about its asymptote $x=2a$.

1698. Find the volume of a paraboloid of revolution whose base has radius R and whose altitude is H.

1699. A right parabolic segment whose base is $2a$ and altitude h is in rotation about the base. Determine the volume of the resulting solid of revolution (Cavalieri's "lemon").

1700. Show that the volume of a part cut by the plane $x=2a$ off a solid formed by the rotation of the equilateral hyperbola $x^2-y^2=a^2$ about the x-axis is equal to the volume of a sphere of radius a.

1701. Find the volume of a solid formed by the rotation of a figure bounded by one arc of the cycloid $x=a(t-\sin t)$, $y=a(1-\cos t)$ and the x-axis, about: a) the x-axis, b) the y-axis, and c) the axis of symmetry of the figure.

1702. Find the volume of a solid formed by the rotation of the astroid $x=a\cos^3 t$, $y=b\sin^3 t$ about the y-axis.

1703. Find the volume of a solid obtained by rotating the cardioid $r=a(1+\cos\varphi)$ about the polar axis.

1704. Find the volume of a solid formed by rotation of the curve $r=a\cos^2\varphi$ about the polar axis.

1705. Find the volume of an obelisk whose parallel bases are rectangles with sides A, B and a, b, and the altitude is h.

1706. Find the volume of a right elliptic cone whose base is an ellipse with semi-axes a and b, and altitude h.

1707. On the chords of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$, which are parallel to the x-axis, are constructed squares whose sides are equal to the lengths of the chords and whose planes are perpendicular to the xy-plane. Find the volume of the solid formed by these squares.

1708. A circle undergoing deformation is moving so that one of the points of its circumference lies on the y-axis, the centre describes an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the plane of the circle is perpendicular to the xy-plane. Find the volume of the solid generated by the circle.

1709. The plane of a moving triangle remains perpendicular to the stationary diameter of a circle of radius a. The base of the triangle is a chord of the circle, while its vertex slides along a straight line parallel to the stationary diameter at a distance h from the plane of the circle. Find the volume of the solid (called a conoid) formed by the motion of this triangle from one end of the diameter to the other.

1710. Find the volume of the solid bounded by the cylinders $x^2+z^2=a^2$ and $y^2+z^2=a^2$.

1711. Find the volume of the segment cut off from the elliptic paraboloid $\frac{y^2}{2p}+\frac{z^2}{2q}=x$ by the plane $x=a$.

1712. Find the volume of the solid bounded by the hyperboloid of one sheet $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ and the planes $z=0$ and $z=h$.

1713. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sec. 10. The Area of a Surface of Revolution

The area of a surface formed by the rotation, about the x-axis, of an arc of the curve $y=f(x)$ between the points $x=a$ and $x=b$, is expressed by the formula

$$S_X=2\pi\int_a^b y\frac{ds}{dx}\,dx=2\pi\int_a^b y\sqrt{1+y'^2}\,dx \qquad (1)$$

(ds is the differential of the arc of the curve).

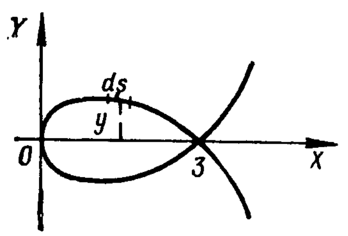

Fig. 54

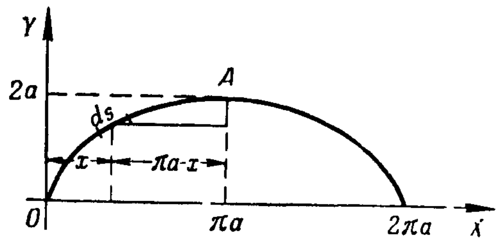

Fig. 55

If the equation of the curve is represented differently, the area of the surface S_X is obtained from formula (1) by an appropriate change of variables.

Example 1. Find the area of a surface formed by rotation, about the x-axis, of a loop of the curve $9y^2=x(3-x)^2$ (Fig. 54).

Solution. For the upper part of the curve, when $0\leqslant x\leqslant 3$, we have $y=\frac{1}{3}(3-x)\sqrt{x}$. Whence the differential of the arc $ds=\frac{x+1}{2\sqrt{x}}dx$. From formula (1) the area of the surface

$$S=2\pi\int_0^3 \frac{1}{3}(3-x)\sqrt{x}\,\frac{x+1}{2\sqrt{x}}\,dx=3\pi.$$

Example 2. Find the area of a surface formed by the rotation of one arc of the cycloid $x=a(t-\sin t)$; $y=a(1-\cos t)$ about its axis of symmetry (Fig. 55).

Solution. The desired surface is formed by rotation of the arc OA about the straight line AB, the equation of which is $x=\pi a$. Taking y as the independent variable and noting that the axis of rotation AB is displaced relative to the y-axis a distance πa, we will have

$$S=2\pi\int_0^{2a}(\pi a-x)\frac{ds}{dy}\cdot dy.$$

Passing to the variable t, we obtain

$$S=2\pi\int_0^{\pi}(\pi a-at+a\sin t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=$$

$$=2\pi\int_0^{\pi}(\pi a-at+a\sin t)\,2a\sin\frac{t}{2}\,dt=$$

$$=4\pi a^2\int_0^{\pi}\left(\pi\sin\frac{t}{2}-t\sin\frac{t}{2}+\sin t\sin\frac{t}{2}\right)dt=$$

$$=4\pi a^2\left[-2\pi\cos\frac{t}{2}+2t\cos\frac{t}{2}-4\sin\frac{t}{2}+\frac{4}{3}\sin^3\frac{t}{2}\right]_0^{\pi}=8\pi\left(\pi-\frac{4}{3}\right)a^2.$$

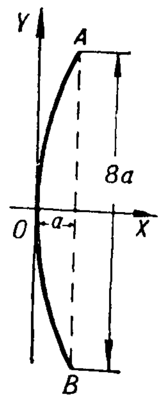

Fig. 56

1714. The dimensions of a parabolic mirror AOB are indicated in Fig. 56. It is required to find the area of its surface.

1715. Find the area of the surface of a spindle obtained by rotation of a lobe of the sinusoidal curve $y=\sin x$ about the x-axis.

1716. Find the area of the surface formed by the rotation of a part of the tangential curve $y=\tan x$ from $x=0$ to $x=\frac{\pi}{4}$, about the x-axis.

1717. Find the area of the surface formed by rotation, about the x-axis, of an arc of the curve $y=e^{-x}$, from $x=0$ to $x=+\infty$.

1718. Find the area of the surface (called a *catenoid*) formed by the rotation of a catenary $y=a\cosh\frac{x}{a}$ about the x-axis from $x=0$ to $x=a$.

1719. Find the area of the surface of rotation of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ about the y-axis.

1720. Find the area of the surface of rotation of the curve $x=\frac{1}{4}y^2-\frac{1}{2}\ln y$ about the x-axis from $y=1$ to $y=e$.

1721*. Find the surface of a torus formed by rotation of the circle $x^2+(y-b)^2=a^2$ about the x-axis ($b>a$).

1722. Find the area of the surface formed by rotation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about: 1) the x-axis, 2) the y-axis ($a>b$).

1723. Find the area of the surface formed by rotation of one arc of the cycloid $x=a(t-\sin t)$ and $y=a(1-\cos t)$ about: a) the x-axis, b) the y-axis, c) the tangent to the cycloid at its highest point.

1724. Find the area of the surface formed by rotation, about the x-axis, of the cardioid

$$\left.\begin{aligned} x&=a(2\cos t-\cos 2t),\\ y&=a(2\sin t-\sin 2t).\end{aligned}\right\}$$

1725. Determine the area of the surface formed by the rotation of the lemniscate $r^2=a^2\cos 2\varphi$ about the polar axis.

1726. Determine the area of the surface formed by the rotation of the cardioid $r=2a(1+\cos\varphi)$ about the polar axis.

Sec. 11. Moments. Centres of Gravity. Guldin's Theorems

1°. Static moment. The *static moment* relative to the l-axis of a material point A having mass m and at a distance d from the l-axis is the quantity $M_l=md$.

The *static moment* relative to the l-axis of a system of n material points with masses $m_1, m_2, \ldots, m_n$ lying in the plane of the axis and at distances $d_1, d_2, \ldots, d_n$ is the sum

$$M_l=\sum_{i=1}^{n} m_i d_i, \tag{1}$$

where the distances of points lying on one side of the l-axis have the plus sign, those on the other side have the minus sign. In a similar manner we define the *static moment of a system of points* relative to a plane.

If the masses continuously fill the line or figure of the xy-plane, then the static moments M_X and M_Y about the x- and y-axes are expressed (respectively) as integrals and not as the sums (1). For the cases of geometric figures, the density is considered equal to unity.

In particular: 1) for the curve $x=x(s)$; $y=y(s)$, where the parameter s is the arc length, we have

$$M_X=\int_0^L y(s)\,ds;\quad M_Y=\int_0^L x(s)\,ds \tag{2}$$

($ds=\sqrt{(dx)^2+(dy)^2}$ is the differential of the arc);

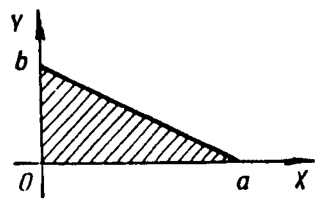

Fig. 57

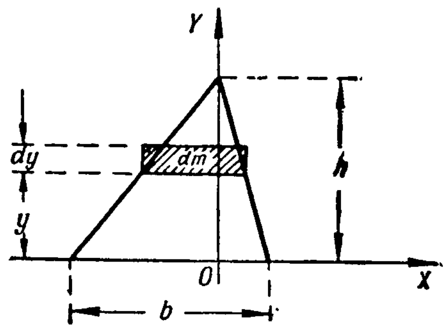

Fig. 58

2) for a plane figure bounded by the curve $y=y(x)$, the x-axis and two vertical lines $x=a$ and $y=b$, we obtain

$$M_X=\frac{1}{2}\int_a^b y\,|y|\,dx;\quad M_Y=\int_a^b x\,|y|\,dx. \tag{3}$$

Example 1. Find the static moments about the x- and y-axes of a triangle bounded by the straight lines: $\frac{x}{a}+\frac{y}{b}=1$, $x=0$, $y=0$ (Fig. 57)

Solution. Here, $y=b\left(1-\frac{x}{a}\right)$. Applying formula (3), we obtain

$$M_X=\frac{b^2}{2}\int_0^a\left(1-\frac{x}{a}\right)^2 dx=\frac{ab^2}{6}$$

and

$$M_Y=b\int_0^a x\left(1-\frac{x}{a}\right)dx=\frac{a^2b}{6}.$$

2°. **Moment of inertia.** The *moment of inertia*, about an l-axis, of a material point of mass m at a distance d from the l-axis, is the number $I_l=md^2$.

The *moment of inertia*, about an l-axis, of a system of n material points with masses $m_1, m_2, \ldots, m_n$ is the sum

$$I_l=\sum_{i=1}^{n} m_i d_i^2,$$

where $d_1, d_2, \ldots, d_n$ are the distances of the points from the l-axis. In the case of a continuous mass, we get an appropriate integral in place of a sum.

Example 2. Find the moment of inertia of a triangle with base b and altitude h about its base.

Solution. For the base of the triangle we take the x-axis, for its altitude, the y-axis (Fig 58).

Divide the triangle into infinitely narrow horizontal strips of width dy, which play the role of elementary masses dm. Utilizing the similarity of triangles, we obtain

$$dm = b\frac{h-y}{h}\,dy$$

and

$$dI_X = y^2 dm = \frac{b}{h}\,y^2(h-y)\,dy.$$

Whence

$$I_X = \frac{b}{h}\int_0^h y^2(h-y)\,dy = \frac{1}{12}bh^3.$$

3°. Centre of gravity. The *coordinates of the centre of gravity* of a plane figure (arc or area) of mass M are computed from the formulas

$$\bar{x} = \frac{M_Y}{M}, \quad \bar{y} = \frac{M_X}{M},$$

where M_X and M_Y are the static moments of the mass. In the case of geometric figures, the mass M is numerically equal to the corresponding arc or area.

For the coordinates of the centre of gravity $(\bar{x}, \bar{y})$ of an arc of the plane curve $y=f(x)$ $(a \leqslant x \leqslant b)$, connecting the points $A[a, f(a)]$ and $B[b, f(b)]$, we have

$$\bar{x} = \frac{\int_A^B x\,ds}{s} = \frac{\int_a^b x\sqrt{1+(y')^2}\,dx}{\int_a^b \sqrt{1+(y')^2}\,dx}, \qquad \bar{y} = \frac{\int_A^B y\,ds}{s} = \frac{\int_a^b y\sqrt{1+(y')^2}\,dx}{\int_a^b \sqrt{1+(y')^2}\,dx}.$$

The coordinates of the centre of gravity $(\bar{x}, \bar{y})$ of the curvilinear trapezoid $a \leqslant x \leqslant b$, $0 \leqslant y \leqslant f(x)$ may be computed from the formulas

$$\bar{x} = \frac{\int_a^b xy\,dx}{S}, \qquad \bar{y} = \frac{\frac{1}{2}\int_a^b y^2\,dx}{S},$$

where $S = \int_a^b y\,dx$ is the area of the figure.

There are similar formulas for the coordinates of the centre of gravity of a volume.

Example 3. Find the centre of gravity of an arc of the semicircle $x^2+y^2=a^2$; $(y \geqslant 0)$ (Fig. 59).

Solution. We have

$$y=\sqrt{a^2-x^2};\quad y'=\frac{-x}{\sqrt{a^2-x^2}}$$

and

$$ds=\sqrt{1+(y')^2}\,dx=\frac{a\,dx}{\sqrt{a^2-x^2}}.$$

Whence

$$M_Y=\int_{-a}^{a}x\,ds=\int_{-a}^{a}\frac{ax}{\sqrt{a^2-x^2}}\,dx=0,$$

$$M_X=\int_{-a}^{a}y\,ds=\int_{-a}^{a}\sqrt{a^2-x^2}\,\frac{a\,dx}{\sqrt{a^2-x^2}}=2a^2,$$

$$M=\int_{-a}^{a}\frac{a\,dx}{\sqrt{a^2-x^2}}=\pi a.$$

Hence,

$$\bar{x}=0;\quad \bar{y}=\frac{2}{\pi}a.$$

4°. Guldin's theorems.

Theorem 1. The area of a surface obtained by the rotation of an arc of a plane curve about some axis lying in the same plane as the curve and not intersecting it is equal to the product of the length of the curve by the circumference of the circle described by the centre of gravity of the arc of the curve.

Theorem 2. The volume of a solid obtained by rotation of a plane figure about some axis lying in the plane of the figure and not intersecting it is equal to the product of the area of this figure by the circumference of the circle described by the centre of gravity of the figure.

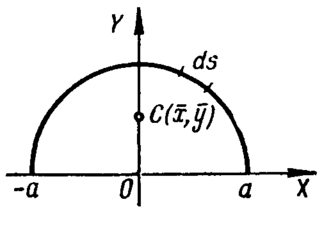

Fig. 59

1727. Find the static moments about the coordinate axes of a segment of the straight line

$$\frac{x}{a}+\frac{y}{b}=1,$$

lying between the axes.

1728. Find the static moments of a rectangle, with sides a and b, about its sides.

1729. Find the static moments, about the x- and y-axes, and the coordinates of the centre of gravity of a triangle bounded by the straight lines $x+y=a$, $x=0$, and $y=0$.

1730. Find the static moments, about the x- and y-axes, and the coordinates of the centre of gravity of an arc of the astroid

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}},$$

lying in the first quadrant.

1731. Find the static moment of the circle

$$r=2a\sin\varphi$$

about the polar axis.

1732. Find the coordinates of the centre of gravity of an arc of the catenary

$$y=a\cosh\frac{x}{a}$$

from $x=-a$ to $x=a$.

1733. Find the centre of gravity of an arc of a circle of radius a subtending an angle 2α.

1734. Find the coordinates of the centre of gravity of the arc of one arch of the cycloid

$$x=a(t-\sin t);\quad y=a(1-\cos t).$$

1735. Find the coordinates of the centre of gravity of an area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the coordinate axes ($x\geqslant 0$, $y\geqslant 0$).

1736. Find the coordinates of the centre of gravity of an area bounded by the curves

$$y=x^2,\quad y=\sqrt{x}.$$

1737. Find the coordinates of the centre of gravity of an area bounded by the first arch of the cycloid

$$x=a(t-\sin t),\quad y=a(1-\cos t)$$

and the x-axis.

1738**. Find the centre of gravity of a hemisphere of radius a lying above the xy-plane with centre at the origin.

1739**. Find the centre of gravity of a homogeneous right circular cone with base radius r and altitude h.

1740**. Find the centre of gravity of a homogeneous hemisphere of radius a lying above the xy-plane with centre at the origin.

1741. Find the moment of inertia of a circle of radius a about its diameter.

1742. Find the moments of inertia of a rectangle with sides a and b about its sides.

1743. Find the moment of inertia of a right parabolic segment with base $2b$ and altitude h about its axis of symmetry.

1744. Find the moments of inertia of the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about its principal axes.

1745.** Find the polar moment of inertia of a circular ring with radii R_1 and R_2 ($R_1<R_2$), that is, the moment of inertia about the axis passing through the centre of the ring and perpendicular to its plane.

1746.** Find the moment of inertia of a homogeneous right circular cone with base radius R and altitude H about its axis.

1747.** Find the moment of inertia of a homogeneous sphere of radius a and of mass M about its diameter.

1748. Find the surface and volume of a torus obtained by rotating a circle of radius a about an axis lying in its plane and at a distance b ($b>a$) from its centre.

1749. a) Determine the position of the centre of gravity of an arc of the astroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ lying in the first quadrant.

b) Find the centre of gravity of an area bounded by the curves $y^2=2px$ and $x^2=2py$.

1750.** a) Find the centre of gravity of a semicircle using Guldin's theorem.

b) Prove by Guldin's theorem that the centre of gravity of a triangle is distant from its base by one third of its altitude

Sec. 12. Applying Definite Integrals to the Solution of Physical Problems

1°. The path traversed by a point. If a point is in motion along some curve and the absolute value of the velocity $v=f(t)$ is a known function of the time t, then the *path* traversed by the point in an interval of time $[t_1, t_2]$ is

$$s=\int_{t_1}^{t_2} f(t)\,dt.$$

Example 1. The velocity of a point is

$$v=0.1t^3 \text{ m/sec.}$$

Find the path s covered by the point in the interval of time $T=10$ sec following the commencement of motion. What is the mean velocity of motion during this interval?

Solution. We have:

$$s=\int_0^{10} 0.1t^3dt=0.1\left.\frac{t^4}{4}\right|_0^{10}=250 \text{ metres}$$

and

$$v_{\text{mean}}=\frac{s}{T}=25 \text{ m/sec.}$$

2°. **The work of a force.** If a variable force $X=f(x)$ acts in the direction of the x-axis, then the *work of this force* over an interval $[x_1, x_2]$ is

$$A=\int_{x_1}^{x_2} f(x)\,dx.$$

Example 2. What work has to be performed to stretch a spring 6 cm, if a force of 1 kgf stretches it by 1 cm?

Solution. According to Hook's law the force X kgf stretching the spring by x_m is equal to $X=kx$, where k is a proportionality constant.

Putting $x=0.01$ m and $X=1$ kgf, we get $k=100$ and, hence, $X=100x$.

Whence the sought-for work is

$$A=\int_0^{0.06} 100\,x\,dx=50\,x^2\Big|_0^{0.06}=0.18 \text{ kgm}$$

3°. **Kinetic energy.** The *kinetic energy* of a material point of mass m and velocity v is defined as

$$K=\frac{mv^2}{2}.$$

The *kinetic energy* of a system of n material points with masses $m_1, m_2, \ldots, m_n$ having respective velocities $v_1, v_2, \ldots, v_n$, is equal to

$$K=\sum_{i=1}^{n}\frac{m_iv_i^2}{2}. \qquad (1)$$

To compute the kinetic energy of a solid, the latter is appropriately partitioned into elementary particles (which play the part of material points); then by summing the kinetic energies of these particles we get, in the limit, an integral in place of the sum (1).

Example 3. Find the kinetic energy of a homogeneous circular cylinder of density δ with base radius R and altitude h rotating about its axis with angular velocity ω.

Solution. For the elementary mass dm we take the mass of a hollow cylinder of altitude h with inner radius r and wall thickness dr (Fig. 60). We have:

$$dm=2\pi r\cdot h\delta\,dr.$$

Since the linear velocity of the mass dm is equal to $v=r\omega$, the elementary kinetic energy is

$$dK=\frac{v^2dm}{2}=\pi r^3\omega^2h\delta\,dr.$$

Whence

$$K=\pi\omega^2 h\delta\int_0^R r^3dr=\frac{\pi\omega^2\delta R^4 h}{4}.$$

4°. Pressure of a liquid. To compute the force of *liquid pressure* we use Pascal's law, which states that the force of pressure of a liquid on an area S at a depth of immersion h is

$$p=\gamma hS,$$

where γ is the specific weight of the liquid.

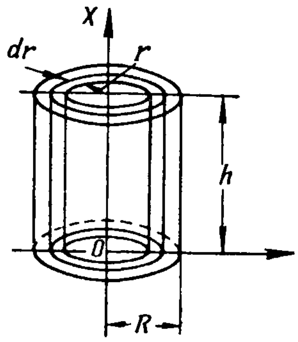

Fig. 60

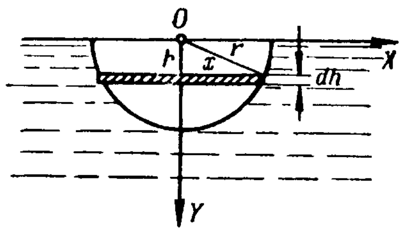

Fig. 61

Example 4. Find the force of pressure experienced by a semicircle of radius r submerged vertically in water so that its diameter is flush with the water surface (Fig 61).

Solution. We partition the area of the semicircle into elements—strips parallel to the surface of the water. The area of one such element (ignoring higher-order infinitesimals) located at a distance h from the surface is

$$ds=2x\,dh=2\sqrt{r^2-h^2}\,dh.$$

The pressure experienced by this element is

$$dP=\gamma h\,ds=2\gamma h\sqrt{r^2-h^2}\,dh,$$

where γ is the specific weight of the water equal to unity.

Whence the entire pressure is

$$P=2\int_0^r h\sqrt{r^2-h^2}\,dh=-\frac{2}{3}(r^2-h^2)^{\frac{3}{2}}\Big|_0^r=\frac{2}{3}r^3.$$

1751. The velocity of a body thrown vertically upwards with initial velocity v_0 (air resistance neglected), is given by the formula

$$v = v_0 - gt,$$

where t is the time that elapses and g is the acceleration of gravity. At what distance from the initial position will the body be in t seconds from the time it is thrown?

1752. The velocity of a body thrown vertically upwards with initial velocity v_0 (air resistance allowed for) is given by the formula

$$v = c\cdot\tan\left(-\frac{g}{c}t + \arctan\frac{v_0}{c}\right),$$

where t is the time, g is the acceleration of gravity, and c is a constant. Find the altitude reached by the body.

1753. A point on the x-axis performs harmonic oscillations about the coordinate origin; its velocity is given by the formula

$$v = v_0 \cos \omega t,$$

where t is the time and v_0, ω are constants.

Find the law of oscillation of a point if when $t = 0$ it had an abscissa $x = 0$. What is the mean value of the absolute magnitude of the velocity of the point during one cycle?

1754. The velocity of motion of a point is $v = te^{-0.01t}$ m/sec. Find the path covered by the point from the commencement of motion to full stop.

1755. A rocket rises vertically upwards. Considering that when the rocket thrust is constant, the acceleration due to decreasing weight of the rocket increases by the law $j = \frac{A}{a-bt}$ $(a - bt > 0)$, find the velocity at any instant of time t, if the initial velocity is zero. Find the altitude reached at time $t = t_1$.

1756*. Calculate the work that has to be done to pump the water out of a vertical cylindrical barrel with base radius R and altitude H.

1757. Calculate the work that has to be done in order to pump the water out of a conical vessel with vertex downwards, the radius of the base of which is R and the altitude H.

1758. Calculate the work to be done in order to pump water out of a semispherical boiler of radius $R = 10$ m.

1759. Calculate the work needed to pump oil out of a tank through an upper opening (the tank has the shape of a cylinder with horizontal axis) if the specific weight of the oil is γ, the length of the tank H and the radius of the base R.

1760.** What work has to be done to raise a body of mass m from the earth's surface (radius R) to an altitude h? What is the work if the body is removed to infinity?

1761.** Two electric charges $e_0 = 100$ CGSE and $e_1 = 200$ CGSE lie on the x-axis at points $x_0 = 0$ and $x_1 = 1$ cm, respectively. What work will be done if the second charge is moved to point $x_2 = 10$ cm?

1762.** A cylinder with a movable piston of diameter $D = 20$ cm and length $l = 80$ cm is filled with steam at a pressure $p = 10$ kgf cm^2. What work must be done to halve the volume of the steam with temperature kept constant (isothermic process)?

1763.** Determine the work performed in the adiabatic expansion of air (having initial volume $v_0 = 1$ m^3 and pressure $p_0 = 1$ kgf/cm^2) to volume $v_1 = 10$ m^3?

1764.** A vertical shaft of weight P and radius a rests on a bearing AB (Fig. 62). The frictional force between a small part σ of the base of the shaft and the surface of the support in contact with it is $F = \mu p \sigma$, where $p = \text{const}$ is the pressure of the shaft on the surface of the support referred to unit area of the support, while μ is the coefficient of friction. Find the work done by the frictional force during one revolution of the shaft.

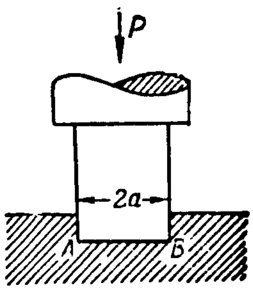

Fig. 62

1765.** Calculate the kinetic energy of a disk of mass M and radius R rotating with angular velocity ω about an axis that passes through its centre perpendicular to its plane.

1766. Calculate the kinetic energy of a right circular cone of mass M rotating with angular velocity ω about its axis, if the radius of the base of the cone is R and the altitude is H.

1767*. What work has to be done to stop an iron sphere of radius $R = 2$ metres rotating with angular velocity $\omega = 1{,}000$ rpm about its diameter? (Specific weight of iron, $\gamma = 7.8$ g/cm^3.)

1768. A vertical triangle with base b and altitude h is submerged vertex downwards in water so that its base is on the surface of the water. Find the pressure of the water.

1769. A vertical dam has the shape of a trapezoid. Calculate the water pressure on the dam if we know that the upper base $a = 70$ m, the lower base $b = 50$ m, and the height $h = 20$ m.

1770. Find the pressure of a liquid, whose specific weight is γ, on a vertical ellipse (with axes $2a$ and $2b$) whose centre is submerged in the liquid to a distance h, while the major axis $2a$ of the ellipse is parallel to the level of the liquid ($h \geqslant b$).

1771. Find the water pressure on a vertical circular cone with radius of base R and altitude H submerged in water vertex downwards so that its base is on the surface of the water.

Miscellaneous Problems

1772. Find the mass of a rod of length $l=100$ cm if the linear density of the rod at a distance x cm from one of its ends is

$$\delta = 2 + 0.001\,x^2 \text{ g/cm}.$$

1773. According to empirical data the specific thermal capacity of water at a temperature t° C ($0 \leqslant t \leqslant 100^\circ$) is

$$c = 0.9983 - 5.184 \times 10^{-5}\,t + 6.912 \times 10^{-7}\,t^2.$$

What quantity of heat has to be expended to heat 1 g of water from 0° C to 100° C?

1774. The wind exerts a uniform pressure p g/cm² on a door of width b cm and height h cm. Find the moment of the pressure of the wind striving to turn the door on its hinges.

1775. What is the force of attraction of a material rod of length l and mass M on a material point of mass m lying on a straight line with the rod at a distance a from one of its ends?

1776.** In the case of steady-state laminar flow of a liquid through a pipe of circular cross-section of radius a, the velocity of flow v at a point distant r from the axis of the pipe is given by the formula

$$v = \frac{p}{4\mu l}(a^2 - r^2),$$

where p is the pressure difference at the ends of the pipe, μ is the coefficient of viscosity, and l is the length of the pipe. Determine the *discharge of liquid* Q (that is, the quantity of liquid flowing through a cross-section of the pipe in unit time).

1777*. The conditions are the same as in Problem 1776, but the pipe has a rectangular cross-section, and the base a is great compared with the altitude $2b$. Here the rate of flow v at a point $M(x,y)$ is defined by the formula

$$v = \frac{p}{2\mu l}[b^2 - (b-y)^2].$$

Determine the discharge of liquid Q.

1778.** In studies of the dynamic qualities of an automobile, use is frequently made of special types of diagrams: the velocities v are laid off on the x-axis, and the reciprocals of corresponding accelerations a, on the y-axis. Show that the area S bounded by an arc of this graph, by two ordinates $v=v_1$ and $v=v_2$, and by the x-axis is numerically equal to the time needed to increase the velocity of motion of the automobile from v_1 to v_2 *(acceleration time).*

1779. A horizontal beam of length l is in equilibrium due to a downward vertical load uniformly distributed over the length of the beam, and of support reactions A and $B\left(A=B=\frac{Q}{2}\right)$, directed vertically upwards. Find the bending moment M_x in a cross-section x, that is, the moment about the point P with abscissa x of all forces acting on the portion of the beam AP.

1780. A horizontal beam of length l is in equilibrium due to support reactions A and B and a load distributed along the length of the beam with intensity $q=kx$, where x is the distance from the left support and k is a constant factor. Find the bending moment M_x in cross-section x.

Note. The intensity of load distribution is the load (force) referred to unit length.

1781*. Find the quantity of heat released by an alternating sinusoidal current

$$I=I_0 \sin\left(\frac{2\pi}{T}t-\varphi\right)$$

during a cycle T in a conductor with resistance R.

Chapter VI

FUNCTIONS OF SEVERAL VARIABLES

Sec. 1. Basic Notions

1°. The concept of a function of several variables. Functional notation. A variable quantity z is called a single-valued function of two variables x, y, if to each set of their values (x, y) in a given range there corresponds a unique value of z. The variables x and y are called *arguments* or *independent variables*. The functional relation is denoted by

$$z = f(x, y).$$

Similarly, we define functions of three or more arguments.

Example 1. Express the volume of a cone V as a function of its generatrix x and of its base radius y.

Solution. From geometry we know that the volume of a cone is

$$V = \frac{1}{3}\pi y^2 h,$$

where h is the altitude of the cone. But $h = \sqrt{x^2 - y^2}$. Hence,

$$V = \frac{1}{3}\pi y^2 \sqrt{x^2 - y^2}.$$

This is the desired functional relation.

The value of the function $z = f(x, y)$ at a point $P(a, b)$, that is, when $x = a$ and $y = b$, is denoted by $f(a, b)$ or $f(P)$. Generally speaking, the geometric representation of a function like $z = f(x, y)$ in a rectangular coordinate system X, Y, Z is a surface (Fig. 63).

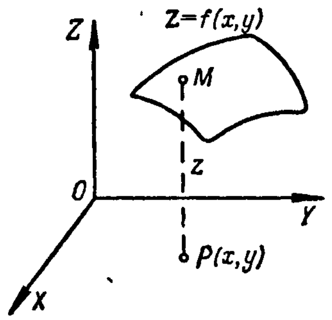

Fig. 63

Example 2. Find $f(2, -3)$ and $f\left(1, \frac{y}{x}\right)$ if

$$f(x, y) = \frac{x^2 + y^2}{2xy}.$$

Solution. Substituting $x = 2$ and $y = -3$, we find

$$f(2, -3) = \frac{2^2 + (-3)^2}{2 \cdot 2 \cdot (-3)} = -\frac{13}{12}.$$

Putting $x=1$ and replacing y by $\frac{y}{x}$, we will have

$$f\left(1, \frac{y}{x}\right)=\frac{1+\left(\frac{y}{x}\right)^2}{2\cdot 1\left(\frac{y}{x}\right)}=\frac{x^2+y^2}{2xy},$$

that is, $f\left(1, \frac{y}{x}\right)=f(x, y)$.

2°. **Domain of definition of a function.** By the *domain of definition* of a function $z=f(x, y)$ we understand a set of points (x, y) in an xy-plane in which the given function is defined (that is to say, in which it takes on definite real values) In the simplest cases, the domain of definition of a function is a finite or infinite part of the xy-plane bounded by one or several curves (the *boundary of the domain*).

Similarly, for a function of three variables $u=f(x, y, z)$ the domain of definition of the function is a volume in xyz-space.

Example 3. Find the domain of definition of the function

$$z=\frac{1}{\sqrt{4-x^2-y^2}}.$$

Solution. The function has real values if $4-x^2-y^2>0$ or $x^2+y^2<4$. The latter inequality is satisfied by the coordinates of points lying inside a circle of radius 2 with centre at the coordinate origin. The domain of definition of the function is the interior of the circle (Fig 64).

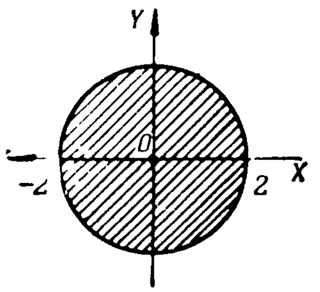

Fig. 64

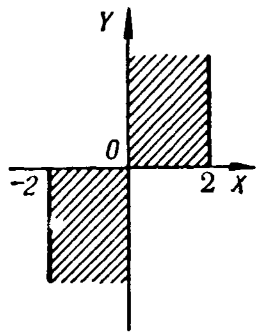

Fig 65

Example 4. Find the domain of definition of the function

$$z=\arcsin\frac{x}{2}+\sqrt{xy}$$

Solution. The first term of the function is defined for $-1\leqslant\frac{x}{2}\leqslant 1$ or $-2\leqslant x\leqslant 2$. The second term has real values if $xy\geqslant 0$, i.e., in two cases: when $\begin{cases} x\geqslant 0, \\ y\geqslant 0\end{cases}$ or when $\begin{cases} x\leqslant 0, \\ y\leqslant 0\end{cases}$. The domain of definition of the entire function is shown in Fig. 65 and includes the boundaries of the domain.

3°. Level lines and level surfaces of a function. The *level line* of a function $z=f(x, y)$ is a line $f(x, y)=C$ (in an xy-plane) at the points of which the function takes on one and the same value $z=C$ (usually labelled in drawings).

The *level surface* of a function of three arguments $u=f(x, y, z)$ is a surface $f(x, y, z)=C$, at the points of which the function takes on a constant value $u=C$.

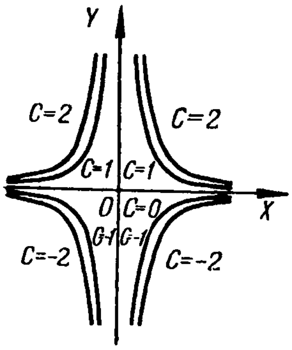

Fig. 66

Example 5. Construct the level lines of the function $z=x^2y$.

Solution. The equation of the level lines has the form $x^2y=C$ or $y=\frac{C}{x^2}$.

Putting $C=0, \pm 1, \pm 2, \ldots$, we get a family of level lines (Fig. 66).

1782. Express the volume V of a regular tetragonal pyramid as a function of its altitude x and lateral edge y.

1783. Express the lateral surface S of a regular hexagonal truncated pyramid as a function of the sides x and y of the bases and the altitude z.

1784. Find $f(1/2, 3)$, $f(1, -1)$, if

$$f(x, y)=xy+\frac{x}{y}.$$

1785 Find $f(y,x)$, $f(-x, -y)$, $f\left(\frac{1}{x}, \frac{1}{y}\right)$, $\frac{1}{f(x, y)}$, if $f(x, y)=\frac{x^2-y^2}{2xy}$.

1786. Find the values assumed by the function

$$f(x, y)=1+x-y$$

at points of the parabola $y=x^2$, and construct the graph of the function

$$F(x)=f(x, x^2).$$

1787. Find the value of the function

$$z=\frac{x^4+2x^2y^2+y^4}{1-x^2-y^2}$$

at points of the circle $x^2+y^2=R^2$.

1788*. Determine $f(x)$, if

$$f\left(\frac{y}{x}\right)=\frac{\sqrt{x^2+y^2}}{y}\ (y>0).$$

1789*. Find $f(x, y)$ if

$$f(x+y, x-y)=xy+y^2.$$

1790*. Let $z=\sqrt{y}+f(\sqrt{x}-1)$. Determine the functions f and z if $z=x$ when $y=1$.

1791.** Let $z=xf\left(\frac{y}{x}\right)$. Determine the functions f and z if $z=\sqrt{1+y^2}$ when $x=1$.

1792. Find and sketch the domains of definition of the following functions:

a) $z=\sqrt{1-x^2-y^2}$;

b) $z=1+\sqrt{-(x-y)^2}$;

c) $z=\ln(x+y)$;

d) $z=x+\arccos y$;

e) $z=\sqrt{1-x^2}+\sqrt{1-y^2}$;

f) $z=\arcsin\frac{y}{x}$;

g) $z=\sqrt{x^2-4}+\sqrt{4-y^2}$;

h) $z=\sqrt{(x^2+y^2-a^2)(2a^2-x^2-y^2)}$ $(a>0)$;

i) $z=\sqrt{y\sin x}$;

j) $z=\ln(x^2+y)$;

k) $z=\text{arc tang}\,\frac{x-y}{1+x^2y^2}$;

l) $z=\frac{1}{x^2+y^2}$;

m) $z=\frac{1}{\sqrt{y-\sqrt{x}}}$;

n) $z=\frac{1}{x-1}+\frac{1}{y}$;

o) $z\sqrt{\sin(x^2+y^2)}$.

1793. Find the domains of the following functions of three arguments:

a) $u=\sqrt{x}+\sqrt{y}+\sqrt{z}$;

b) $u=\ln(xyz)$;

c) $u=\arcsin x+\arcsin y+\arcsin z$;

d) $u=\sqrt{1-x^2-y^2-z^2}$.

1794. Construct the level lines of the given functions and determine the character of the surfaces depicted by these functions:

a) $z=x+y$;

b) $z=x^2+y^2$;

c) $z=x^2-y^2$;

d) $z=\sqrt{xy}$;

e) $z=(1+x+y)^2$;

f) $z=1-|x|-|y|$;

g) $z=\frac{y}{x^2}$;

h) $z=\frac{y}{\sqrt{x}}$;

i) $z=\frac{2x}{x^2+y^2}$.

1795. Find the level lines of the following functions:

a) $z=\ln(x^2+y)$;

b) $z=\arcsin xy$;

c) $z=f(\sqrt{x^2+y^2})$;

d) $z=f(y-ax)$;

e) $z=f\left(\frac{y}{x}\right)$.

1796. Find the level surfaces of the functions of three independent variables:

a) $u=x+y+z$;

b) $u=x^2+y^2+z^2$;

c) $u=x^2+y^2-z^2$.

Sec. 2. Continuity

1°. The limit of a function. A number A is called the *limit of a function* $z=f(x, y)$ as the point $P'(x, y)$ approaches the point $P(a, b)$, if for any $\varepsilon>0$ there is a $\delta>0$ such that when $0<\rho<\delta$, where $\rho=\sqrt{(x-a)^2+(y-b)^2}$ is the distance between P and P', we have the inequality

$$|f(x, y)-A|<\varepsilon.$$

In this case we write

$$\lim_{\substack{x\to a\\ y\to b}} f(x, y)=A.$$

2°. Continuity and points of discontinuity. A function $z=f(x, y)$ is called *continuous at a point* $P(a, b)$ if

$$\lim_{\substack{x\to a\\ y\to b}} f(x, y)=f(a, b).$$

A function that is continuous at all points of a given range is called *continuous over this range*

A function $f(x, y)$ may cease to be continuous either at separate points *(isolated point of discontinuity)* or at points that form one or several lines *(lines of discontinuity)* or (at times) more complex geometric objects.

Example 1. Find the discontinuities of the function

$$z=\frac{xy+1}{x^2-y}.$$

Solution. The function will be meaningless if the denominator becomes zero. But $x^2-y=0$ or $y=x^2$ is the equation of a parabola. Hence, the given function has for its discontinuity the parabola $y=x^2$.

1797*. Find the following limits of functions:

a) $\lim\limits_{\substack{x\to 0\\ y\to 0}}(x^2+y^2)\sin\frac{1}{xy}$; c) $\lim\limits_{\substack{x\to 0\\ y\to 2}}\frac{\sin xy}{x}$; e) $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x}{x+y}$;

b) $\lim\limits_{\substack{x\to\infty\\ y\to\infty}}\frac{x+y}{x^2+y^2}$; d) $\lim\limits_{\substack{x\to\infty\\ y\to k}}\left(1+\frac{y}{x}\right)^x$; f) $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x^2-y^2}{x^2+y^2}$.

1798. Test the following function for continuity:

$$f(x, y)=\begin{cases}\sqrt{1-x^2-y^2} & \text{when } x^2+y^2\leqslant 1,\\ 0 & \text{when } x^2+y^2>1.\end{cases}$$

1799. Find points of discontinuity of the functions:

a) $z=\ln\sqrt{x^2+y^2}$; c) $z=\frac{1}{1-x^2-y^2}$;

b) $z=\frac{1}{(x-y)^2}$; d) $z=\cos\frac{1}{xy}$.

1800*. Show that the function

$$z=\begin{cases} \dfrac{2xy}{x^2+y^2} & \text{when } x^2+y^2\neq 0, \\ 0 & \text{when } x=y=0 \end{cases}$$

is continuous with respect to each of the variables x and y separately, but is not continuous at the point (0, 0) with respect to these variables together.

Sec. 3. Partial Derivatives

1°. Definition of a partial derivative. If $z=f(x, y)$, then assuming, for example, y constant, we get the derivative

$$\frac{\partial z}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x, y)-f(x, y)}{\Delta x}=f'_x(x, y),$$

which is called the *partial derivative* of the function z with respect to the variable x. In similar fashion we define and denote the partial derivative of the function z with respect to the variable y. It is obvious that to find partial derivatives, one can use the ordinary formulas of differentiation.

Example 1. Find the partial derivatives of the function

$$z=\ln\tan\frac{x}{y}.$$

Solution. Regarding y as constant, we get

$$\frac{\partial z}{\partial x}=\frac{1}{\tan\frac{x}{y}}\,\frac{1}{\cos^2\frac{x}{y}}\cdot\frac{1}{y}=\frac{2}{y\sin\frac{2x}{y}}.$$

Similarly, holding x constant, we will have

$$\frac{\partial z}{\partial y}=\frac{1}{\tan\frac{x}{y}}\cdot\frac{1}{\cos^2\frac{x}{y}}\left(-\frac{x}{y^2}\right)=-\frac{2x}{y^2\sin\frac{2x}{y}}.$$

Example 2. Find the partial derivatives of the following function of three arguments:

$$u=x^3y^2z+2x-3y+z+5.$$

Solution.

$$\frac{\partial u}{\partial x}=3x^2y^2z+2,$$

$$\frac{\partial u}{\partial y}=2x^3yz-3,$$

$$\frac{\partial u}{\partial z}=x^3y^2+1.$$

2°. Euler's theorem. A function $f(x, y)$ is called a *homogeneous* function of degree n if for every real factor k we have the equality

$$f(kx, ky)=k^nf(x, y)$$

A rational integral function will be homogeneous if all its terms are of one and the same degree.

The following relationship holds for a homogeneous differentiable function of degree n (*Euler's theorem*):

$$xf'_x(x, y)+yf'_y(x, y)=nf(x, y).$$

Find the partial derivatives of the following functions:

1801. $z=x^3+y^3-3axy$.

1802. $z=\frac{x-y}{x+y}$.

1803. $z=\frac{y}{x}$.

1804. $z=\sqrt{x^2-y^2}$.

1805. $z=\frac{x}{\sqrt{x^2+y^2}}$.

1806. $z=\ln(x+\sqrt{x^2+y^2})$.

1807. $z=\arctan\frac{y}{x}$.

1808. $z=x^y$.

1809. $z=e^{\sin\frac{y}{x}}$.

1810. $z=\arcsin\sqrt{\frac{x^2-y^2}{x^2+y^2}}$.

1811. $z=\ln\sin\frac{x+a}{\sqrt{y}}$.

1812. $u=(xy)^z$.

1813. $u=z^{xy}$.

1814. Find $f'_x(2, 1)$ and $f'_y(2, 1)$ if $f(x, y)=\sqrt{xy+\frac{x}{y}}$.

1815. Find $f'_x(1, 2, 0)$, $f'_y(1, 2, 0)$, $f'_z(1, 2, 0)$ if

$$f(x, y, z)=\ln(xy+z).$$

Verify Euler's theorem on homogeneous functions in Examples 1816 to 1819:

1816. $f(x, y)=Ax^2+2Bxy-Cy^2$.

1817. $z=\frac{x}{x^2+y^2}$.

1818. $f(x, y)=\frac{x+y}{\sqrt[3]{x^2+y^2}}$.

1819. $f(x, y)=\ln\frac{y}{x}$.

1820. Find $\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$, where $r=\sqrt{x^2+y^2+z^2}$.

1821. Calculate $\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix}$, if $x=r\cos\varphi$ and $y=r\sin\varphi$.

1822. Show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2$, if $z=\ln(x^2+xy+y^2)$.

1823. Show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=xy+z$, if $z=xy+xe^{\frac{y}{x}}$.

1824. Show that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$, if $u=(x-y)(y-z)(z-x)$.

1825. Show that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=1$, if $u=x+\frac{x-y}{y-z}$.

1826. Find $z=z(x, y)$, if $\frac{\partial z}{\partial y}=\frac{x}{x^2+y^2}$.

1827. Find $z=z(x, y)$ knowing that

$$\frac{\partial z}{\partial x}=\frac{x^2+y^2}{x} \text{ and } z(x, y)=\sin y \text{ when } x=1.$$

1828. Through the point $M(1, 2, 6)$ of a surface $z=2x^2+y^2$ are drawn planes parallel to the coordinate surfaces XOZ and YOZ. Determine the angles formed with the coordinate axes by the tangent lines (to the resulting cross-sections) drawn at their common point M.

1829. The area of a trapezoid with bases a and b and altitude h is equal to $S={}^1/_2(a+b)h$. Find $\frac{\partial S}{\partial a}$, $\frac{\partial S}{\partial b}$, $\frac{\partial S}{\partial h}$ and, using the drawing, determine their geometrical meaning.

1830*. Show that the function

$$f(x, y)=\begin{cases} \frac{2xy}{x^2+y^2}, & \text{if } x^2+y^2\neq 0, \\ 0, & \text{if } x=y=0, \end{cases}$$

has partial derivatives $f'_x(x, y)$ and $f'_y(x, y)$ at the point (0, 0), although it is discontinuous at this point. Construct the geometric image of this function near the point (0, 0).

Sec. 4. Total Differential of a Function

1°. **Total increment of a function.** The *total increment* of a function $z=f(x, y)$ is the difference

$$\Delta z=\Delta f(x, y)=f(x+\Delta x, y+\Delta y)-f(x, y).$$

2°. **The total differential of a function.** The *total* (or *exact*) *differential* of a function $z=f(x, y)$ is the principal part of the total increment Δz, which is linear with respect to the increments in the arguments Δx and Δy.

The difference between the total increment and the total differential of the function is an infinitesimal of higher order compared with $\rho=\sqrt{\Delta x^2+\Delta y^2}$.

A function definitely has a total differential if its partial derivatives are continuous. If a function has a total differential, then it is called *differentiable*. The differentials of independent variables coincide with their increments, that is, $dx=\Delta x$ and $dy=\Delta y$. The total differential of the function $z=f(x, y)$ is computed by the formula

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy.$$

Similarly, the total differential of a function of three arguments $u=f(x, y, z)$ is computed from the formula

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz.$$

Example 1. For the function

$$f(x, y)=x^2+xy-y^2$$

find the total increment and the total differential.

Solution. $f(x+\Delta x, y+\Delta y) = (x+\Delta x)^2 + (x+\Delta x)(y+\Delta y) - (y+\Delta y)^2$;

$$\Delta f(x, y) = [(x+\Delta x)^2 + (x+\Delta x)(y+\Delta y) - (y+\Delta y)^2] - (x^2+xy-y^2) =$$
$$= 2x\cdot\Delta x + \Delta x^2 + x\cdot\Delta y + y\cdot\Delta x + \Delta x\cdot\Delta y - 2y\cdot\Delta y - \Delta y^2 =$$
$$= [(2x+y)\,\Delta x + (x-2y)\,\Delta y] + (\Delta x^2 + \Delta x\cdot\Delta y - \Delta y^2).$$

Here, the expression $df = (2x+y)\,\Delta x + (x-2y)\,\Delta y$ is the total differential of the function, while $(\Delta x^2 + \Delta x\cdot\Delta y - \Delta y^2)$ is an infinitesimal of higher order compared with $\sqrt{\Delta x^2 + \Delta y^2}$.

Example 2. Find the total differential of the function

$$z = \sqrt{x^2+y^2}.$$

Solution. $\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$; $\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$.

$$dz = \frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy = \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}.$$

3°. Applying the total differential of a function to approximate calculations. For sufficiently small $|\Delta x|$ and $|\Delta y|$ and, hence, for sufficiently small $\varrho = \sqrt{\Delta x^2 + \Delta y^2}$, we have for a differentiable function $z = f(x, y)$ the approximate equality $\Delta z \approx dz$ or

$$\Delta z \approx \frac{\partial z}{\partial x}\,\Delta x + \frac{\partial z}{\partial y}\,\Delta y.$$

Example 3. The altitude of a cone is $H = 30$ cm, the radius of the base $R = 10$ cm. How will the volume of the cone change, if we increase H by 3 mm and diminish R by 1 mm?

Solution. The volume of the cone is $V = \frac{1}{3}\pi R^2 H$. The change in volume we replace approximately by the differential

$$\Delta V \approx dV = \frac{1}{3}\pi\,(2RH\,dR + R^2\,dH) =$$
$$= \frac{1}{3}\pi\,(-2\cdot 10\cdot 30\cdot 0.1 + 100\cdot 0.3) = -10\pi \approx -31.4\ \text{cm}^3.$$

Example 4. Compute $1.02^{3.01}$ approximately.

Solution. We consider the function $z = x^y$. The desired number may be considered the increased value of this function when $x=1$, $y=3$, $\Delta x = 0.02$, $\Delta y = 0.01$. The initial value of the function $z = 1^3 = 1$,

$$\Delta z \approx dz = yx^{y-1}\,\Delta x + x^y \ln x\,\Delta y = 3\cdot 1\cdot 0.02 + 1\cdot \ln 1\cdot 0.01 = 0.06.$$

Hence, $1.02^{3.01} \approx 1 + 0.06 = 1.06$.

1831. For the function $f(x, y) = x^2 y$ find the total increment and the total differential at the point (1, 2); compare them if

a) $\Delta x = 1$, $\Delta y = 2$; b) $\Delta x = 0.1$, $\Delta y = 0.2$.

1832. Show that for the functions u and v of several (for example, two) variables the ordinary rules of differentiation hold:

a) $d(u+v) = du + dv$; b) $d(uv) = u\,dv + v\,du$;

c) $d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$.

Find the total differentials of the following functions:

1833. $z = x^3 + y^3 - 3xy$.

1834. $z = x^2 y^3$.

1835. $z = \frac{x^2 - y^2}{x^2 + y^2}$.

1836. $z = \sin^2 x + \cos^2 y$.

1837. $z = yx^y$.

1838. $z = \ln(x^2 + y^2)$.

1839. $f(x, y) = \ln\left(1 + \frac{x}{y}\right)$.

1840. $z = \arctan\frac{y}{x} + \arctan\frac{x}{y}$.

1841. $z = \ln\tan\frac{y}{x}$.

1842. Find $df(1, 1)$, if $f(x, y) = \frac{x}{y^2}$.

1843. $u = xyz$.

1844. $u = \sqrt{x^2 + y^2 + z^2}$.

1845. $u = \left(xy + \frac{x}{y}\right)^z$.

1846. $u = \arctan\frac{xy}{z^2}$.

1847. Find $df(3, 4, 5)$ if $f(x, y, z) = \frac{z}{\sqrt{x^2 + y^2}}$.

1848. One side of a rectangle is $a = 10$ cm, the other $b = 24$ cm. How will a diagonal l of the rectangle change if the side a is increased by 4 mm and b is shortened by 1 mm? Approximate the change and compare it with the exact value.

1849. A closed box with outer dimensions 10 cm, 8 cm, and 6 cm is made of 2-mm-thick plywood. Approximate the volume of material used in making the box.

1850*. The central angle of a circular sector is 80°; it is desired to reduce it by 1°. By how much should the radius of the sector be increased so that the area will remain unchanged, if the original length of the radius is 20 cm?

1851. Approximate:

a) $(1.02)^3 \cdot (0.97)^2$; b) $\sqrt{(4.05)^2 + (2.93)^2}$;

c) $\sin 32° \cdot \cos 59°$ (when converting degrees into radius and calculating $\sin 60°$ take three significant figures; round off the last digit).

1852. Show that the relative error of a product is approximately equal to the sum of the relative errors of the factors.

1853. Measurements of a triangle ABC yielded the following data: side $a = 100\text{m} \pm 2\text{m}$, side $b = 200\text{m} \pm 3$ m, angle $C = 60° \pm 1°$. To what degree of accuracy can we compute the side c?

1854. The oscillation period T of a pendulum is computed from the formula

$$T = 2\pi\sqrt{\frac{l}{g}},$$

where l is the length of the pendulum and g is the acceleration of gravity. Find the error, when determining T, obtained as a result of small errors $\Delta l=\alpha$ and $\Delta g=\beta$ in measuring l and g.

1855. The distance between the points $P_0(x_0, y_0)$ and $P(x, y)$ is equal to ϱ, while the angle formed by the vector $\overline{P_0P}$ with the x-axis is α. By how much will the angle α change if the point P (P_0 is fixed) moves to $P_1(x+dx, y+dy)$?

Sec. 5. Differentiation of Composite Functions

1°. The case of one independent variable. If $z=f(x, y)$ is a differentiable function of the arguments x and y, which in turn are differentiable functions of an independent variable t,

$$x=\varphi(t), \quad y=\psi(t),$$

then the derivative of the composite function $z=f[\varphi(t), \psi(t)]$ may be computed from the formula

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{1}$$

In particular, if t coincides with one of the arguments, for instance x, then the "total" derivative of the function z with respect to x will be:

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\frac{dy}{dx}. \tag{2}$$

Example 1. Find $\frac{dz}{dt}$, if

$$z=e^{3x+2y}, \text{ where } x=\cos t, \ y=t^2.$$

Solution. From formula (1) we have:

$$\frac{dz}{dt}=e^{3x+2y}\cdot 3(-\sin t)+e^{3x+2y}\cdot 2\cdot 2t=e^{3x+2y}(4t-3\sin t)=e^{3\cos t+2t^2}(4t-3\sin t).$$

Example 2. Find the partial derivative $\frac{\partial z}{\partial x}$ and the total derivative $\frac{dz}{dx}$, if

$$z=e^{xy}, \text{ where } y=\varphi(x).$$

Solution. $\frac{\partial z}{\partial x}=ye^{xy}$.

From formula (2) we obtain

$$\frac{dz}{dx}=ye^{xy}+xe^{xy}\varphi'(x).$$

2°. The case of several independent variables. If z is a composite function of several independent variables, for instance, $z=f(x,y)$, where $x=\varphi(u,v)$, $y=\psi(u, v)$ (u and v are independent variables), then the partial derivatives z **with respect** to u and v are expressed as

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \tag{3}$$

and

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}. \qquad (4)$$

In all the cases considered the following formula holds:

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$

(the invariance property of a total differential).

Example 3. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$, if

$$z=f(x, y), \quad \text{where} \quad x=uv, \ y=\frac{u}{v}.$$

Solution. Applying formulas (3) and (4), we get:

$$\frac{\partial z}{\partial u}=f'_x(x, y)\cdot v+f'_y(x, y)\frac{1}{v}$$

and

$$\frac{\partial z}{\partial v}=f'_x(x, y)\ u-f'_y(x, y)\frac{u}{v^2}.$$

Example 4. Show that the function $z=\varphi(x^2+y^2)$ satisfies the equation

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=0.$$

Solution. The function φ depends on x and y via the intermediate argument $x^2+y^2=t$, therefore,

$$\frac{\partial z}{\partial x}=\frac{dz}{dt}\frac{\partial t}{\partial x}=\varphi'(x^2+y^2)\,2x$$

and

$$\frac{\partial z}{\partial y}=\frac{dz}{dt}\frac{\partial t}{\partial y}=\varphi'(x^2+y^2)\,2y.$$

Substituting the partial derivatives into the left-hand side of the equation, we get

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=y\varphi'(x^2+y^2)\,2x-x\varphi'(x^2+y^2)\,2y=2xy\,\varphi'(x^2+y^2)-2xy\,\varphi'(x^2+y^2)\equiv 0,$$

that is, the function z satisfies the given equation.

1856. Find $\frac{dz}{dt}$ if

$$z=\frac{x}{y}, \text{ where } x=e^t, \ y=\ln t.$$

1857. Find $\frac{du}{dt}$ if

$$u=\ln\sin\frac{x}{\sqrt{y}}, \text{ where } x=3t^2, \ y=\sqrt{t^2+1}.$$

1858. Find $\frac{du}{dt}$ if

$$u=xyz, \text{ where } x=t^2+1, \ y=\ln t, \ z=\tan t.$$

1859. Find $\frac{du}{dt}$ if

$$u=\frac{z}{\sqrt{x^2+y^2}}, \text{ where } x=R\cos t,\ y=R\sin t,\ z=H.$$

1860. Find $\frac{dz}{dx}$ if

$$z=u^v, \text{ where } u=\sin x,\ v=\cos x.$$

1861. Find $\frac{\partial z}{\partial x}$ and $\frac{dz}{dx}$ if

$$z=\arctan\frac{y}{x} \text{ and } y=x^2.$$

1862. Find $\frac{\partial z}{\partial x}$ and $\frac{dz}{dx}$ if

$$z=x^y, \text{ where } y=\varphi(x).$$

1863. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$z=f(u,\ v), \text{ where } u=x^2-y^2,\ v=e^{xy}.$$

1864. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ if

$$z=\arctan\frac{x}{y}, \text{ where } x=u\sin v,\ y=u\cos v.$$

1865. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$z=f(u), \text{ where } u=xy+\frac{y}{x}.$$

1866. Show that if

$$u=\Phi(x^2+y^2+z^2), \text{ where } x=R\cos\varphi\cos\psi,$$
$$y=R\cos\varphi\sin\psi,\quad z=R\sin\varphi,$$

then

$$\frac{\partial u}{\partial\varphi}=0 \text{ and } \frac{\partial u}{\partial\psi}=0.$$

1867. Find $\frac{du}{dx}$ if

$$u=f(x,\ y,\ z), \text{ where } y=\varphi(x),\ z=\psi(x,\ y).$$

1868. Show that if

$$z=f(x+ay),$$

where f is a differentiable function, then

$$\frac{\partial z}{\partial y}=a\frac{\partial z}{\partial x}.$$

1869. Show that the function

$$w=f(u, v),$$

where $u=x+at$, $v=y+bt$ satisfy the equation

$$\frac{\partial w}{\partial t}=a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}.$$

1870. Show that the function

$$z=y\varphi(x^2-y^2)$$

satisfies the equation $\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{z}{y^2}$.

1871. Show that the function

$$z=xy+x\varphi\left(\frac{y}{x}\right)$$

satisfies the equation $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=xy+z$.

1872. Show that the function

$$z=e^y\varphi\left(ye^{\frac{x^2}{2y^2}}\right)$$

satisfies the equation $(x^2-y^2)\frac{\partial z}{\partial x}+xy\frac{\partial z}{\partial y}=xyz$.

1873. The side of a rectangle $x=20$ m increases at the rate of 5 m/sec, the other side $y=30$ m decreases at 4 m/sec. What is the rate of change of the perimeter and the area of the rectangle?

1874. The equations of motion of a material point are

$$x=t,\ y=t^2,\ z=t^3.$$

What is the rate of recession of this point from the coordinate origin?

1875. Two boats start out from A at one time; one moves northwards, the other in a northeasterly direction. Their velocities are respectively 20 km/hr and 40 km/hr. At what rate does the distance between them increase?

Sec. 6. Derivative in a Given Direction and the Gradient of a Function

1°. The derivative of a function in a given direction. The *derivative* of a function $z=f(x, y)$ *in a given direction* $l=\overrightarrow{PP_1}$ is

$$\frac{\partial z}{\partial l}=\lim_{P_1P\to 0}\frac{f(P_1)-f(P)}{P_1P},$$

where $f(P)$ and $f(P_1)$ are values of the function at the points P and P_1. If the function z is differentiable, then the following formula holds:

$$\frac{\partial z}{\partial l}=\frac{\partial z}{\partial x}\cos\alpha+\frac{\partial z}{\partial y}\sin\alpha, \tag{1}$$

where α is the angle formed by the vector l with the x-axis (Fig. 67).

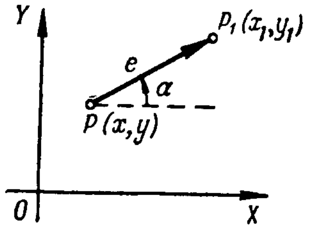

Fig. 67

In similar fashion we define the derivative in a given direction l for a function of three arguments $u=f(x, y, z)$. In this case

$$\frac{\partial u}{\partial l}=\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta+\frac{\partial u}{\partial z}\cos\gamma, \tag{2}$$

where α, β, γ are the angles between the direction l and the corresponding coordinate axes. The directional derivative characterises the rate of change of the function in the given direction.

Example 1. Find the derivative of the function $z=2x^2-3y^2$ at the point $P(1, 0)$ in a direction that makes a 120° angle with the x-axis.

Solution. Find the partial derivatives of the given function and their values at the point P:

$$\frac{\partial z}{\partial x}=4x; \quad \left(\frac{\partial z}{\partial x}\right)_P=4;$$

$$\frac{\partial z}{\partial y}=-6y; \quad \left(\frac{\partial z}{\partial y}\right)_P=0.$$

Here,

$$\cos\alpha=\cos 120^\circ=-\frac{1}{2},$$

$$\sin\alpha=\sin 120^\circ=\frac{\sqrt{3}}{2}.$$

Applying formula (1), we get

$$\frac{\partial z}{\partial l}=4\left(-\frac{1}{2}\right)+0\cdot\frac{\sqrt{3}}{2}=-2.$$

The minus sign indicates that the function diminishes at the given point and in the given direction.

2°. **The gradient of a function.** The *gradient* of a function $z=f(x, y)$ is a vector whose projections on the coordinate axes are the corresponding partial derivatives of the given function:

$$\operatorname{grad} z=\frac{\partial z}{\partial x}\boldsymbol{i}+\frac{\partial z}{\partial y}\boldsymbol{j}. \tag{3}$$

The derivative of the given function in the direction $\boldsymbol{l}$ is connected with the gradient of the function by the following formula:

$$\frac{\partial z}{\partial l}=\operatorname{pro}\boldsymbol{j}_{\boldsymbol{l}}\operatorname{grad} z.$$

That is, the derivative in a given direction is equal to the projection of the gradient of the function on the direction of differentiation.

The gradient of a function at each point is directed along the normal to the corresponding level line of the function. The direction of the gradient of the function at a given point is the direction of the maximum rate of increase of the function at this point, that is, when $\boldsymbol{l}=\operatorname{grad} z$ the derivative $\frac{\partial z}{\partial l}$ takes on its greatest value, equal to

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}.$$

In similar fashion we define the gradient of a function of three variables, $u=f(x, y, z)$:

$$\operatorname{grad} u=\frac{\partial u}{\partial x}\boldsymbol{i}+\frac{\partial u}{\partial y}\boldsymbol{j}+\frac{\partial u}{\partial z}\boldsymbol{k}. \tag{4}$$

The gradient of a function of three variables at each point is directed along the normal to the level surface passing through this point.

Example 2. Find and construct the gradient of the function $z=x^2y$ at the point $P(1, 1)$.

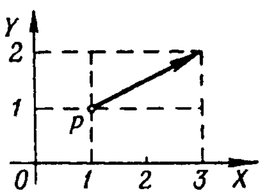

Fig. 68

Solution. Compute the partial derivatives and their values at the point P.

$$\frac{\partial z}{\partial x}=2xy; \quad \left(\frac{\partial z}{\partial x}\right)_P=2;$$

$$\frac{\partial z}{\partial y}=x^2; \quad \left(\frac{\partial z}{\partial y}\right)_P=1.$$

Hence, $\operatorname{grad} z=2\boldsymbol{i}+\boldsymbol{j}$ (Fig. 68).

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1876. Find the derivative of the function $z=x^2-xy-2y^2$ at the point $P(1, 2)$ in the direction that produces an angle of 60° with the x-axis.

1877. Find the derivative of the function $z=x^3-2x^2y+xy^2+1$ at the point $M(1, 2)$ in the direction from this point to the point $N(4, 6)$.

1878. Find the derivative of the function $z=\ln\sqrt{x^2+y^2}$ at the point $P(1, 1)$ in the direction of the bisector of the first quadrantal angle.

1879. Find the derivative of the function $u=x^2-3yz+5$ at the point $M(1, 2, -1)$ in the direction that forms identical angles with all the coordinate axes.

1880. Find the derivative of the function $u=xy+yz+zx$ at the point $M(2, 1, 3)$ in the direction from this point to the point $N(5, 5, 15)$.

1881. Find the derivative of the function $u=\ln(e^x+e^y+e^z)$ at the origin in the direction which forms with the coordinate axes x, y, z the angles α, β, γ, respectively.

1882. The point at which the derivative of a function in any direction is zero is called the *stationary point* of this function. Find the stationary points of the following functions:

a) $z=x^2+xy+y^2-4x-2y$;
b) $z=x^3+y^3-3xy$;
c) $u=2y^2+z^2-xy-yz+2x$.

1883. Show that the derivative of the function $z=\frac{y^2}{x}$ taken at any point of the ellipse $2x^2+y^2=C^2$ along the normal to the ellipse is equal to zero.

1884. Find grad z at the point (2, 1) if

$$z=x^3+y^3-3xy.$$

1885. Find grad z at the point (5, 3) if

$$z=\sqrt{x^2-y^2}.$$

1886. Find grad u at the point (1, 2, 3), if $u=xyz$.

1887. Find the magnitude and direction of grad u at the point (2, −2, 1) if

$$u=x^2+y^2+z^2.$$

1888. Find the angle between the gradients of the function $z=\ln\frac{y}{x}$ at the points $A(1/2, 1/4)$ and $B(1, 1)$.

1889. Find the steepest slope of the surface

$$z = x^2 + 4y^2$$

at the point (2, 1, 8).

1890. Construct a vector field of the gradient of the following functions:

a) $z = x + y$; c) $z = x^2 + y^2$;

b) $z = xy$; d) $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$.

Sec. 7. Higher-Order Derivatives and Differentials

1°. Higher-order partial derivatives. The *second partial derivatives* of a function $z = f(x, y)$ are the partial derivatives of its first partial derivatives.

For second derivatives we use the notations

$$\frac{\partial}{\partial x}\left(\frac{z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f''_{xx}(x, y);$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x \partial y} = f''_{xy}(x, y) \text{ and so forth.}$$

Derivatives of order higher than second are similarly defined and denoted.

If the partial derivatives to be evaluated are continuous, then the *result of repeated differentiation is independent of the order in which the differentiation is performed.*

Example 1. Find the second partial derivatives of the function

$$z = \arctan \frac{x}{y}.$$

Solution. First find the first partial derivatives:

$$\frac{\partial z}{\partial x} = \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y} = \frac{y}{x^2+y^2},$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right) = -\frac{x}{x^2+y^2}.$$

Now differentiate a second time:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = -\frac{2xy}{(x^2+y^2)^2},$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{x}{x^2+y^2}\right) = \frac{2xy}{(x^2+y^2)^2}.$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{1\cdot(x^2+y^2)-2y\cdot y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

We note that the so-called "mixed" partial derivative may be found in a different way, namely:

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial x}\left(-\frac{x}{x^2+y^2}\right) = -\frac{1\cdot(x^2+y^2)-2x\cdot x}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

2°. **Higher-order differentials.** The *second differential* of a function $z=f(x, y)$ is the differential of the differential (first-order) of this function:

$$d^2z=d(dz)$$

We similarly define the differentials of a function z of order higher than two, for instance:

$$d^3z=d(d^2z)$$

and, generally,

$$d^nz=d(d^{n-1}z).$$

If $z=f(x, y)$, where x and y are independent variables, then the second differential of the function z is computed from the formula

$$d^2z=\frac{\partial^2 z}{\partial x^2}dx^2+2\frac{\partial^2 z}{\partial x\,\partial y}dx\,dy+\frac{\partial^2 z}{\partial y^2}dy^2. \qquad (1)$$

Generally, the following symbolic formula holds true:

$$d^nz=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}\right)^n z;$$

it is formally expanded by the binomial law.

If $z=f(x, y)$, where the arguments x and y are functions of one or several independent variables, then

$$d^2z=\frac{\partial^2 z}{\partial x^2}dx^2+2\frac{\partial^2 z}{\partial x\,\partial y}dx\,dy+\frac{\partial^2 z}{\partial y^2}dy^2+\frac{\partial z}{\partial x}d^2x+\frac{\partial z}{\partial y}d^2y. \qquad (2)$$

If x and y are independent variables, then $d^2x=0$, $d^2y=0$, and formula (2) becomes identical with formula (1).

Example 2. Find the total differentials of the first and second orders of the function

$$z=2x^2-3xy-y^2.$$

Solution. First method. We have

$$\frac{\partial z}{\partial x}=4x-3y,\quad \frac{\partial z}{\partial y}=-3x-2y.$$

Therefore,

$$\partial z=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=(4x-3y)\,dx-(3x+2y)\,dy.$$

Further we have

$$\frac{\partial^2 z}{\partial x^2}=4,\quad \frac{\partial^2 z}{\partial x\,\partial y}=-3,\quad \frac{\partial^2 z}{\partial y^2}=-2,$$

whence it follows that

$$d^2z=\frac{\partial^2 z}{\partial x^2}dx^2+2\frac{\partial^2 z}{\partial x\,\partial y}dx\,dy+\frac{\partial^2 z}{\partial y^2}dy^2=4dx^2-6\,dx\,dy-2\,dy^2.$$

Second method. Differentiating we find

$$dz=4x\,dx-3\,(y\,dx+x\,dy)-2y\,dy=(4x-3y)\,dx-(3x+2y)\,dy.$$

Differentiating again and remembering that dx and dy are not dependent on x and y, we get

$$d^2z=(4dx-3dy)\,dx-(3dx+2dy)\,dy=4dx^2-6dx\,dy-2dy^2.$$

1891. Find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial^2 z}{\partial y^2}$ if

$$z = c\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}.$$

1892. Find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial^2 z}{\partial y^2}$ if

$$z = \ln(x^2 + y).$$

1893. Find $\frac{\partial^2 z}{\partial x \partial y}$ if

$$z = \sqrt{2xy + y^2}.$$

1894. Find $\frac{\partial^2 z}{\partial x \partial y}$ if

$$z = \arctan \frac{x+y}{1-xy}.$$

1895. Find $\frac{\partial^2 r}{\partial x^2}$ if

$$r = \sqrt{x^2 + y^2 + z^2}.$$

1896. Find all second partial derivatives of the function

$$u = xy + yz + zx.$$

1897. Find $\frac{\partial^3 u}{\partial x \partial y \partial z}$ if

$$u = x^\alpha y^\beta z^\gamma.$$

1898. Find $\frac{\partial^3 z}{\partial x \partial y^2}$ if

$$z = \sin(xy).$$

1899. Find $f''_{xx}(0, 0)$, $f''_{xy}(0, 0)$, $f''_{yy}(0, 0)$ if

$$f(x, y) = (1 + x)^m (1 + y)^n.$$

1900. Show that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ if

$$z = \arcsin \sqrt{\frac{x-y}{x}}.$$

1901. Show that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ if

$$z = x^y.$$

1902*. Show that for the function

$$f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2}$$

[provided that $f(0, 0)=0$] we have

$$f''_{xy}(0, 0)=-1, \quad f''_{yx}(0, 0)=+1.$$

1903. Find $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ if

$$z=f(u, v),$$

where $u=x^2+y^2$, $v=xy$.

1904. Find $\frac{\partial^2 u}{\partial x^2}$ if $u=f(x, y, z)$, where $z=\varphi(x, y)$.

1905. Find $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ if

$$z=f(u, v), \text{ where } u=\varphi(x, y), \; v=\psi(x, y).$$

1906. Show that the function

$$u=\arctan\frac{y}{x}$$

satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

1907. Show that the function

$$u=\ln\frac{1}{r},$$

where $r=\sqrt{(x-a)^2+(y-b)^2}$, satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

1908. Show that the function

$$u(x, t)=A\sin(a\lambda t+\varphi)\sin\lambda x$$

satisfies the equation of oscillations of a string

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}.$$

1909. Show that the function

$$u(x, y, z, t)=\frac{1}{(2a\sqrt{\pi t})^3}e^{-\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4a^2 t}}$$

(where x_0, y_0, z_0, a are constants) satisfies the equation of heat conduction

$$\frac{\partial u}{\partial t}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right).$$

1910. Show that the function

$$u=\varphi(x-at)+\psi(x+at),$$

where φ and ψ are arbitrary twice differentiable functions, satisfies the equation of oscillations of a string

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}.$$

1911. Show that the function

$$z=x\varphi\left(\frac{y}{x}\right)+\psi\left(\frac{y}{x}\right)$$

satisfies the equation

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0.$$

1912. Show that the function

$$u=\varphi(xy)+\sqrt{xy}\,\psi\left(\frac{y}{x}\right)$$

satisfies the equation

$$x^2\frac{\partial^2 u}{\partial x^2}-y^2\frac{\partial^2 u}{\partial y^2}=0.$$

1913. Show that the function $z=f[x+\varphi(y)]$ satisfies the equation

$$\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x^2}.$$

1914. Find $u=u(x, y)$ if

$$\frac{\partial^2 u}{\partial x\,\partial y}=0.$$

1915. Determine the form of the function $u=u(x, y)$, which satisfies the equation

$$\frac{\partial^2 u}{\partial x^2}=0.$$

1916. Find d^2z if

$$z=e^{xy}.$$

1917. Find d^2u if

$$u=xyz.$$

1918. Find d^2z if

$$z=\varphi(t),\ \text{where}\ t=x^2+y^2.$$

1919. Find dz and d^2z if

$$z=u^v\ \text{where}\ u=\frac{x}{y},\ v=xy.$$

1920. Find d^2z if

$$z=f(u,\ v),\ \text{where}\ u=ax,\ v=by.$$

1921. Find d^2z if

$$z=f(u,\ v),\ \text{where}\ u=xe^y,\ v=ye^x.$$

1922. Find d^3z if

$$z=e^x\cos y.$$

1923. Find the third differential of the function

$$z=x\cos y+y\sin x.$$

Determine all third partial derivatives.

1924. Find $df(1,\ 2)$ and $d^2f(1,\ 2)$ if

$$f(x,\ y)=x^2+xy+y^2-4\ln x-10\ln y.$$

1925. Find $d^2f(0,\ 0,\ 0)$ if

$$f(x,\ y,\ z)=x^2+2y^2+3z^2-2xy+4xz+2yz.$$

Sec. 8. Integration of Total Differentials

1°. The condition for a total differential. For an expression $P(x,\ y)\,dx+Q(x,\ y)\,dy$, where the functions $P(x,\ y)$ and $Q(x,\ y)$ are continuous in a simply connected region D together with their first partial derivatives, to be (in D) the total differential of some function $u(x,\ y)$, it is necessary and sufficient that

$$\frac{\partial Q}{\partial x}\equiv\frac{\partial P}{\partial y}.$$

Example 1. Make sure that the expression

$$(2x+y)\,dx+(x+2y)\,dy$$

is a total differential of some function, and find that function.

Solution. In the given case, $P=2x+y$, $Q=x+2y$. Therefore, $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}=1$, and, hence,

$$(2x+y)\,dx+(x+2y)\,dy=du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy,$$

where u is the desired function.

It is given that $\frac{\partial u}{\partial x}=2x+y$; therefore,

$$u=\int(2x+y)\,dx=x^2+xy+\varphi(y).$$

But on the other hand $\frac{\partial u}{\partial y}=x+\varphi'(y)=x+2y$, whence $\varphi'(y)=2y$, $\varphi(y)=y^2+C$ and

$$u=x^2+xy+y^2+C.$$

Finally we have

$$(2x+y)\,dx+(x+2y)\,dy=d\,(x^2+xy+y^2+C).$$

2°. The case of three variables. Similarly, the expression

$$P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz,$$

where $P(x, y, z)$, $Q(x, y, z)$, $R(x, y, z)$ are, together with their first partial derivatives, continuous functions of the variables x, y and z, is the total differential of some function $u(x, y, z)$ if and only if the following conditions are fulfilled:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}.$$

Example 2. Be sure that the expression

$$(3x^2 + 3y - 1)\,dx + (z^2 + 3x)\,dy + (2yz + 1)\,dz$$

is the total differential of some function, and find that function.

Solution. Here, $P = 3x^2 + 3y - 1$, $Q = z^2 + 3x$, $R = 2yz + 1$. We establish the fact that

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = 3, \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} = 2z, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = 0$$

and, hence,

$$(3x^2 + 3y - 1)\,dx + (z^2 + 3x)\,dy + (2yz + 1)\,dz = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz,$$

where u is the sought-for function.

We have

$$\frac{\partial u}{\partial x} = 3x^2 + 3y - 1,$$

hence,

$$u = \int (3x^2 + 3y - 1)\,dx = x^3 + 3xy - x + \varphi(y, z).$$

On the other hand,

$$\frac{\partial u}{\partial y} = 3x + \frac{\partial \varphi}{\partial y} = z^2 + 3x,$$
$$\frac{\partial u}{\partial z} = \frac{\partial \varphi}{\partial z} = 2yz + 1,$$

whence $\frac{\partial \varphi}{\partial y} = z^2$ and $\frac{\partial \varphi}{\partial z} = 2yz + 1$. The problem reduces to finding the function of two variables $\varphi(y, z)$ whose partial derivatives are known and the condition for total differential is fulfilled.

We find φ:

$$\varphi(y, z) = \int z^2\,dy = yz^2 + \psi(z),$$
$$\frac{\partial \varphi}{\partial z} = 2yz + \psi'(z) = 2yz + 1,$$
$$\psi'(z) = 1, \quad \psi(z) = z + C,$$

that is, $\varphi(y, z) = yz^2 + z + C$. And finally,

$$u = x^3 + 3xy - x + yz^2 + z + C.$$

Having convinced yourself that the expressions given below are total differentials of certain functions, find these functions.

1926. $y\,dx+x\,dy$.

1927. $(\cos x+3x^2y)\,dx+(x^3-y^2)\,dy$.

1928. $\frac{(x+2y)\,dx+y\,dy}{(x+y)^2}$.

1929. $\frac{x+2y}{x^2+y^2}\,dx-\frac{2x-y}{x^2+y^2}\,dy$.

1930. $\frac{1}{y}\,dx-\frac{x}{y^2}\,dy$.

1931. $\frac{x}{\sqrt{x^2+y^2}}\,dx+\frac{y}{\sqrt{x^2+y^2}}\,dy$.

1932. Determine the constants a and b in such a manner that the expression

$$\frac{(ax^2+2xy+y^2)\,dx-(x^2+2xy+by^2)\,dy}{(x^2+y^2)^2}$$

should be a total differential of some function z, and find that function.

Convince yourself that the expressions given below are total differentials of some functions and find these functions.

1933. $(2x+y+z)\,dx+(x+2y+z)\,dy+(x+y+2z)\,dz$.

1934. $(3x^2+2y^2+3z)\,dx+(4xy+2y-z)\,dy+(3x-y-2)\,dz$.

1935. $(2xyz-3y^2z+8xy^2+2)\,dx+$
$+(x^2z-6xyz+8x^2y+1)\,dy+(x^2y-3xy^2+3)\,dz$.

1936. $\left(\frac{1}{y}-\frac{z}{x^2}\right)dx+\left(\frac{1}{z}-\frac{x}{y^2}\right)dy+\left(\frac{1}{x}-\frac{y}{z^2}\right)dz$.

1937. $\frac{x\,dx+y\,dy+z\,dz}{\sqrt{x^2+y^2+z^2}}$.

1938*. Given the projections of a force on the coordinate axes

$$X=\frac{y}{(x+y)^2},\quad Y=\frac{\lambda x}{(x+y)^2},$$

where λ is a constant. What must the coefficient λ be for the force to have a potential?

1939. What condition must the function $f(x, y)$ satisfy for the expression

$$f(x, y)\,(dx+dy)$$

to be a total differential?

1940. Find the function u if

$$du=f(xy)\,(y\,dx+x\,dy).$$

Sec. 9. Differentiation of Implicit Functions

1°. The case of one independent variable. If the equation $f(x, y)=0$, where $f(x, y)$ is a differentiable function of the variables x and y, defines y as a function of x, then the derivative of this implicitly defined function, provided that $f'_y(x, y) \neq 0$, may be found from the formula

$$\frac{dy}{dx}=-\frac{f'_x(x, y)}{f'_y(x, y)}. \tag{1}$$

Higher-order derivatives are found by successive differentiation of formula (1)

Example 1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if

$$(x^2+y^2)^3-3(x^2+y^2)+1=0.$$

Solution. Denoting the left-hand side of this equation by $f(x, y)$, we find the partial derivatives

$$f'_x(x, y)=3(x^2+y^2)^2\cdot 2x-3\cdot 2x=6x[(x^2+y^2)^2-1],$$
$$f'_y(x, y)=3(x^2+y^2)^2\cdot 2y-3\cdot 2y=6y[(x^2+y^2)^2-1].$$

Whence, applying formula (1), we get

$$\frac{dy}{dx}=-\frac{f'_x(x, y)}{f'_y(x, y)}=-\frac{6x[(x^2+y^2)^2-1]}{6y[(x^2+y^2)^2-1]}=-\frac{x}{y}.$$

To find the second derivative, differentiate with respect to x the first derivative which we have found, taking into consideration the fact that y is a function of x:

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(-\frac{x}{y}\right)=-\frac{1\cdot y-x\frac{dy}{dx}}{y^2}=-\frac{y-x\left(-\frac{x}{y}\right)}{y^2}=-\frac{y^2+x^2}{y^3}.$$

2°. The case of several independent variables. Similarly, if the equation $F(x, y, z)=0$, where $F(x, y, z)$ is a differentiable function of the variables x, y and z, defines z as a function of the independent variables x and y and $F'_z(x, y, z) \neq 0$, then the partial derivatives of this implicitly represented function can, generally speaking, be found from the formulas

$$\frac{\partial z}{\partial x}=-\frac{F'_x(x, y, z)}{F'_z(x, y, z)}, \quad \frac{\partial z}{\partial y}=-\frac{F'_y(x, y, z)}{F'_z(x, y, z)}. \tag{2}$$

Here is another way of finding the derivatives of the function z: differentiating the equation $F(x, y, z)=0$, we find

$$\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial z}dz=0.$$

Whence it is possible to determine dz, and, therefore,

$$\frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}.$$

Example 2. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$x^2-2y^2+3z^2-yz+y=0.$$

Solution. First method. Denoting the left side of this equation by $F(x, y, z)$, we find the partial derivatives

$$F'_x(x, y, z)=2x,\ F'_y(x, y, z)=-4y-z+1,\ F'_z(x, y, z)=6z-y.$$

Applying formulas (2), we get

$$\frac{\partial z}{\partial x}=-\frac{F'_x(x, y, z)}{F'_z(x, y, z)}=-\frac{2x}{6z-y};\quad \frac{\partial z}{\partial y}=-\frac{F'_y(x, y, z)}{F'_z(x, y, z)}=-\frac{1-4y-z}{6z-y}.$$

Second method. Differentiating the given equation, we obtain

$$2x\,dx-4y\,dy+6z\,dz-y\,dz-z\,dy+dy=0.$$

Whence we determine dz, that is, the total differential of the implicit function:

$$dz=\frac{2x\,dx+(1-4y-z)\,dy}{y-6z}.$$

Comparing with the formula $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$, we see that

$$\frac{\partial z}{\partial x}=\frac{2x}{y-6z},\quad \frac{\partial z}{\partial y}=\frac{1-4y-z}{y-6z}.$$

3°. A system of implicit functions. If a system of two equations

$$\begin{cases} F(x, y, u, v)=0, \\ G(x, y, u, v)=0 \end{cases}$$

defines u and v as functions of the variables x and y and the Jacobian

$$\frac{D(F, G)}{D(u, v)}=\begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}\neq 0,$$

then the differentials of these functions (and hence their partial derivatives as well) may be found from the following set of equations

$$\begin{cases} \frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial u}du+\frac{\partial F}{\partial v}dv=0, \\ \frac{\partial G}{\partial x}dx+\frac{\partial G}{\partial y}dy+\frac{\partial G}{\partial u}du+\frac{\partial G}{\partial v}dv=0. \end{cases} \tag{3}$$

Example 3. The equations

$$u+v=x+y,\quad xu+yv=1$$

define u and v as functions of x and y; find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$.

Solution. First method. Differentiating both equations with respect to x, we obtain

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}=1,$$

$$u+x\frac{\partial u}{\partial x}+y\frac{\partial v}{\partial x}=0,$$

whence

$$\frac{\partial u}{\partial x}=-\frac{u+y}{x-y},\quad \frac{\partial v}{\partial x}=\frac{u+x}{x-y}.$$

Similarly we find

$$\frac{\partial u}{\partial y}=-\frac{v+y}{x-y},\quad \frac{\partial v}{\partial y}=\frac{v+x}{x-y}.$$

Second method. By differentiation we find two equations that connect the differentials of all four variables:

$$du+dv=dx+dy,$$
$$x\,du+u\,dx+y\,dv+v\,dy=0.$$

Solving this system for the differentials du and dv, we obtain

$$du=-\frac{(u+y)\,dx+(v+y)\,dy}{x-y},\quad dv=\frac{(u+x)\,dx+(v+x)\,dy}{x-y}.$$

Whence

$$\frac{\partial u}{\partial x}=-\frac{u+y}{x-y},\quad \frac{\partial u}{\partial y}=-\frac{v+y}{x-y},$$
$$\frac{\partial v}{\partial x}=\frac{u+x}{x-y},\quad \frac{\partial v}{\partial y}=\frac{v+x}{x-y}.$$

4°. Parametric representation of a function. If a function z of the variables x and y is represented parametrically by the equations

$$x=x(u,\ v),\quad y=y(u,\ v),\quad z=z(u,\ v)$$

and

$$\frac{D(x,\ y)}{D(u,\ v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}\neq 0,$$

then the differential of this function may be found from the following system of equations

$$\begin{cases} dx=\frac{\partial x}{\partial u}\,du+\frac{\partial x}{\partial v}\,dv,\\ dy=\frac{\partial y}{\partial u}\,du+\frac{\partial y}{\partial v}\,dv,\\ dz=\frac{\partial z}{\partial u}\,du+\frac{\partial z}{\partial v}\,dv.\end{cases}\tag{4}$$

Knowing the differential $dz=p\,dx+q\,dy$, we find the partial derivatives $\frac{\partial z}{\partial x}=p$ and $\frac{\partial z}{\partial y}=q$.

Example 4. The function z of the arguments x and y is defined by the equations

$$x=u+v, \quad y=u^2+v^2, \quad z=u^3+v^3 \quad (u\neq v).$$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution. First method. By differentiation we find three equations that connect the differentials of all five variables:

$$\begin{cases} dx=du+dv, \\ dy=2u\,du+2v\,dv, \\ dz=3u^2du+3v^2dv. \end{cases}$$

From the first two equations we determine du and dv:

$$du=\frac{2v\,dx-dy}{2(v-u)}, \quad dv=\frac{dy-2u\,dx}{2(v-u)}.$$

Substituting into the third equation the values of du and dv just found, we have:

$$dz=3u^2\frac{2v\,dx-dy}{2(v-u)}+3v^2\frac{dy-2u\,dx}{2(v-u)}=$$

$$=\frac{6uv(u-v)\,dx+3(v^2-u^2)\,dy}{2(v-u)}=-3uv\,dx+\frac{3}{2}(u+v)\,dy.$$

Whence

$$\frac{\partial z}{\partial x}=-3uv, \quad \frac{\partial z}{\partial y}=\frac{3}{2}(u+v).$$

Second method. From the third given equation we can find

$$\frac{\partial z}{\partial x}=3u^2\frac{\partial u}{\partial x}+3v^2\frac{\partial v}{\partial x}; \quad \frac{\partial z}{\partial y}=3u^2\frac{\partial u}{\partial y}+3v^2\frac{\partial v}{\partial y}. \tag{5}$$

Differentiate the first two equations first with respect to x and then with respect to y:

$$\begin{cases} 1=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}, & 0=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}, \\ 0=2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}, & 1=2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}. \end{cases}$$

From the first system we find

$$\frac{\partial u}{\partial x}=\frac{v}{v-u}, \quad \frac{\partial v}{\partial x}=\frac{u}{u-v}.$$

From the second system we find

$$\frac{\partial u}{\partial y}=\frac{1}{2(u-v)}, \quad \frac{\partial v}{\partial y}=\frac{1}{2(v-u)}.$$

Substituting the expressions $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ into formula (5), we obtain

$$\frac{\partial z}{\partial x}=3u^2\frac{v}{v-u}+3v^2\frac{u}{u-v}=-3uv,$$

$$\frac{\partial z}{\partial y}=3u^2\frac{1}{2(u-v)}+3v^2\cdot\frac{1}{2(v-u)}=\frac{3}{2}(u+v).$$

1941. Let y be a function of x defined by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

1942. y is a function defined by the equation

$$x^2+y^2+2a\,xy=0 \quad (a>1).$$

Show that $\frac{d^2y}{dx^2}=0$ and explain the result obtained.

1943. Find $\frac{dy}{dx}$ if $y=1+y^x$.

1944. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $y=x+\ln y$.

1945. Find $\left(\frac{dy}{dx}\right)_{x=1}$ and $\left(\frac{d^2y}{dx^2}\right)_{x=1}$ if

$$x^2-2xy+y^2+x+y-2=0.$$

Taking advantage of the results obtained, show approximately the portions of the given curve in the neighbourhood of the point $x=1$.

1946. The function y is defined by the equation

$$\ln\sqrt{x^2+y^2}=a\arctan\frac{y}{x}\ (a\neq 0).$$

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

1947. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if

$$1+xy-\ln(e^{xy}+e^{-xy})=0.$$

1948. The function z of the variables x and y is defined by the equation

$$x^3+2y^3+z^3-3xyz-2y+3=0.$$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

1949. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$x\cos y+y\cos z+z\cos x=1.$$

1950. The function z is defined by the equation

$$x^2+y^2-z^2-xy=0.$$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the system of values $x=-1$, $y=0$, $z=1$.

1951. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial^2 z}{\partial y^2}$ if

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

1952. $f(x, y, z)=0$. Show that $\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x}=-1$.

1953. $z=\varphi(x, y)$, where y is a function of x defined by the equation $\psi(x, y)=0$. Find $\frac{dz}{dx}$.

1954. Find dz and d^2z, if

$$x^2+y^2+z^2=a^2.$$

1955. z is a function of the variables x and y defined by the equation

$$2x^2+2y^2+z^2-8xz-z+8=0.$$

Find dz and d^2z for the values $x=2$, $y=0$, $z=1$.

1956. Find dz and d^2z, if $\ln z=x+y+z-1$. What are the first- and second-order derivatives of the function z?

1957. Let the function z be defined by the equation

$$x^2+y^2+z^2=\varphi(ax+by+cz),$$

where φ is an arbitrary differentiable function and a, b, c are constants. Show that

$$(cy-bz)\frac{\partial z}{\partial x}+(az-cx)\frac{\partial z}{\partial y}=bx-ay.$$

1958. Show that the function z defined by the equation

$$F(x-az,\ y-bz)=0,$$

where F is an arbitrary differentiable function of two arguments, satisfies the equation

$$a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=1.$$

1959. $F\left(\frac{x}{z}, \frac{y}{z}\right)=0$. Show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$.

1960. Show that the function z defined by the equation $y=x\varphi(z)+\psi(z)$ satisfies the equation

$$\frac{\partial^2 z}{\partial x^2}\left(\frac{\partial z}{\partial y}\right)^2-2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial^2 z}{\partial y^2}\left(\frac{\partial z}{\partial x}\right)^2=0.$$

1961. The functions y and z of the independent variable x are defined by a system of equations $x^2+y^2-z^2=0$, $x^2+2y^2+3z^2=4$. Find $\frac{dy}{dx}$, $\frac{dz}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^2z}{dx^2}$ for $x=1$, $y=0$, $z=1$.

1962. The functions y and z of the independent variable x are defined by the following system of equations:

$$xyz=a,\ x+y+z=b.$$

Find dy, dz, d^2y, d^2z.

1963. The functions u and v of the independent variables x and y are defined implicitly by the system of equations

$$u=x+y,\quad uv=y.$$

Calculate

$$\frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y},\ \frac{\partial^2 u}{\partial x^2},\ \frac{\partial^2 u}{\partial x\,\partial y},\ \frac{\partial^2 u}{\partial y^2},\ \frac{\partial v}{\partial x},\ \frac{\partial v}{\partial y},\ \frac{\partial^2 v}{\partial x^2},\ \frac{\partial^2 v}{\partial x\,\partial y},\ \frac{\partial^2 v}{\partial y^2}$$

for $x=0$, $y=1$.

1964. The functions u and v of the independent variables x and y are defined implicitly by the system of equations

$$u+v=x,\quad u-yv=0.$$

Find du, dv, d^2u, d^2v.

1965. The functions u and v of the variables x and y are defined implicitly by the system of equations

$$x=\varphi(u,\ v),\ \ y=\psi(u,\ v).$$

Find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$.

1966. a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, if $x=u\cos v$, $y=u\sin v$, $z=cv$.

b) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, if $x=u+v$, $y=u-v$, $z=uv$.

c) Find dz, if $x=e^{u+v}$, $y=e^{u-v}$, $z=uv$.

1967. $z=F(r,\ \varphi)$ where r and φ are functions of the variables x and y defined by the system of equations

$$x=r\cos\varphi,\ \ y=r\sin\varphi.$$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

1968. Regarding z as a function of x and y, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, if

$$x=a\cos\varphi\cos\psi,\ \ y=b\sin\varphi\cos\psi,\ \ z=c\sin\psi.$$

Sec. 10. Change of Variables

When changing variables in differential expressions, the derivatives in them should be expressed in terms of other derivatives by the rules of differentiation of a composite function.

1°. Change of variables in expressions containing ordinary derivatives.

Example 1. Transform the equation

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+\frac{a^2}{x^2}y=0$$

putting $x=\frac{1}{t}$.

Solution. Express the derivatives of y with respect to x in terms of the derivatives of y with respect to t. We have

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{dy}{dt}}{-\frac{1}{t^2}}=-t^2\frac{dy}{dt},$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}=-\left(2t\frac{dy}{dt}+t^2\frac{d^2y}{dt^2}\right)(-t^2)=2t^3\frac{dy}{dt}+t^4\frac{d^2y}{dt^2}.$$

Substituting the expressions of the derivatives just found into the given equation and replacing x by $\frac{1}{t}$, we get

$$\frac{1}{t^2}\cdot t^3\left(2\frac{dy}{dt}+t\frac{d^2y}{dt^2}\right)+2\cdot\frac{1}{t}\left(-t^2\frac{dy}{dt}\right)+a^2t^2y=0$$

or

$$\frac{d^2y}{dt^2}+a^2y=0.$$

Example 2. Transform the equation

$$x\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3-\frac{dy}{dx}=0,$$

taking y for the argument and x for the function.

Solution. Express the derivatives of y with respect to x in terms of the derivatives of x with respect to y.

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}};$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{\frac{dx}{dy}}\right)=\frac{d}{dy}\left(\frac{1}{\frac{dx}{dy}}\right)\frac{dy}{dx}=-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2}\cdot\frac{1}{\frac{dx}{dy}}=-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}.$$

Substituting these expressions of the derivatives into the given equation, we will have

$$x\left[-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}\right]+\frac{1}{\left(\frac{dx}{dy}\right)^3}-\frac{1}{\frac{dx}{dy}}=0,$$

or, finally,

$$x\frac{d^2x}{dy^2}-1+\left(\frac{dx}{dy}\right)^2=0.$$

Example 3. Transform the equation

$$\frac{dy}{dx}=\frac{x+y}{x-y},$$

by passing to the polar coordinates

$$x=r\cos\varphi,\quad y=r\sin\varphi. \tag{1}$$

Solution. Considering r as a function of φ, from formula (1) we have

$$dx=\cos\varphi\, dr-r\sin\varphi\, d\varphi,\quad dy=\sin\varphi\, dr+r\cos\varphi\, d\varphi,$$

whence

$$\frac{dy}{dx}=\frac{\sin\varphi\, dr+r\cos\varphi\, d\varphi}{\cos\varphi\, dr-r\sin\varphi\, d\varphi}=\frac{\sin\varphi\frac{dr}{d\varphi}+r\cos\varphi}{\cos\varphi\frac{dr}{d\varphi}-r\sin\varphi}.$$

Putting into the given equation the expressions for x, y, and $\frac{dy}{dx}$, we will have

$$\frac{\sin\varphi\frac{dr}{d\varphi}+r\cos\varphi}{\cos\varphi\frac{dr}{d\varphi}-r\sin\varphi}=\frac{r\cos\varphi+r\sin\varphi}{r\cos\varphi-r\sin\varphi},$$

or, after simplifications,

$$\frac{dr}{d\varphi}=r.$$

2° Change of variables in expressions containing partial derivatives.
Example 4. Take the equation of oscillations of a string

$$\frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2}\qquad (a\neq 0)$$

and change it to the new independent variables α and β, where $\alpha=x-at$, $\beta=x+at$.

Solution. Let us express the partial derivatives of u with respect to x and t in terms of the partial derivatives of u with respect to α and β. Applying the formulas for differentiating a composite function

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial\alpha}\frac{\partial\alpha}{\partial t}+\frac{\partial u}{\partial\beta}\frac{\partial\beta}{\partial t},\quad \frac{\partial u}{\partial x}=\frac{\partial u}{\partial\alpha}\frac{\partial\alpha}{\partial x}+\frac{\partial u}{\partial\beta}\frac{\partial\beta}{\partial x},$$

we get

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial\alpha}(-a)+\frac{\partial u}{\partial\beta}a=a\left(\frac{\partial u}{\partial\beta}-\frac{\partial u}{\partial\alpha}\right),$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial\alpha}\cdot 1+\frac{\partial u}{\partial\beta}\cdot 1=\frac{\partial u}{\partial\alpha}+\frac{\partial u}{\partial\beta}.$$

Differentiate again using the same formulas:

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)=\frac{\partial}{\partial \alpha}\left(\frac{\partial u}{\partial t}\right)\frac{\partial \alpha}{\partial t}+\frac{\partial}{\partial \beta}\left(\frac{\partial u}{\partial t}\right)\frac{\partial \beta}{\partial t}=$$

$$=a\left(\frac{\partial^2 u}{\partial \alpha\,\partial \beta}-\frac{\partial^2 u}{\partial \alpha^2}\right)(-a)+a\left(\frac{\partial^2 u}{\partial \beta^2}-\frac{\partial^2 u}{\partial \alpha\,\partial \beta}\right)a=$$

$$=a^2\left(\frac{\partial^2 u}{\partial \alpha^2}-2\frac{\partial^2 u}{\partial \alpha\,\partial \beta}+\frac{\partial^2 u}{\partial \beta^2}\right);$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial \alpha}\left(\frac{\partial u}{\partial x}\right)\frac{\partial \alpha}{\partial x}+\frac{\partial}{\partial \beta}\left(\frac{\partial u}{\partial x}\right)\frac{\partial \beta}{\partial x}=$$

$$=\left(\frac{\partial^2 u}{\partial \alpha^2}+\frac{\partial^2 u}{\partial \alpha\,\partial \beta}\right)\cdot 1+\left(\frac{\partial^2 u}{\partial \alpha\,\partial \beta}+\frac{\partial^2 u}{\partial \beta^2}\right)\cdot 1=$$

$$=\frac{\partial^2 u}{\partial \alpha^2}+2\frac{\partial^2 u}{\partial \alpha\,\partial \beta}+\frac{\partial^2 u}{\partial \beta^2}.$$

Substituting into the given equation, we will have

$$a^2\left(\frac{\partial^2 u}{\partial \alpha^2}-2\frac{\partial^2 u}{\partial \alpha\,\partial \beta}+\frac{\partial^2 u}{\partial \beta^2}\right)=a^2\left(\frac{\partial^2 u}{\partial \alpha^2}+2\frac{\partial^2 u}{\partial \alpha\,\partial \beta}+\frac{\partial^2 u}{\partial \beta^2}\right)$$

or

$$\frac{\partial^2 u}{\partial \alpha\,\partial \beta}=0.$$

Example 5. Transform the equation $x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=z^2$, taking $u=x$, $v=\frac{1}{y}-\frac{1}{x}$ for the new independent variables, and $w=\frac{1}{z}-\frac{1}{x}$ for the new function.

Solution. Let us express the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of the partial derivatives $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$. To do this, differentiate the given relationships between the old and new variables:

$$du=dx,\quad dv=\frac{dx}{x^2}-\frac{dy}{y^2},\quad dw=\frac{dx}{x^2}-\frac{dz}{z^2}.$$

On the other hand,

$$dw=\frac{\partial w}{\partial u}du+\frac{\partial w}{\partial v}dv.$$

Therefore,

$$\frac{\partial w}{\partial u}du+\frac{\partial w}{\partial v}dv=\frac{dx}{x^2}-\frac{dz}{z^2}$$

or

$$\frac{\partial w}{\partial u}dx+\frac{\partial w}{\partial v}\left(\frac{dx}{x^2}-\frac{dy}{y^2}\right)=\frac{dx}{x^2}-\frac{dz}{z^2}.$$

Whence

$$dz=z^2\left(\frac{1}{x^2}-\frac{\partial w}{\partial u}-\frac{1}{x^2}\frac{\partial w}{\partial v}\right)dx+\frac{z^2}{y^2}\frac{\partial w}{\partial v}dy$$

and, consequently,

$$\frac{\partial z}{\partial x}=z^2\left(\frac{1}{x^2}-\frac{\partial w}{\partial u}-\frac{1}{x^2}\frac{\partial w}{\partial v}\right)$$

and

$$\frac{\partial z}{\partial y}=\frac{z^2}{y^2}\frac{\partial w}{\partial v}.$$

Substituting these expressions into the given equation, we get

$$x^2z^2\left(\frac{1}{x^2}-\frac{\partial w}{\partial u}-\frac{1}{x^2}\frac{\partial w}{\partial v}\right)+z^2\frac{\partial w}{\partial v}=z^2$$

or

$$\frac{\partial w}{\partial u}=0.$$

1969. Transform the equation

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+y=0,$$

putting $x=e^t$.

1970. Transform the equation

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0,$$

putting $x=\cos t$.

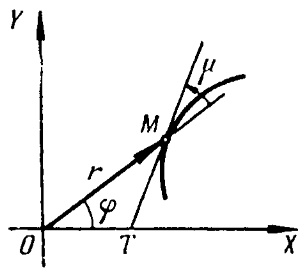

Fig 69

1971 Transform the following equations, taking y as the argument:

a) $\frac{d^2y}{dx^2}+2y\left(\frac{dy}{dx}\right)^2=0,$

b) $\frac{dy}{dx}\frac{d^3y}{dx^3}-3\left(\frac{d^2y}{dx^2}\right)^2=0.$

1972. The tangent of the angle μ formed by the tangent line MT and the radius vector OM of the point of tangency (Fig. 69) is expressed as follows:

$$\tan\mu=\frac{y'-\frac{y}{x}}{1+\frac{y}{x}y'}.$$

Transform this expression by passing to polar coordinates: $x=r\cos\varphi$, $y=r\sin\varphi$.

1973. Express, in the polar coordinates $x=r\cos\varphi$, $y=r\sin\varphi$, the formula of the curvature of the curve

$$K=\frac{y''}{[1+(y')^2]^{3/2}}.$$

1974. Transform the following equation to new independent variables u and v:

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=0,$$

if $u=x$, $v=x^2+y^2$.

1975. Transform the following equation to new independent variables u and v:

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}-z=0,$$

if $u=x$, $v=\frac{y}{x}$.

1976. Transform the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

to the polar coordinates

$$x=r\cos\varphi,\quad y=r\sin\varphi.$$

1977. Transform the equation

$$x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=0,$$

putting $u=xy$ and $v=\frac{x}{y}$.

1978. Transform the equation

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=(y-x)z,$$

by introducing new independent variables

$$u=x^2+y^2,\quad v=\frac{1}{x}+\frac{1}{y}$$

and the new function $w=\ln z-(x+y)$.

1979. Transform the equation

$$\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial^2 z}{\partial y^2}=0,$$

taking $u=x+y$, $v=\frac{y}{x}$ for the new independent variables and $w=\frac{z}{x}$ for the new function.

1980. Transform the equation

$$\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial^2 z}{\partial y^2}=0,$$

putting $u=x+y$, $v=x-y$, $w=xy-z$, where $w=w(u, v)$.

Sec. 11. The Tangent Plane and the Normal to a Surface

1°. The equations of a tangent plane and a normal for the case of explicit representation of a surface. The *tangent plane* to a surface at a point M (point of tangency) is a plane in which lie all the tangents at the point M to various curves drawn on the surface through this point.

The *normal* to the surface is the perpendicular to the tangent plane at the point of tangency

If the equation of a surface, in a rectangular coordinate system, is given in explicit form, $z=f(x, y)$, where $f(x, y)$ is a differentiable function, then the equation of the tangent plane at the point $M(x_0, y_0, z_0)$ of the surface is

$$Z-z_0=f'_x(x_0, y_0)(X-x_0)+f'_y(x_0, y_0)(Y-y_0). \qquad (1)$$

Here, $z_0=f(x_0, y_0)$ and X, Y, Z are the current coordinates of the point of the tangent plane.

The equations of the normal are of the form

$$\frac{X-x_0}{f'_x(x_0, y_0)}=\frac{Y-y_0}{f'_y(x_0, y_0)}=\frac{Z-z_0}{-1}, \qquad (2)$$

where X, Y, Z are the current coordinates of the point of the normal.

Example 1. Write the equations of the tangent plane and the normal to the surface $z=\frac{x^2}{2}-y^2$ at the point $M(2, -1, 1)$.

Solution. Let us find the partial derivatives of the given function and their values at the point M

$$\frac{\partial z}{\partial x}=x, \qquad \left(\frac{\partial z}{\partial x}\right)_M=2,$$

$$\frac{\partial z}{\partial y}=-2y, \quad \left(\frac{\partial z}{\partial y}\right)_M=2.$$

Whence, applying formulas (1) and (2), we will have $z-1=2(x-2)+2(y+1)$ or $2x+2y-z-1=0$ which is the equation of the tangent plane and $\frac{x-2}{2}=\frac{y+1}{2}=\frac{z-1}{-1}$, which is the equation of the normal.

2°. Equations of the tangent plane and the normal for the case of implicit representation of a surface. When the equation of a surface is represented implicitly,

$$F(x, y, z)=0,$$

and $F(x_0, y_0, z_0)=0$, the corresponding equations will have the form

$$F'_x(x_0, y_0, z_0)(X-x_0)+F'_y(x_0, y_0, z_0)(Y-y_0)+F'_z(x_0, y_0, z_0)(Z-z_0)=0 \quad (3)$$

which is the equation of the tangent plane, and

$$\frac{X-x_0}{F'_x(x_0, y_0, z_0)}=\frac{Y-y_0}{F'_y(x_0, y_0, z_0)}=\frac{Z-z_0}{F'_z(x_0, y_0, z_0)} \qquad (4)$$

which are the equations of the normal.

Example 2. Write the equations of the tangent plane and the normal to the surface $3xyz-z^3=a^3$ at a point for which $x=0$, $y=a$.

Solution. Find the z-coordinate of the point of tangency, putting $x=0$, $y=a$ into the equation of the surface: $-z^3=a^3$, whence $z=-a$. Thus, the point of tangency is $M(0, a, -a)$.

Denoting by $F(x, y, z)$ the left-hand side of the equation, we find the partial derivatives and their values at the point M:

$$F'_x=3yz, \qquad (F'_x)_M=-3a^2,$$
$$F'_y=3xz, \qquad (F'_y)_M=0,$$
$$F'_z=3xy-3z^2, \qquad (F'_z)_M=-3a^2.$$

Applying formulas (3) and (4), we get

$$-3a^2(x-0)+0(y-a)-3a^2(z+a)=0$$

or $x+z+a=0$, which is the equation of the tangent plane,

$$\frac{x-0}{-3a^2}=\frac{y-a}{0}=\frac{z+a}{-3a^2}$$

or $\frac{x}{1}=\frac{y-a}{0}=\frac{z+a}{1}$, which are the equations of the normal.

1981. Write the equation of the tangent plane and the equations of the normal to the following surfaces at the indicated points:

a) to the paraboloid of revolution $z=x^2+y^2$ at the point $(1, -2, 5)$;

b) to the cone $\frac{x^2}{16}+\frac{y^2}{9}-\frac{z^2}{8}=0$ at the point (4, 3, 4);

c) to the sphere $x^2+y^2+z^2=2Rz$ at the point $(R\cos\alpha, R\sin\alpha, R)$.

1982. At what point of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

does the normal to it form equal angles with the coordinate axes?

1983. Planes perpendicular to the x- and y-axes are drawn through the point $M(3, 4, 12)$ of the sphere $x^2+y^2+z^2=169$. Write the equation of the plane passing through the tangents to the obtained sections at their common point M.

1984. Show that the equation of the tangent plane to the central surface (of order two)

$$ax^2+by^2+cz^2=k$$

at the point $M(x_0, y_0, z_0)$ has the form

$$ax_0x + by_0y + cz_0z = k.$$

1985. Draw to the surface $x^2 + 2y^2 + 3z^2 = 21$ tangent planes parallel to the plane $x + 4y + 6z = 0$.

1986. Draw to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ a tangent plane which cuts off equal segments on the coordinate axes.

1987. On the surface $x^2 + y^2 - z^2 - 2x = 0$ find points at which the tangent planes are parallel to the coordinate planes.

1988. Prove that the tangent planes to the surface $xyz = m^3$ form a tetrahedron of constant volume with the planes of the coordinates.

1989. Show that the tangent planes to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ cut off, on the coordinate axes, segments whose sum is constant.

1990. Show that the cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$ and the sphere

$$x^2 + y^2 + \left(z - \frac{b^2 + c^2}{c}\right)^2 = \frac{b^2}{c^2}(b^2 + c^2)$$

are tangent at the points $(0, \pm b, c)$.

1991. The angle between the tangent planes drawn to given surfaces at a point under consideration is called the *angle between two surfaces* at the point of their intersection.

At what angle does the cylinder $x^2 + y^2 = R^2$ and the sphere $(x - R)^2 + y^2 + z^2 = R^2$ intersect at the point $M\left(\frac{R}{2}, \frac{R\sqrt{3}}{2}, 0\right)$?

1992. Surfaces are called *orthogonal* if they intersect at right angles at each point of the line of their intersection.

Show that the surfaces $x^2 + y^2 + z^2 = r^2$ (sphere), $y = x \tan\varphi$ (plane), and $z^2 = (x^2 + y^2)\tan^2\psi$ (cone), which are the coordinate surfaces of the spherical coordinates r, φ, ψ, are mutually orthogonal.

1993. Show that all the planes tangent to the conical surface $z = xf\left(\frac{y}{x}\right)$ at the point $M(x_0, y_0, z_0)$, where $x_0 \neq 0$, pass through the coordinate origin.

1994*. Find the projections of the ellipsoid

$$x^2 + y^2 + z^2 - xy - 1 = 0$$

on the coordinate planes.

1995. Prove that the normal at any point of the surface of revolution $z = f(\sqrt{x^2 + y^2})$ $(f' \neq 0)$ intersect the axis of rotation.

Sec. 12. Taylor's Formula for a Function of Several Variables

Let a function $f(x, y)$ have continuous partial derivatives of all orders up to the $(n+1)$ th inclusive in the neighbourhood of a point (a, b). Then *Taylor's formula* will hold in the neighbourhood under consideration:

$$f(x, y)=f(a, b)+\frac{1}{1!}[f'_x(a, b)(x-a)+f'_y(a, b)(y-b)]+$$
$$+\frac{1}{2!}[f''_{xx}(a, b)(x-a)^2+2f''_{xy}(a, b)(x-a)(y-b)+f''_{yy}(a, b)(y-b)^2]+\dots$$
$$\dots+\frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^n f(a, b)+R_n(x, y), \quad (1)$$

where

$$R_n(x, y)=\frac{1}{(n+1)!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^{n+1} f[a+\theta(x-a), b+\theta(y-b)]$$
$$(0<\theta<1).$$

In other notation,

$$f(x+h, y+k)=f(x, y)+\frac{1}{1!}[hf'_x(x, y)+kf'_y(x, y)]+\frac{1}{2!}[h^2f''_{xx}(x, y)+$$
$$+2hkf''_{xy}(x, y)+k^2f''_{yy}(x, y)]+\dots+\frac{1}{n!}\left[h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right]^n f(x, y)+$$
$$+\frac{1}{(n+1)!}\left[h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right]^{n+1} f(x+\theta h; y+\theta k), \quad (2)$$

or

$$\Delta f(x, y)=\frac{1}{1!}df(x, y)+\frac{1}{2!}d^2f(x, y)+\dots$$
$$\dots+\frac{1}{n!}d^nf(x, y)+\frac{1}{(n+1)!}d^{n+1}f(x+\theta h; y+\theta k) \quad (3)$$

The particular case of formula (1), when $a=b=0$, is called *Maclaurin's formula*.

Similar formulas hold for functions of three and a larger number of variables.

Example. Find the increment obtained by the function $f(x, y)=x^3-2y^3+3xy$ when passing from the values $x=1$, $y=2$ to the values $x_1=1+h$, $y_1=2+k$.

Solution. The desired increment may be found by applying formula (2). First calculate the successive partial derivatives and their values at the given point (1, 2):

$$f'_x(x, y)=3x^2+3y, \qquad f'_x(1, 2)=3\cdot 1+3\cdot 2=9,$$
$$f'_y(x, y)=-6y^2+3x, \qquad f'_y(1, 2)=-6\cdot 4+3\cdot 1=-21,$$
$$f''_{xx}(x, y)=6x, \qquad f''_{xx}(1, 2)=6\cdot 1=6,$$
$$f''_{xy}(x, y)=3, \qquad f''_{xy}(1, 2)=3,$$
$$f''_{yy}(x, y)=-12y, \qquad f''_{yy}(1, 2)=-12\cdot 2=-24,$$
$$f'''_{xxx}(x, y)=6, \qquad f'''_{xxx}(1, 2)=6,$$
$$f'''_{xxy}(x, y)=0, \qquad f'''_{xxy}(1, 2)=0,$$
$$f'''_{xyy}(x, y)=0, \qquad f'''_{xyy}(1, 2)=0,$$
$$f'''_{yyy}(x, y)=-12, \qquad f'''_{yyy}(1, 2)=-12.$$

All subsequent derivatives are identically zero. Putting these results into formula (2), we obtain:

$$\Delta f(x, y) = f(1+h, 2+k) - f(1, 2) = \frac{1}{1!}[h\cdot 9 + k(-21)] +$$

$$+\frac{1}{2!}[h^2\cdot 6 + 2hk\cdot 3 + k^2(-24)] + \frac{1}{3!}[h^3\cdot 6 + 3h^2k\cdot 0 + 3hk^2\cdot 0 + k^3(-12)] =$$

$$= 9h - 21k + 3h^2 + 3hk - 12k^2 + h^3 - 2k^3.$$

1996. Expand $f(x+h, y+k)$ in a series of positive integral powers of h and k if

$$f(x, y) = ax^2 + 2bxy + cy^2.$$

1997. Expand the function $f(x, y) = -x^2 + 2xy + 3y^2 - 6x - 2y - 4$ by Taylor's formula in the neighbourhood of the point $(-2, 1)$.

1998. Find the increment received by the function $f(x, y) = x^2y$ when passing from the values $x=1$, $y=1$ to

$$x_1 = 1+h, \quad y_1 = 1+k.$$

1999. Expand the function $f(x, y, z) = x^2 + y^2 + z^2 + 2xy - yz - 4x - 3y - z + 4$ by Taylor's formula in the neighbourhood of the point $(1, 1, 1)$.

2000. Expand $f(x+h, y+k, z+l)$ in a series of positive integral powers of h, k, and l, if

$$f(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz.$$

2001. Expand the following function in a Maclaurin's series up to terms of the third order inclusive:

$$f(x, y) = e^x \sin y.$$

2002. Expand the following function in a Maclaurin's series up to terms of order four inclusive:

$$f(x, y) = \cos x \cos y.$$

2003. Expand the following function in a Taylor's series in the neighbourhood of the point $(1, 1)$ up to terms of order two inclusive:

$$f(x, y) = y^x.$$

2004. Expand the following function in a Taylor's series in the neighbourhood of the point $(1, -1)$ up to terms of order three inclusive:

$$f(x, y) = e^{x+y}.$$

2005. Derive approximate formulas (accurate to second-order terms in α and β) for the expressions

a) $\arctan\frac{1+\alpha}{1-\beta}$; b) $\sqrt{\frac{(1+\alpha)^m+(1+\beta)^n}{2}}$,

if $|\alpha|$ and $|\beta|$ are small compared with unity.

2006*. Using Taylor's formulas up to second-order terms, approximate

a) $\sqrt{1.03}$; $\sqrt[3]{0.98}$; b) $(0.95)^{2.01}$.

2007. z is an implicit function of x and y defined by the equation $z^3-2xz+y=0$, which takes on the value $z=1$ for $x=1$ and $y=1$. Write several terms of the expansion of the function z in increasing powers of the differences $x-1$ and $y-1$.

Sec. 13. The Extremum of a Function of Several Variables

1°. Definition of an extremum of a function. We say that a function $f(x, y)$ has a *maximum* (*minimum*) $f(a, b)$ at the point $P(a, b)$, if for all points $P'(x, y)$ different from P in a sufficiently small neighbourhood of P the inequality $f(a, b) > f(x, y)$ [or, accordingly, $f(a, b) < f(x, y)$] is fulfilled. The generic term for maximum and minimum of a function is *extremum*. In similar fashion we define the extremum of a function of three or more variables.

2°. Necessary conditions for an extremum. The points at which a differentiable function $f(x, y)$ may attain an extremum (so-called *stationary points*) are found by solving the following system of equations:

$$f'_x(x, y)=0, \quad f'_y(x, y)=0 \qquad (1)$$

(*necessary conditions* for an extremum). System (1) is equivalent to a single equation, $df(x, y)=0$. In the general case, at the point of the extremum $P(a, b)$, the function $f(x, y)$, or $df(a, b)=0$, or $df(a, b)$ does not exist.

3°. Sufficient conditions for an extremum. Let $P(a, b)$ be a stationary point of the function $f(x, y)$, that is, $df(a, b)=0$. Then: a) if $d^2f(a, b)<0$ for $dx^2+dy^2>0$, then $f(a, b)$ is the *maximum* of the function $f(x, y)$; b) if $d^2f(a, b)>0$ for $dx^2+dy^2>0$, then $f(a, b)$ is the *minimum* of the function $f(x, y)$; c) if $d^2f(a, b)$ changes sign, then $f(a, b)$ is not an extremum of $f(x, y)$.

The foregoing conditions are equivalent to the following: let $f'_x(a, b)=f'_y(a, b)=0$ and $A=f''_{xx}(a, b)$, $B=f''_{xy}(a, b)$, $C=f''_{yy}(a, b)$. We form the *discriminant*

$$\Delta=AC-B^2.$$

Then: 1) if $\Delta>0$, then the function has an extremum at the point $P(a, b)$, namely a maximum, if $A<0$ (or $C<0$), and a minimum, if $A>0$ (or $C>0$); 2) if $\Delta<0$, then there is no extremum at $P(a, b)$; 3) if $\Delta=0$, then the question of an extremum of the function at $P(a, b)$ remains open (which is to say, it requires further investigation).

4°. The case of a function of many variables. For a function of three or more variables, the necessary conditions for the existence of an extremum are similar to conditions (1), while the sufficient conditions are analogous to the conditions a), b), and c) 3°.

Example 1. Test the following function for an extremum:

$$z = x^3 + 3xy^2 - 15x - 12y.$$

Solution. Find the partial derivatives and form a system of equations (1):

$$\frac{\partial z}{\partial x} = 3x^2 + 3y^2 - 15 = 0; \quad \frac{\partial z}{\partial y} = 6xy - 12 = 0$$

or

$$\begin{cases} x^2 + y^2 - 5 = 0, \\ xy - 2 = 0. \end{cases}$$

Solving the system we get four stationary points:

$$P_1(1, 2); \quad P_2(2, 1); \quad P_3(-1, -2); \quad P_4(-2, -1).$$

Let us find the second derivatives

$$\frac{\partial^2 z}{\partial x^2} = 6x, \quad \frac{\partial^2 z}{\partial x\,\partial y} = 6y, \quad \frac{\partial^2 z}{\partial y^2} = 6x$$

and form the discriminant $\Delta = AC - B^2$ for each stationary point.

1) For the point P_1: $A = \left(\frac{\partial^2 z}{\partial x^2}\right)_{P_1} = 6$, $B = \left(\frac{\partial^2 z}{\partial x\,\partial y}\right)_{P_1} = 12$, $C = \left(\frac{\partial^2 z}{\partial y^2}\right)_{P_1} = 6$, $\Delta = AC - B^2 = 36 - 144 < 0$. Thus, there is no extremum at the point P_1.

2) For the point P_2: $A = 12$, $B = 6$, $C = 12$; $\Delta = 144 - 36 > 0$, $A > 0$. At P_2 the function has a minimum. This minimum is equal to the value of the function for $x = 2$, $y = 1$:

$$z_{\text{min}} = 8 + 6 - 30 - 12 = -28.$$

3) For the point P_3: $A = -6$, $B = -12$, $C = -6$; $\Delta = 36 - 144 < 0$. There is no extremum.

4) For the point P_4: $A = -12$, $B = -6$, $C = -12$; $\Delta = 144 - 36 > 0$, $A < 0$. At the point P_4 the function has a maximum equal to $z_{\text{max}} = -8 - 6 + 30 + 12 = 28$.

5°. Conditional extremum. In the simplest case, the *conditional extremum* of a function $f(x, y)$ is a maximum or minimum of this function which is attained on the condition that its arguments are related by the equation $\varphi(x, y) = 0$ (*coupling equation*). To find the conditional extremum of a function $f(x, y)$, given the relationship $\varphi(x, y) = 0$ we form the so-called *Lagrange function*

$$F(x, y) = f(x, y) + \lambda \cdot \varphi(x, y),$$

where λ is an undetermined multiplier, and we seek the ordinary extremum of this auxiliary function. The necessary conditions for the extremum reduce to a system of three equations:

$$\begin{cases} \dfrac{\partial F}{\partial x} = \dfrac{\partial f}{\partial x} + \lambda \dfrac{\partial \varphi}{\partial x} = 0, \\ \dfrac{\partial F}{\partial y} = \dfrac{\partial f}{\partial y} + \lambda \dfrac{\partial \varphi}{\partial y} = 0, \\ \varphi(x, y) = 0 \end{cases} \qquad (2)$$

with three unknowns x, y, λ, from which it is, generally speaking, possible to determine these unknowns.

The question of the existence and character of a conditional extremum is solved on the basis of a study of the sign of the second differential of the Lagrange function:

$$d^2F(x, y)=\frac{\partial^2F}{\partial x^2}dx^2+2\frac{\partial^2F}{\partial x\,\partial y}dx\,dy+\frac{\partial^2F}{\partial y^2}dy^2$$

for the given system of values of x, y, λ obtained from (2) or the condition that dx and dy are related by the equation

$$\frac{\partial\varphi}{\partial x}dx+\frac{\partial\varphi}{\partial y}dy=0\quad(dx^2+dy^2\neq 0).$$

Namely, the function $f(x, y)$ has a conditional maximum, if $d^2F<0$, and a conditional minimum, if $d^2F>0$. As a particular case, if the discriminant Δ of the function $F(x, y)$ at a stationary point is positive, then at this point there is a conditional maximum of the function $f(x, y)$, if $A<0$ (or $C<0$), and a conditional minimum, if $A>0$ (or $C>0$)

In similar fashion we find the conditional extremum of a function of three or more variables provided there is one or several coupling equations (the number of which, however, must be less than the number of the variables) Here, we have to introduce into the Lagrange function as many undetermined multipliers factors as there are coupling equations.

Example 2. Find the extremum of the function

$$z=6-4x-3y$$

provided the variables x and y satisfy the equation

$$x^2+y^2=1$$

Solution. Geometrically, the problem reduces to finding the greatest and least values of the z-coordinate of the plane $z=6-4x-3y$ for points of its intersection with the cylinder $x^2+y^2=1$

We form the Lagrange function

$$F(x, y)=6-4x-3y+\lambda(x^2+y^2-1).$$

We have $\frac{\partial F}{\partial x}=-4+2\lambda x$, $\frac{\partial F}{\partial y}=-3+2\lambda y$. The necessary conditions yield the following system of equations:

$$\begin{cases} -4+2\lambda x=0, \\ -3+2\lambda y=0, \\ x^2+y^2=1. \end{cases}$$

Solving this system we find

$$\lambda_1=\frac{5}{2},\quad x_1=\frac{4}{5},\quad y_1=\frac{3}{5},$$

and

$$\lambda_2=-\frac{5}{2},\quad x_2=-\frac{4}{5},\quad y_2=-\frac{3}{5}.$$

Since

$$\frac{\partial^2F}{\partial x^2}=2\lambda,\quad \frac{\partial^2F}{\partial x\,\partial y}=0,\quad \frac{\partial^2F}{\partial y^2}=2\lambda,$$

it follows that

$$d^2F=2\lambda(dx^2+dy^2).$$

If $\lambda=\frac{5}{2}$, $x=\frac{4}{5}$ and $y=\frac{3}{5}$, then $d^2F>0$, and, consequently, the function has a conditional minimum at this point. If $\lambda=-\frac{5}{2}$, $x=-\frac{4}{5}$ and $y=-\frac{3}{5}$, then $d^2F<0$, and, consequently, the function at this point has a conditional maximum.

Thus,

$$z_{\max}=6+\frac{16}{5}+\frac{9}{5}=11,$$
$$z_{\min}=6-\frac{16}{5}-\frac{9}{5}=1.$$

6°. **Greatest and smallest values of a function.** A function that is differentiable in a limited closed region attains its greatest (smallest) value either at a stationary point or at a point of the boundary of the region.

Example 3. Determine the greatest and smallest values of the function

$$z=x^2+y^2-xy+x+y$$

in the region

$$x\leqslant 0,\ y\leqslant 0,\ x+y\geqslant -3$$

Solution. The indicated region is a triangle (Fig. 70).

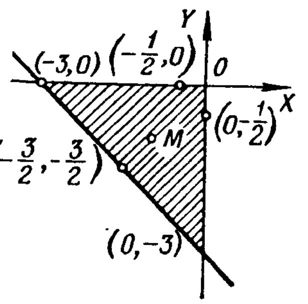

Fig. 70

1) Let us find the stationary points:

$$\begin{cases} z'_x=2x-y+1=0, \\ z'_y=2y-x+1=0; \end{cases}$$

whence $x=-1$, $y=-1$; and we get the point $M(-1, -1)$.

At M the value of the function $z_M=-1$. It is not absolutely necessary to test for an extremum.

2) Let us investigate the function on the boundaries of the region.

When $x=0$ we have $z=y^2+y$, and the problem reduces to seeking the greatest and smallest values of this function of one argument on the interval $-3\leqslant y\leqslant 0$. Investigating, we find that $(z_{gr})_{x=0}=6$ at the point $(0, -3)$; $(z_{sm})_{x=0}=-\frac{1}{4}$ at the point $(0, -1/2)$.

When $y=0$ we get $z=x^2+x$. Similarly, we find that $(z_{gr})_{y=0}=6$ at the point $(-3, 0)$; $(z_{sm})_{y=0}=-\frac{1}{4}$ at the point $(-1/2, 0)$.

When $x+y=-3$ or $y=-3-x$ we will have $z=3x^2+9x+6$. Similarly we find that $(z_{sm})_{x+y=-3}=-\frac{3}{4}$ at the point $\left(-\frac{3}{2}, -\frac{3}{2}\right)$; $(z_{gr})_{x+y=-3}=6$ metres coincides with $(z_{gr})_{x=0}$ and $(z_{gr})_{y=0}$. On the straight line $x+y=-3$ we could test the function for a conditional extremum without reducing to a function of one argument.

3) Correlating all the values obtained of the function z, we conclude that $z_{gr}=6$ at the points $(0, -3)$ and $(-3, 0)$; $z_{sm}=-1$ at the stationary point M.

8—1900

Test for maximum and minimum the following functions of two variables:

2008. $z=(x-1)^2+2y^2$.

2009. $z=(x-1)^2-2y^2$.

2010. $z=x^2+xy+y^2-2x-y$.

2011. $z=x^3y^2(6-x-y)\ (x>0,\ y>0)$.

2012. $z=x^4+y^4-2x^2+4xy-2y^2$.

2013. $z=xy\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$.

2014. $z=1-(x^2+y^2)^{2/3}$.

2015. $z=(x^2+y^2)e^{-(x^2+y^2)}$.

2016. $z=\frac{1+x-y}{\sqrt{1+x^2+y^2}}$.

Find the extrema of the following functions of three variables:

2017. $u=x^2+y^2+z^2-xy+x-2z$.

2018. $u=x+\frac{y^2}{4x}+\frac{z^2}{y}+\frac{2}{z}\ (x>0,\ y>0,\ z>0)$.

Find the extrema of the following implicitly represented functions:

2019*. $x^2+y^2+z^2-2x+4y-6z-11=0$.

2020. $x^3-y^2-3x+4y+z^2+z-8=0$.

Determine the conditional extrema of the following functions:

2021. $z=xy$ for $x+y=1$.

2022. $z=x+2y$ for $x^2+y^2=5$.

2023. $z=x^2+y^2$ for $\frac{x}{2}+\frac{y}{3}=1$.

2024. $z=\cos^2 x+\cos^2 y$ for $y-x=\frac{\pi}{4}$.

2025. $u=x-2y+2z$ for $x^2+y^2+z^2=9$.

2026. $u=x^2+y^2+z^2$ for $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\ (a>b>c>0)$.

2027. $u=xy^2z^3$ for $x+y+z=12\ (x>0, y>0, z>0)$.

2028. $u=xyz$ provided $x+y+z=5$, $xy+yz+zx=8$.

2029. Prove the inequality

$$\frac{x+y+z}{3}\geqslant\sqrt[3]{xyz},$$

if $x\geqslant 0,\ y\geqslant 0,\ z\geqslant 0$.

Hint: Seek the maximum of the function $u=xyz$ provided $x+y+z=S$.

2030. Determine the greatest value of the function $z=1+x+2y$ in the regions: a) $x\geqslant 0$, $y\geqslant 0$, $x+y\leqslant 1$; b) $x\geqslant 0$, $y\leqslant 0$, $x-y\leqslant 1$.

2031. Determine the greatest and smallest values of the functions a) $z=x^2y$ and b) $z=x^2-y^2$ in the region $x^2+y^2\leqslant 1$.

2032. Determine the greatest and smallest values of the function $z=\sin x+\sin y+\sin (x+y)$ in the region $0\leqslant x\leqslant \frac{\pi}{2}$, $0\leqslant y\leqslant \frac{\pi}{2}$.

2033. Determine the greatest and smallest values of the function $z=x^3+y^3-3xy$ in the region $0\leqslant x\leqslant 2$, $-1\leqslant y\leqslant 2$.

Sec. 14. Finding the Greatest and Smallest Values of Functions

Example 1. It is required to break up a positive number a into three nonnegative numbers so that their product should be the greatest possible.

Solution. Let the desired numbers be x, y, $a-x-y$. We seek the maximum of the function $f(x, y)=xy(a-x-y)$.

According to the problem, the function $f(x, y)$ is considered inside a closed triangle $x\geqslant 0$, $y\geqslant 0$, $x+y\leqslant a$ (Fig. 71).

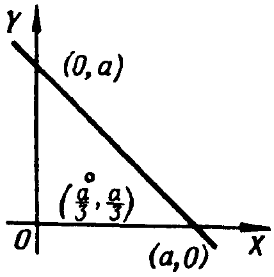

Fig. 71

Solving the system of equations

$$\begin{cases} f'_x(x, y)\equiv y(a-2x-y)=0, \\ f'_y(x, y)\equiv x(a-x-2y)=0, \end{cases}$$

we will have the unique stationary point $\left(\frac{a}{3}, \frac{a}{3}\right)$ for the interior of the triangle. Let us test the sufficiency conditions. We have

$$f''_{xx}(x, y)=-2y,\ f''_{xy}(x, y)=a-2x-2y,\ f''_{yy}(x, y)=-2x.$$

Consequently,

$$A=f''_{xx}\left(\frac{a}{3},\ \frac{a}{3}\right)=-\frac{2}{3}a,$$
$$B=f''_{xy}\left(\frac{a}{3},\ \frac{a}{3}\right)=-\frac{1}{3}a,$$
$$C=f''_{yy}\left(\frac{a}{3},\ \frac{a}{3}\right)=-\frac{2}{3}a \text{ and}$$
$$\Delta=AC-B^2>0,\ A<0.$$

And so at $\left(\frac{a}{3},\ \frac{a}{3}\right)$ the function reaches a maximum. Since $f(x,\ y)=0$ on the contour of the triangle, this maximum will be the greatest value, which is to say that the product will be greatest, if $x=y=a-x-y=\frac{a}{3}$, and the greatest value is equal to $\frac{a^3}{27}$.

Note The ploblem can also be solved by the methods of a conditional extremum, by seeking the maximum of the function $u=xyz$ on the condition that $x+y+z=a$.

2034. From among all rectangular parallelepipeds with a given volume V, find the one whose total surface is the least.

2035. For what dimensions does an open rectangular bathtub of a given capacity V have the smallest surface?

2036. Of all triangles of a given perimeter $2p$, find the one that has the greatest area.

2037. Find a rectangular parallelepiped of a given surface S with greatest volume.

2038. Represent a positive number a in the form of a product of four positive factors which have the least possible sum.

2039. Find a point $M(x,\ y)$, on an xy-plane, the sum of the squares of the distances of which from three straight lines ($x=0$, $y=0$, $x-y+1=0$) is the least possible.

2040. Find a triangle of a given perimeter $2p$, which, upon being revolved about one of its sides, generates a solid of greatest volume.

2041. Given in a plane are three material points $P_1(x_1,\ y_1)$, $P_2(x_2,\ y_2)$, $P_3(x_3,\ y_3)$ with masses m_1, m_2, m_3. For what position of the point $P(x,\ y)$ will the quadratic moment (the moment of inertia) of the given system of points relative to the point P (i.e., the sum $m_1P_1P^2+m_2P_2P^2+m_3P_3P^2$) be the least?

2042. Draw a plane through the point $M(a,\ b,\ c)$ to form a tetrahedron of least volume with the planes of the coordinates.

2043. Inscribe in an ellipsoid a rectangular parallelepiped of greatest volume.

2044. Determine the outer dimensions of an open box with a given wall thickness δ and capacity (internal) V so that the smallest quantity of material is used to make it.

2045. At what point of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

does the tangent line to it form with the coordinate axes a triangle of smallest area?

2046*. Find the axes of the ellipse

$$5x^2+8xy+5y^2=9.$$

2047. Inscribe in a given sphere a cylinder having the greatest total surface.

2048. The beds of two rivers (in a certain region) approximately represent a parabola $y=x^2$ and a straight line $x-y-2=0$. It is required to connect these rivers by a straight canal of least length. Through what points will it pass?

2049. Find the shortest distance from the point $M(1, 2, 3)$ to the straight line

$$\frac{x}{1}=\frac{y}{-3}=\frac{z}{2}.$$

2050*. The points A and B are situated in different optical media separated by a straight line (Fig. 72). The velocity of

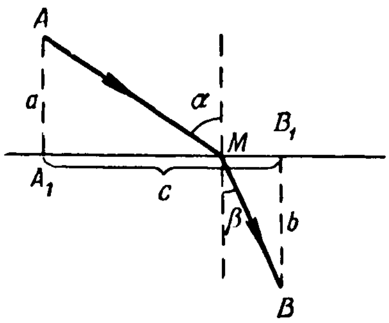

Fig. 72

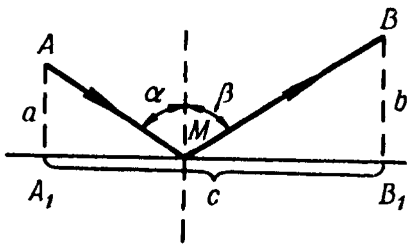

Fig. 73

light in the first medium is v_1, in the second, v_2. Applying the Fermat principle, according to which a light ray is propagated along a line AMB which requires the least time to cover, derive the law of refraction of light rays.

2051. Using the Fermat principle, derive the law of reflection of a light ray from a plane in a homogeneous medium (Fig. 73).

2052*. If a current I flows in an electric circuit containing a resistance R, then the quantity of heat released in unit time is proportional to I^2R. Determine how to divide the current I into currents I_1, I_2, I_3 by means of three wires, whose resistances are R_1, R_2, R_3, so that the generation of heat would be the least possible?

Sec. 15. Singular Points of Plane Curves

1°. Definition of a singular point. A point $M(x_0, y_0)$ of a plane curve $f(x, y)=0$ is called a *singular point* if its coordinates satisfy three equations at once:

$$f(x_0, y_0)=0, \quad f'_x(x_0, y_0)=0, \quad f'_y(x_0, y_0)=0.$$

2°. Basic types of singular points. At a singular point $M(x_0, y_0)$, let the second derivatives

$$A=f''_{xx}(x_0, y_0),$$
$$B=f''_{xy}(x_0, y_0),$$
$$C=f''_{yy}(x_0, y_0)$$

be not all equal to zero and

$$\Delta=AC-B^2,$$

then:

a) if $\Delta>0$, then M is an *isolated point* (Fig. 74);
b) if $\Delta<0$, then M is a *node (double point)* (Fig. 75);
c) if $\Delta=0$, then M is either a *cusp* of the first kind (Fig. 76) or of the second kind (Fig. 77), or an *isolated point*, or a *tacnode* (Fig. 78).

Fig. 74

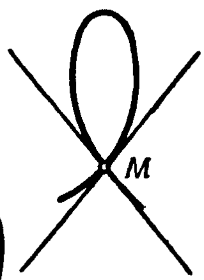

Fig. 75

When solving the problems of this section it is always necessary to draw the curves.

Example 1. Show that the curve $y^2=ax^2+x^3$ has a node if $a>0$; an isolated point if $a<0$; a cusp of the first kind if $a=0$.

Solution. Here, $f(x, y)\equiv ax^2+x^3-y^2$. Let us find the partial derivatives and equate them to zero:

$$f'_x(x, y)=2ax+3x^2=0,$$
$$f'_y(x, y)=-2y=0.$$

This system has two solutions: $O(0, 0)$ and $N\left(-\frac{2}{3}a, 0\right)$; but the coordinates of the point N do not satisfy the equation of the given curve. Hence, there is a unique singular point O (0, 0).

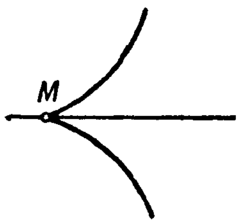

Fig. 76

Fig. 77

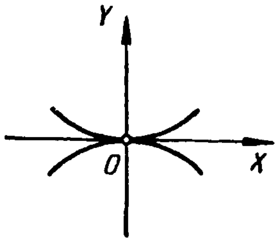

Fig. 78

Let us find the second derivatives and their values at the point O:

$$f''_{xx}(x, y) = 2a + 6x, \quad A = 2a,$$
$$f''_{xy}(x, y) = 0, \quad B = 0,$$
$$f''_{yy}(x, y) = -2, \quad C = -2,$$
$$\Delta = AC - B^2 = -4a.$$

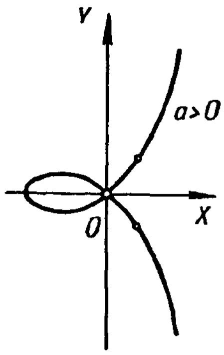

Fig. 79

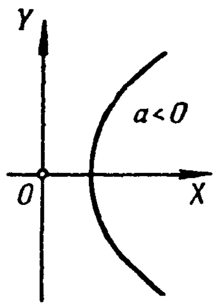

Fig. 80

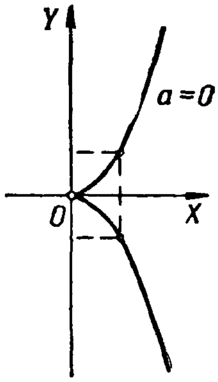

Fig. 81

Hence,

if $a > 0$, then $\Delta < 0$ and the point O is a node (Fig. 79);
if $a < 0$, then $\Delta > 0$ and O is an isolated point (Fig. 80);
if $a = 0$, then $\Delta = 0$. The equation of the curve in this case will be $y^2 = x^3$ or $y = \pm\sqrt{x^3}$; $y =$ exists only when $x \geqslant 0$; the curve is symmetric about the x-axis, which is a tangent. Hence, the point M is a cusp of the first kind (Fig. 81).

Determine the character of the singular points of the following curves:

2053. $y^2 = -x^2 + x^4$.

2054. $(y-x^2)^2 = x^5$.

2055. $a^4y^2 = a^2x^4 - x^6$.

2056. $x^2y^2 - x^2 - y^2 = 0$.

2057. $x^3 + y^3 - 3axy = 0$ *(folium of Descartes)*.

2058. $y^2(a-x) = x^3$ *(cissoid)*.

2059. $(x^2+y^2)^2 = a^2(x^2-y^2)$ *(lemniscate)*.

2060. $(a+x)y^2 = (a-x)x^2$ *(strophoid)*.

2061. $(x^2+y^2)(x-a)^2 = b^2x^2$ $(a>0,\ b>0)$ *(conchoid)*.
Consider three cases:

1) $a>b$, 2) $a=b$, 3) $a<b$.

2062. Determine the change in character of the singular point of the curve $y^2 = (x-a)(x-b)(x-c)$ depending on the values of a, b, c ($a \leqslant b \leqslant c$ are real).

Sec. 16. Envelope

1°. Definition of an envelope. The *envelope of a family of plane curves* is a curve (or a set of several curves) which is tangent to all lines of the given family, and at each point is tangent to some line of the given family.

2°. Equations of an envelope. If a family of curves

$$f(x, y, \alpha) = 0$$

dependent on a single variable parameter α has an envelope, then the parametric equations of the latter are found from the system of equations

$$\left\{\begin{array}{l} f(x, y, \alpha) = 0, \\ f'_\alpha(x, y, \alpha) = 0. \end{array}\right. \qquad (1)$$

Eliminating the parameter α from the system (1), we get an equation of the form

$$D(x, y) = 0. \qquad (2)$$

It should be pointed out that the formally obtained curve (2) (the so-called "*discriminant curve*") may contain, in addition to an envelope (if there is one), a locus of singular points of the given family, which locus is not part of the envelope of this family.

When solving the problems of this section it is advisable to make drawings.

Example. Find the envelope of the family of curves

$$x\cos\alpha + y\sin\alpha - p = 0 \ (p = \text{const},\ p > 0).$$

Solution. The given family of curves depends on the parameter α. Form the system of equations (1):

$$\begin{cases} x\cos\alpha + y\sin\alpha - p = 0, \\ -x\sin\alpha + y\cos\alpha = 0. \end{cases}$$

Solving the system for x and y, we obtain parametric equations of the envelope

$$x = p\cos\alpha, \quad y = p\sin\alpha.$$

Squaring both equations and adding, we eliminate the parameter α:

$$x^2 + y^2 = p^2.$$

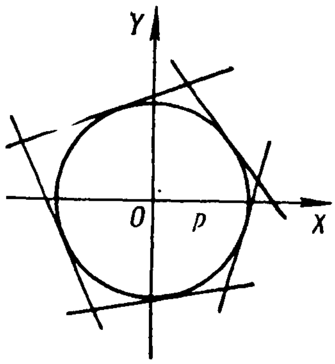

Fig. 82

Thus, the envelope of this family of straight lines is a circle of radius p with centre at the origin. This particular family of straight lines is a family of tangent lines to this circle (Fig. 82).

2063. Find the envelope of the family of circles

$$(x-a)^2 + y^2 = \frac{a^2}{2}.$$

2064. Find the envelope of the family of straight lines

$$y = kx + \frac{p}{2k}$$

(k is a variable parameter).

2065. Find the envelope of a family of circles of the same radius R whose centres lie on the x-axis.

2066. Find a curve which forms an envelope of a section of length l when its end-points slide along the coordinate axes.

2067. Find the envelope of a family of straight lines that form with the coordinate axes a triangle of constant area S.

2068. Find the envelope of ellipses of constant area S whose axes of symmetry coincide.

2069. Investigate the character of the "discriminant curves" of families of the following lines (C is a constant parameter):

a) cubic parabolas $y=(x-C)^3$;
b) semicubical parabolas $y^2=(x-C)^3$;
c) Neile parabolas $y^3=(x-C)^2$;
d) strophoids $(a+x)(y-C)^2=x^2(a-x)$.

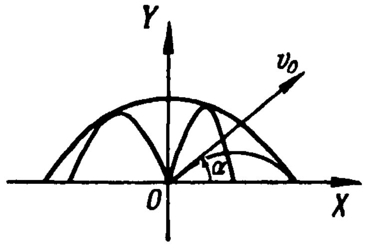

Fig. 83

2070. The equation of the trajectory of a shell fired from a point O with initial velocity v_0 at an angle α to the horizon (air resistance disregarded) is

$$y=x\tan\alpha-\frac{gx^2}{2v_0^2\cos^2\alpha}.$$

Taking the angle α as the parameter, find the envelope of all trajectories of the shell located in one and the same vertical plane ("*safety parabola*") (Fig. 83).

Sec. 17. Arc Length of a Space Curve

The *differential of an arc* of a space curve in rectangular Cartesian coordinates is equal to

$$ds=\sqrt{dx^2+dy^2+dz^2},$$

where x, y, z are the current coordinates of a point of the curve.

If

$$x=x(t), \quad y=y(t), \quad z=z(t)$$

are parametric equations of the space curve, then the arc length of a section of it from $t=t_1$ to $t=t_2$ is

$$s=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt.$$

In Problems 2071-2076 find the arc length of the curve:

2071. $x=t,\ y=t^2,\ z=\frac{2t^3}{3}$ from $t=0$ to $t=2$.

2072. $x=2\cos t,\ y=2\sin t,\ z=\frac{3}{\pi}t$ from $t=0$ to $t=\pi$.

2073. $x=e^t\cos t,\ y=e^t\sin t,\ z=e^t$ from $t=0$ to arbitrary t.

2074. $y=\frac{x^2}{2},\ z=\frac{x^3}{6}$ from $x=0$ to $x=6$.

2075. $x^2=3y,\ 2xy=9z$ from the point $O(0, 0, 0)$ to $M(3, 3, 2)$.

2076. $y=a\arcsin\frac{x}{a},\ z=\frac{a}{4}\ln\frac{a+x}{a-x}$ from the point $O(0, 0, 0)$ to the point $M(x_0, y_0, z_0)$.

2077. The position of a point for any time $t\ (t>0)$ is defined by the equations

$$x=2t,\quad y=\ln t,\quad z=t^2.$$

Find the mean velocity of motion between times $t=1$ and $t=10$.

Sec. 18. The Vector Function of a Scalar Argument

1°. The derivative of the vector function of a scalar argument. The *vector function* $\boldsymbol{a}=\boldsymbol{a}(t)$ may be defined by specifying three scalar functions $a_x(t)$, $a_y(t)$ and $a_z(t)$, which are its projections on the coordinate axes:

$$\boldsymbol{a}=a_x(t)\,\boldsymbol{i}+a_y(t)\,\boldsymbol{j}+a_z(t)\,\boldsymbol{k}.$$

The derivative of the vector function $\boldsymbol{a}=\boldsymbol{a}(t)$ with respect to the scalar argument t is a new vector function defined by the equality

$$\frac{d\boldsymbol{a}}{dt}=\lim_{\Delta t\to 0}\frac{\boldsymbol{a}(t+\Delta t)-\boldsymbol{a}(t)}{\Delta t}=\frac{da_x(t)}{dt}\boldsymbol{i}+\frac{da_y(t)}{dt}\boldsymbol{j}+\frac{da_z(t)}{dt}\boldsymbol{k}.$$

The modulus of the derivative of the vector function is

$$\left|\frac{d\boldsymbol{a}}{dt}\right|=\sqrt{\left(\frac{da_x}{dt}\right)^2+\left(\frac{da_y}{dt}\right)^2+\left(\frac{da_z}{dt}\right)^2}.$$

The end-point of the variable of the radius vector $\boldsymbol{r}=\boldsymbol{r}(t)$ describes in space the curve

$$\boldsymbol{r}=x(t)\,\boldsymbol{i}+y(t)\,\boldsymbol{j}+z(t)\,\boldsymbol{k},$$

which is called the ***hodograph*** of the vector $\boldsymbol{r}$.

The derivative $\frac{d\boldsymbol{r}}{dt}$ is a vector, tangent to the hodograph at the corresponding point; here,

$$\left|\frac{d\boldsymbol{r}}{dt}\right|=\frac{ds}{dt},$$

where s is the arc length of the hodograph reckoned from some initial point. For example, $\left|\frac{d\boldsymbol{r}}{ds}\right|=1$.

If the parameter t is the time, then $\frac{d\boldsymbol{r}}{dt}=\boldsymbol{v}$ is the *velocity vector* of the extremity of the vector $\boldsymbol{r}$, and $\frac{d^2\boldsymbol{r}}{dt^2}=\frac{d\boldsymbol{v}}{dt}=\boldsymbol{w}$ is the *acceleration vector* of the extremity of the vector $\boldsymbol{r}$.

2°. Basic rules for differentiating the vector function of a scalar argument.

1) $\frac{d}{dt}(\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c})=\frac{d\boldsymbol{a}}{dt}+\frac{d\boldsymbol{b}}{dt}-\frac{d\boldsymbol{c}}{dt}$;

2) $\frac{d}{dt}(m\boldsymbol{a})=m\frac{d\boldsymbol{a}}{dt}$, where m is a constant scalar;

3) $\frac{d}{dt}(\varphi\boldsymbol{a})=\frac{d\varphi}{dt}\boldsymbol{a}+\varphi\frac{d\boldsymbol{a}}{dt}$, where $\varphi(t)$ is a scalar function of t;

4) $\frac{d}{dt}(\boldsymbol{a}\boldsymbol{b})=\frac{d\boldsymbol{a}}{dt}\boldsymbol{b}+\boldsymbol{a}\frac{d\boldsymbol{b}}{dt}$;

5) $\frac{d}{dt}(\boldsymbol{a}\times\boldsymbol{b})=\frac{d\boldsymbol{a}}{dt}\times\boldsymbol{b}+\boldsymbol{a}\times\frac{d\boldsymbol{b}}{dt}$;

6) $\frac{d}{dt}\boldsymbol{a}[\varphi(t)]=\frac{d\boldsymbol{a}}{d\varphi}\cdot\frac{d\varphi}{dt}$;

7) $\boldsymbol{a}\frac{d\boldsymbol{a}}{dt}=0$, if $|\boldsymbol{a}|=\text{const}$.

Example 1. The radius vector of a moving point is at any instant of time defined by the equation

$$\boldsymbol{r}=\boldsymbol{i}-4t^2\boldsymbol{j}+3t^2\boldsymbol{k}. \qquad (1)$$

Determine the trajectory of motion, the velocity and acceleration.

Solution. From (1) we have:

$$x=1, \quad y=-4t^2, \quad z=3t^2.$$

Eliminating the time t, we find that the trajectory of motion is a straight line:

$$\frac{x-1}{0}=\frac{y}{-4}=\frac{z}{3}.$$

From equation (1), differentiating, we find the velocity

$$\frac{d\boldsymbol{r}}{dt}=-8t\boldsymbol{j}+6t\boldsymbol{k}$$

and the acceleration

$$\frac{d^2\boldsymbol{r}}{dt^2}=-8\boldsymbol{j}+6\boldsymbol{k}.$$

The magnitude of the velocity is

$$\left|\frac{d\boldsymbol{r}}{dt}\right|=\sqrt{(-8t)^2+(6t)^2}=10\,|t|.$$

We note that the acceleration is constant and is

$$\left|\frac{d^2\boldsymbol{r}}{dt^2}\right|=\sqrt{(-8)^2+6^2}=10.$$

2078. Show that the vector equation $\boldsymbol{r}-\boldsymbol{r}_1=(\boldsymbol{r}_2-\boldsymbol{r}_1)t$, where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are radius vectors of two given points, is the equation of a straight line.

2079. Determine which lines are hodographs of the following vector functions:

a) $\boldsymbol{r}=\boldsymbol{a}t+\boldsymbol{c}$; c) $\boldsymbol{r}=\boldsymbol{a}\cos t+\boldsymbol{b}\sin t$;

b) $\boldsymbol{r}=\boldsymbol{a}t^2+\boldsymbol{b}t$; d) $\boldsymbol{r}=\boldsymbol{a}\cosh t+\boldsymbol{b}\sinh t$,

where $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are constant vectors; the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular to each other.

2080. Find the derivative vector-function of the function $\boldsymbol{a}(t)=a(t)\boldsymbol{a}^\circ(t)$, where $a(t)$ is a scalar function, while $\boldsymbol{a}^\circ(t)$ is a unit vector, for cases when the vector $\boldsymbol{a}(t)$ varies: 1) in length only, 2) in direction only, 3) in length and in direction (general case). Interpret geometrically the results obtained.

2081. Using the rules of differentiating a vector function with respect to a scalar argument, derive a formula for differentiating a mixed product of three vector functions $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

2082. Find the derivative, with respect to the parameter t, of the volume of a parallelepiped constructed on three vectors:

$$\boldsymbol{a}=\boldsymbol{i}+t\boldsymbol{j}+t^2\boldsymbol{k};$$
$$\boldsymbol{b}=2t\boldsymbol{i}-\boldsymbol{j}+t^3\boldsymbol{k};$$
$$\boldsymbol{c}=-t^2\boldsymbol{i}+t^3\boldsymbol{j}+\boldsymbol{k}.$$

2083. The equation of motion is

$$\boldsymbol{r}=3\boldsymbol{i}\cos t+4\boldsymbol{j}\sin t,$$

where t is the time. Determine the trajectory of motion, the velocity and the acceleration. Construct the trajectory of motion and the vectors of velocity and acceleration for times, $t=0$, $t=\frac{\pi}{4}$ and $t=\frac{\pi}{2}$.

2084. The equation of motion is

$$\boldsymbol{r}=2\boldsymbol{i}\cos t+2\boldsymbol{j}\sin t+3\boldsymbol{k}t.$$

Determine the trajectory of motion, the velocity and the acceleration. What are the magnitudes of velocity and acceleration and what directions have they for time $t=0$ and $t=\frac{\pi}{2}$?

2085. The equation of motion is

$$\boldsymbol{r}=\boldsymbol{i}\cos\alpha\cos\omega t+\boldsymbol{j}\sin\alpha\cos\omega t+\boldsymbol{k}\sin\omega t,$$

where α and ω are constants and t is the time. Determine the trajectory of motion and the magnitudes and directions of the velocity and the acceleration.

2086. The equation of motion of a shell (neglecting air resistance) is

$$\boldsymbol{r}=\boldsymbol{v}_0 t-\frac{gt^2}{2}\boldsymbol{k},$$

where $\boldsymbol{v}_0\{v_{ox}, v_{oy}, v_{oz}\}$ is the initial velocity. Find the velocity and the acceleration at any instant of time.

2087. Prove that if a point is in motion along the parabola $y=\frac{x^2}{a}$, $z=0$ in such a manner that the projection of velocity on the x-axis remains constant $\left(\frac{dx}{dt}=\text{const}\right)$, then the acceleration remains constant as well.

2088. A point lying on the thread of a screw being screwed into a beam describes the spiral

$$x=a\cos\theta, \quad y=a\sin\theta, \quad z=h\theta,$$

where θ is the turning angle of the screw, a is the radius of the screw, and h is the height of rise in a rotation of one radian. Determine the velocity of the point.

2089. Find the velocity of a point on the circumference of a wheel of radius a rotating with constant angular velocity ω so that its centre moves in a straight line with constant velocity v_0.

Sec. 19. The Natural Trihedron of a Space Curve

At any nonsingular point $M(x, y, z)$ of a space curve $\boldsymbol{r}=\boldsymbol{r}(t)$ it is possible to construct a *natural trihedron* consisting of three mutually perpendicular planes (Fig. 84):

1) *osculating* plane MM_1M_2, containing the vectors $\frac{d\boldsymbol{r}}{dt}$ and $\frac{d^2\boldsymbol{r}}{dt^2}$;

2) *normal* plane MM_2M_3, which is perpendicular to the vector $\frac{d\boldsymbol{r}}{dt}$ and

3) *rectifying* plane MM_1M_3, which is perpendicular to the first two planes.

At the intersection we obtain three straight lines;

1) the *tangent* MM_1; 2) the *principal normal* MM_2; 3) the *binormal* MM_3, all of which are defined by the appropriate vectors:

1) $\boldsymbol{T}=\frac{d\boldsymbol{r}}{dt}$ (the *vector of the tangent line*);

2) $\boldsymbol{B}=\frac{d\boldsymbol{r}}{dt}\times\frac{d^2\boldsymbol{r}}{dt^2}$ (the *vector of the binormal*);

3) $\boldsymbol{N}=\boldsymbol{B}\times\boldsymbol{T}$ (the *vector of the principal normal*);

The corresponding unit vectors

$$\boldsymbol{\tau}=\frac{\boldsymbol{T}}{|\boldsymbol{T}|}; \quad \boldsymbol{\beta}=\frac{\boldsymbol{B}}{|\boldsymbol{B}|}; \quad \boldsymbol{\nu}=\frac{\boldsymbol{N}}{|\boldsymbol{N}|}$$

may be computed from the formulas

$$\boldsymbol{\tau}=\frac{d\boldsymbol{r}}{ds}; \quad \boldsymbol{\nu}=\frac{\frac{d\boldsymbol{\tau}}{ds}}{\left|\frac{d\boldsymbol{\tau}}{ds}\right|}; \quad \boldsymbol{\beta}=\boldsymbol{\tau}\times\boldsymbol{\nu}.$$

If X, Y, Z are the current coordinates of the point of the tangent, then the equations of the tangent have the form

$$\frac{X-x}{T_x}=\frac{Y-y}{T_y}=\frac{Z-z}{T_z}, \tag{1}$$

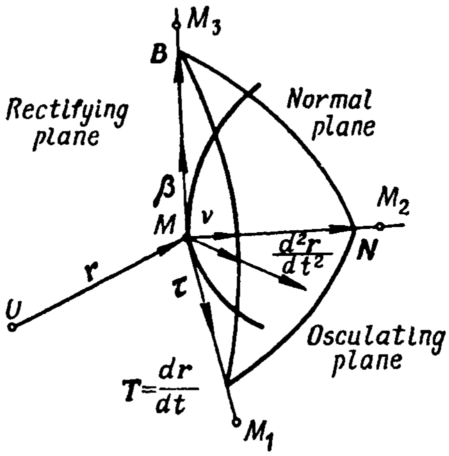

Fig. 84

where $T_x=\frac{dx}{dt}$; $T_y=\frac{dy}{dt}$, $T_z=\frac{dz}{dt}$; from the condition of perpendicularity of the line and the plane we get an equation of the normal plane:

$$T_x(X-x)+T_y(Y-y)+T_z(Z-z)=0. \tag{2}$$

If in equations (1) and (2), we replace T_x, T_y, T_z by B_x, B_y, B_z and N_x, N_y, N_z, we get the equations of the binormal and the principal normal and, respectively, the osculating plane and the rectifying plane.

Example 1. Find the basic unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$ of the curve

$$x=t, \quad y=t^2, \quad z=t^3$$

at the point $t=1$.

Write the equations of the tangent, the principal normal and the binormal at this point.

Solution. We have

$$\boldsymbol{r}=t\boldsymbol{i}+t^2\boldsymbol{j}+t^3\boldsymbol{k}$$

and

$$\frac{d\boldsymbol{r}}{dt}=\boldsymbol{i}+2t\boldsymbol{j}+3t^2\boldsymbol{k},$$

$$\frac{d^2\boldsymbol{r}}{dt^2}=2\boldsymbol{j}+6t\boldsymbol{k}.$$

Whence, when $t=1$, we get

$$T=\frac{d\mathbf{r}}{dt}=\mathbf{i}+2\mathbf{j}+3\mathbf{k};$$

$$B=\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 3\\ 0 & 2 & 6\end{vmatrix}=6\mathbf{i}-6\mathbf{j}+2\mathbf{k};$$

$$N=B\times T=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 6 & -6 & 2\\ 1 & 2 & 3\end{vmatrix}=-22\mathbf{i}-16\mathbf{j}+18\mathbf{k}.$$

Consequently,

$$\tau=\frac{\mathbf{i}+2\mathbf{j}+3\mathbf{k}}{\sqrt{14}},\quad \beta=\frac{3\mathbf{i}-3\mathbf{j}+\mathbf{k}}{\sqrt{19}},\quad \nu=\frac{-11\mathbf{i}-8\mathbf{j}+9\mathbf{k}}{\sqrt{266}}.$$

Since for $t=1$ we have $x=1$, $y=1$, $z=1$, it follows that

$$\frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{3}$$

are the equations of the tangent,

$$\frac{x-1}{3}=\frac{y-1}{-3}=\frac{z-1}{1}$$

are the equations of the binormal and

$$\frac{x-1}{-11}=\frac{y-1}{-8}=\frac{z-1}{9}$$

are the equations of the principal normal.

If a space curve is represented as an intersection of two surfaces

$$F(x, y, z)=0,\quad G(x, y, z)=0,$$

then in place of the vectors $\frac{d\mathbf{r}}{dt}$ and $\frac{d^2\mathbf{r}}{dt^2}$ we can take the vectors $d\mathbf{r}\{dx, dy, dz\}$ and $d^2\mathbf{r}\{d^2x, d^2y, d^2z\}$; and one of the variables x, y, z may be considered independent and we can put its second differential equal to zero.

Example 2. Write the equation of the osculating plane of the circle

$$x^2+y^2+z^2=6,\quad x+y+z=0 \tag{3}$$

at its point $M(1, 1, -2)$.

Solution. Differentiating the system (3) and considering x an independent variable, we will have

$$x\,dx+y\,dy+z\,dz=0,$$
$$dx+dy+dz=0$$

and

$$dx^2+dy^2+y\,d^2y+dz^2+z\,d^2z=0,$$
$$d^2y+d^2z=0.$$

Putting $x=1$, $y=1$, $z=-2$, we get

$$dy=-dx;\quad dz=0;$$

$$d^2y=-\frac{2}{3}dx^2;\quad d^2z=\frac{2}{3}dx^2.$$

Hence, the osculating plane is defined by the vectors

$$\{dx, \ -dx, \ 0\} \quad \text{and} \quad \left\{0, \ -\frac{2}{3}dx^2, \ \frac{2}{3}dx^2\right\}$$

or

$$\{1, \ -1, \ 0\} \quad \text{and} \quad \{0, \ -1, \ 1\}.$$

Whence the normal vector of the osculating plane is

$$\boldsymbol{B}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{vmatrix}=-\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}$$

and, therefore, its equation is

$$-1(x-1)-(y-1)-(z+2)=0,$$

that is,

$$x+y+z=0,$$

as it should be, since our curve is located in this plane.

2090. Find the basic unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$ of the curve

$$x=1-\cos t, \quad y=\sin t, \quad z=t$$

at the point $t=\frac{\pi}{2}$.

2091. Find the unit vectors of the tangent and the principal normal of the conic spiral

$$\boldsymbol{r}=e^t(\boldsymbol{i}\cos t+\boldsymbol{j}\sin t+\boldsymbol{k})$$

at an arbitrary point. Determine the angles that these lines make with the z-axis.

2092. Find the basic unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$ of the curve

$$y=x^2, \quad z=2x$$

at the point $x=2$.

2093. For the screw line

$$x=a\cos t, \quad y=a\sin t, \quad z=bt$$

write the equations of the straight lines that form a natural trihedron at an arbitrary point of the line. Determine the direction cosines of the tangent line and the principal normal.

2094. Write the equations of the planes that form the natural trihedron of the curve

$$x^2+y^2+z^2=6, \quad x^2-y^2+z^2=4$$

at one of its points $M(1, 1, 2)$.

2095. Form the equations of the tangent line, the normal plane and the osculating plane of the curve $x=t$, $y=t^2$, $z=t^3$ at the point $M(2, 4, 8)$.

2096. Form the equations of the tangent, principal normal, and binormal at an arbitrary point of the curve

$$x=\frac{t^4}{4}, \quad y=\frac{t^3}{3}, \quad z=\frac{t^2}{2}.$$

Find the points at which the tangent to this curve is parallel to the plane $x+3y+2z-10=0$.

2097. Form equations of the tangent, the osculating plane, the principal normal and the binormal of the curve

$$x=t, \quad y=-t, \quad z=\frac{t^2}{2}$$

at the point $t=2$. Compute the direction cosines of the binormal at this point.

2098. Write the equations of the tangent and the normal plane to the following curves:

a) $x=R\cos^2 t,\ y=R\sin t\cos t,\ z=R\sin t$ for $t=\frac{\pi}{4}$;

b) $z=x^2+y^2,\ x=y$ at the point (1, 1, 2);

c) $x^2+y^2+z^2=25,\ x+z=5$ at the point $(2, 2\sqrt{3}, 3)$.

2099 Find the equation of the normal plane to the curve $z=x^2-y^2,\ y=x$ at the coordinate origin.

2100. Find the equation of the osculating plane to the curve $x=e^t,\ y=e^{-t},\ z=t\sqrt{2}$ at the point $t=0$.

2101. Find the equations of the osculating plane to the curves:

a) $x^2+y^2+z^2=9,\ x^2-y^2=3$ at the point (2, 1, 2);

b) $x^2=4y,\ x^3=24z$ at the point (6, 9, 9);

c) $x^2+z^2=a^2,\ y^2+z^2=b^2$ at any point of the curve (x_0, y_0, z_0).

2102. Form the equations of the osculating plane, the principal normal and the binormal to the curve

$$y^2=x, \quad x^2=z \text{ at the point } (1, 1, 1).$$

2103. Form the equations of the osculating plane, the principal normal and the binormal to the conical screw-line $x=t\cos t$, $y=t\sin t$, $z=bt$ at the origin. Find the unit vectors of the tangent, the principal normal, and the binormal at the origin.

Sec. 20. Curvature and Torsion of a Space Curve

1°. Curvature. By the *curvature* of a curve at a point M we mean the number

$$K=\frac{1}{R}=\lim_{\Delta s\to 0}\frac{\varphi}{\Delta s},$$

where φ is the angle of turn of the tangent line (*angle of contingence*) on a segment of the curve $\overset{\frown}{MN}$, Δs is the arc length of this segment of the curve. R is called the *radius of curvature*. If a curve is defined by the equation $\boldsymbol{r}=\boldsymbol{r}(s)$, where s is the arc length, then

$$\frac{1}{R}=\left|\frac{d^2\boldsymbol{r}}{ds^2}\right|.$$

For the case of a general parametric representation of the curve we have

$$\frac{1}{R}=\frac{\left|\frac{d\boldsymbol{r}}{dt}\times\frac{d^2\boldsymbol{r}}{dt^2}\right|}{\left|\frac{d\boldsymbol{r}}{dt}\right|^3}. \qquad (1)$$

2°. **Torsion.** By *torsion* (*second curvature*) of a curve at a point M we mean the number

$$T=\frac{1}{\varrho}=\lim_{\Delta s\to 0}\frac{\theta}{\Delta s},$$

where θ is the angle of turn of the binormal (*angle of contingence of the second kind*) on the segment of the curve $\overset{\frown}{MN}$. The quantity ϱ is called the *radius of torsion* or the *radius of second curvature*. If $\boldsymbol{r}=\boldsymbol{r}(s)$, then

$$\frac{1}{\varrho}=\pm\left|\frac{d\boldsymbol{\beta}}{ds}\right|=\frac{\frac{d\boldsymbol{r}}{ds}\frac{d^2\boldsymbol{r}}{ds^2}\frac{d^3\boldsymbol{r}}{ds^3}}{\left(\frac{d^2\boldsymbol{r}}{ds^2}\right)^2},$$

where the minus sign is taken when the vectors $\frac{d\boldsymbol{\beta}}{ds}$ and $\boldsymbol{\nu}$ have the same direction, and the plus sign, when not the same.

If $\boldsymbol{r}=\boldsymbol{r}(t)$, where t is an arbitrary parameter, then

$$\frac{1}{\varrho}=\frac{\frac{d\boldsymbol{r}}{dt}\frac{d^2\boldsymbol{r}}{dt^2}\frac{d^3\boldsymbol{r}}{dt^3}}{\left(\frac{d\boldsymbol{r}}{dt}\times\frac{d^2\boldsymbol{r}}{dt^2}\right)^2}. \qquad (2)$$

Example 1. Find the curvature and the torsion of the screw-line

$$\boldsymbol{r}=\boldsymbol{i}\,a\cos t+\boldsymbol{j}\,a\sin t+\boldsymbol{k}\,bt \qquad (a>0).$$

Solution. We have

$$\frac{d\boldsymbol{r}}{dt}=-\boldsymbol{i}\,a\sin t+\boldsymbol{j}\,a\cos t+\boldsymbol{k}b,$$

$$\frac{d^2\boldsymbol{r}}{dt^2}=-\boldsymbol{i}\,a\cos t-\boldsymbol{j}\,a\sin t,$$

$$\frac{d^3\boldsymbol{r}}{dt^3}=-\boldsymbol{i}\,a\sin t-\boldsymbol{j}\,a\cos t.$$

Whence

$$\frac{d\boldsymbol{r}}{dt}\times\frac{d^2\boldsymbol{r}}{dt^2}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ -a\sin t & a\cos t & b\\ -a\cos t & -a\sin t & 0\end{vmatrix}=\boldsymbol{i}\,ab\sin t-\boldsymbol{j}\,ab\cos t+a^2\boldsymbol{k}$$

and

$$\frac{d\mathbf{r}}{dt}\frac{d^2\mathbf{r}}{dt^2}\frac{d^3\mathbf{r}}{dt^3}=\begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}=a^2b.$$

Hence, on the basis of formulas (1) and (2), we get

$$\frac{1}{R}=\frac{a\sqrt{a^2+b^2}}{(a^2+b^2)^{3/2}}=\frac{a}{a^2+b^2}$$

and

$$\frac{1}{\varrho}=\frac{a^2b}{a^2(a^2+b^2)}=\frac{b}{a^2+b^2}.$$

Thus, for a screw-line, the curvature and torsion are constants.

3° **Frenet formulas:**

$$\frac{d\boldsymbol{\tau}}{ds}=\frac{\boldsymbol{\nu}}{R}, \quad \frac{d\boldsymbol{\nu}}{ds}=-\frac{\boldsymbol{\tau}}{R}+\frac{\boldsymbol{\beta}}{\varrho}, \quad \frac{d\boldsymbol{\beta}}{ds}=-\frac{\boldsymbol{\nu}}{\varrho}.$$

2104. Prove that if the curvature at all points of a line is zero, then the line is a straight line.

2105. Prove that if the torsion at all points of a curve is zero, then the curve is a plane curve.

2106. Prove that the curve

$$x=1+3t+2t^2, \quad y=2-2t+5t^2, \quad z=1-t^2$$

is a plane curve; find the plane in which it lies.

2107. Compute the curvature of the following curves:

a) $x=\cos t$, $y=\sin t$, $z=\cosh t$ at the point $t=0$;
b) $x^2-y^2+z^2=1$, $y^2-2x+z=0$ at the point (1, 1, 1).

2108. Compute the curvature and torsion at any point of the curves:

a) $x=e^t\cos t$, $y=e^t\sin t$, $z=e^t$;
b) $x=a\cosh t$, $y=a\sinh t$, $z=at$ (hyperbolic screw-line).

2109. Find the radii of curvature and torsion at an arbitrary point (x, y, z) of the curves:

a) $x^2=2ay$, $x^3=6a^2z$;
b) $x^3=3p^2y$, $2xz=p^2$.

2110. Prove that the tangential and normal components of acceleration $\boldsymbol{w}$ are expressed by the formulas

$$\boldsymbol{w}_{\tau}=\frac{dv}{dt}\boldsymbol{\tau}, \quad \boldsymbol{w}_{\nu}=\frac{v^2}{R}\boldsymbol{\nu},$$

where v is the velocity, R is the radius of curvature of the trajectory, $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are unit vectors of the tangent and principal normal to the curve.

2111. A point is in uniform motion along a screw-line $r = ia\cos t + ja\sin t + btk$ with velocity v. Compute its acceleration w.

2112. The equation of motion is

$$r = ti + t^2 j + t^3 k.$$

Determine, at times $t = 0$ and $t = 1$: 1) the curvature of the trajectory and 2) the tangential and normal components of the acceleration.

Chapter VII

MULTIPLE AND LINE INTEGRALS

Sec. 1. The Double Integral in Rectangular Coordinates

1°. **Direct computation of double integrals.** The *double integral* of a continuous function $f(x, y)$ over a bounded closed region S is the limit of the corresponding two-dimensional integral sum

$$\int\limits_{(S)}\int f(x, y)\,dx\,dy = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_k \to 0}} \sum_i \sum_k f(x_i, y_k)\,\Delta x_i\,\Delta y_k, \qquad (1)$$

where $\Delta x_i = x_{i+1} - x_i$, $\Delta y_k = y_{k+1} - y_k$ and the sum is extended over those values of i and k for which the points (x_i, y_k) belong to S.

2°. **Setting up the limits of integration in a double integral.** We distinguish two basic types of region of integration.

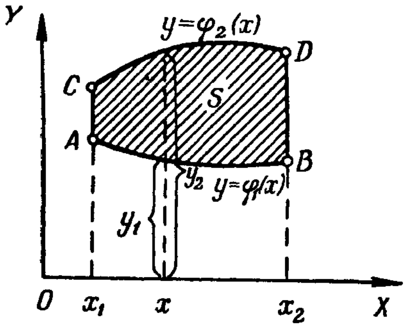

Fig. 85

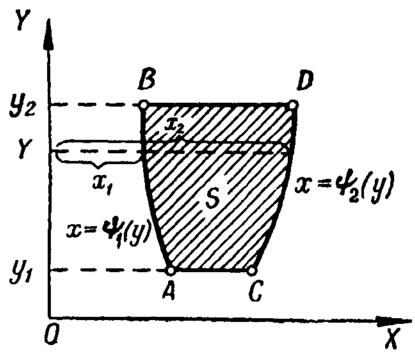

Fig. 86

1) The region of integration S (Fig. 85) is bounded on the left and right by the straight lines $x = x_1$ and $x = x_2$ $(x_2 > x_1)$, from below and from above by the continuous curves $y = \varphi_1(x)$ (AB) and $y = \varphi_2(x)$ (CD) $[\varphi_2(x) \geqslant \varphi_1(x)]$, each of which intersects the vertical $x = X$ $(x_1 \leqslant X \leqslant x_2)$ at only one point (see Fig. 85). In the region S, the variable x varies from x_1 to x_2, while the variable y (for x constant) varies from $y_1 = \varphi_1(x)$ to $y_2 = \varphi_2(x)$. The integral (1) may

be computed by reducing to an iterated integral by the formula

$$\iint\limits_{(S)} f(x, y)\,dx\,dy = \int\limits_{x_1}^{x_2} dx \int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy,$$

where x is held constant when calculating $\int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy$.

2) The region of integration S is bounded from below and from above by the straight lines $y=y_1$ and $y=y_2\,(y_2>y_1)$, and from the left and the right by the continuous curves $x=\psi_1(y)$ (AB) and $x=\psi_2(y)$ (CD) $[\psi_2(y)\geqslant\psi_1(y)]$, each of which intersects the parallel $y=Y\,(y_1\leqslant Y\leqslant y_2)$ at only one point (Fig. 86).

As before, we have

$$\iint\limits_{(S)} f(x, y)\,dx\,dy = \int\limits_{y_1}^{y_2} dy \int\limits_{\psi_1(y)}^{\psi_2(y)} f(x, y)\,dx,$$

here, in the integral $\int\limits_{\psi_1(y)}^{\psi_2(y)} f(x, y)\,dx$ we consider y constant.

If the region of integration does not belong to any of the above-discussed types, then an attempt is made to break it up into parts, each of which does belong to one of these two types.

Example 1. Evaluate the integral

$$I = \int\limits_0^1 dx \int\limits_x^1 (x+y)\,dy.$$

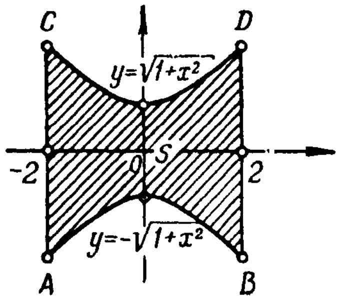

Fig. 87

Solution.

$$I = \int\limits_0^1 \left(xy+\frac{y^2}{2}\right)\Bigg|_{y=x}^{y=1} dx = \int\limits_0^1 \left[\left(x+\frac{1}{2}\right)-\left(x^2+\frac{x^2}{2}\right)\right] dx = \frac{1}{2}.$$

Example 2. Determine the limits of integration of the integral

$$\iint\limits_{(S)} f(x, y)\,dx\,dy$$

if the region of integration S (Fig. 87) is bounded by the hyperbola $y^2-x^2=1$ and by two straight lines $x=2$ and $x=-2$ (we have in view the region containing the coordinate origin).

Solution. The region of integration $ABCD$ (Fig. 87) is bounded by the straight lines $x=-2$ and $x=2$ and by two branches of the hyperbola

$$y=\sqrt{1+x^2} \quad \text{and} \quad y=-\sqrt{1+x^2};$$

that is, it belongs to the first type. We have:

$$\iint_{(S)} f(x, y)\,dx\,dy = \int_{-2}^{2} dx \int_{-\sqrt{1+x^2}}^{\sqrt{1+x^2}} f(x, y)\,dy.$$

Evaluate the following iterated integrals:

2113. $\int_0^2 dy \int_0^1 (x^2+2y)\,dx.$

2114 $\int_3^4 dx \int_1^2 \frac{dy}{(x+y)^2}.$

2115. $\int_0^1 dx \int_0^1 \frac{x^2dy}{1+y^2}.$

2116. $\int_1^2 dx \int_{\frac{1}{x}}^{x} \frac{x^2dy}{y^2}.$

2117. $\int_{-3}^{3} dy \int_{y^2-4}^{5} (x+2y)\,dx.$

2118. $\int_0^{2\pi} d\varphi \int_{a\sin\varphi}^{a} r\,dr.$

2119. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{3\cos\varphi} r^2\sin^2\varphi\,dr.$

2120. $\int_0^1 dx \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dy.$

Write the equations of curves bounding regions over which the following double integrals are extended, and draw these regions:

2121. $\int_{-6}^{2} dy \int_{\frac{y^2}{4}-1}^{2-y} f(x, y)\,dx.$

2122. $\int_1^3 dx \int_{x^2}^{x+9} f(x, y)\,dy.$

2123. $\int_0^4 dy \int_y^{10-y} f(x, y)\,dx.$

2124. $\int_1^3 dx \int_{\frac{x}{3}}^{2x} f(x, y)\,dy.$

2125. $\int_0^3 dx \int_0^{\sqrt{25-x^2}} f(x, y)\,dy.$

2126. $\int_{-1}^{2} dx \int_{x^2}^{x+2} f(x, y)\,dy.$

Set up the limits of integration in one order and then in the other in the double integral

$$\iint_{(S)} f(x, y)\,dx\,dy$$

for the indicated regions S.

2127. S is a rectangle with vertices $O(0, 0)$, $A(2, 0)$, $B(2, 1)$, $C(0, 1)$.

2128. S is a triangle with vertices $O(0, 0)$, $A(1, 0)$, $B(1, 1)$.

2129. S is a trapezoid with vertices $O(0, 0)$, $A(2, 0)$, $B(1, 1)$, $C(0, 1)$.

2130. S is a parallelogram with vertices $A(1, 2)$, $B(2, 4)$, $C(2, 7)$, $D(1, 5)$.

2131. S is a circular sector OAB with centre at the point $O(0, 0)$, whose arc end-points are $A(1, 1)$ and $B(-1, 1)$ (Fig. 88).

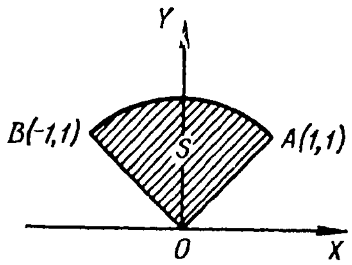

Fig. 88

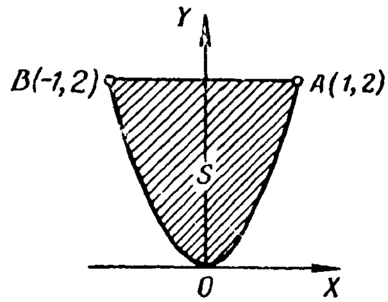

Fig. 89

2132. S is a right parabolic segment AOB bounded by the parabola BOA and a segment of the straight line BA connecting the points $B(-1, 2)$ and $A(1, 2)$ (Fig. 89).

2133. S is a circular ring bounded by circles with radii $r=1$ and $R=2$ and with common centre $O(0, 0)$.

2134. S is bounded by the hyperbola $y^2-x^2=1$ and the circle $x^2+y^2=9$ (the region containing the origin is meant).

2135. Set up the limits of integration in the double integral

$$\iint\limits_{(S)} f(x, y)\,dx\,dy$$

if the region S is defined by the inequalities

a) $x \geqslant 0$; $y \geqslant 0$; $x+y \leqslant 1$;

b) $x^2+y^2 \leqslant a^2$;

c) $x^2+y^2 \leqslant x$;

d) $y \geqslant x$; $x \geqslant -1$; $y \leqslant 1$;

e) $y \leqslant x \leqslant y+2a$; $0 \leqslant y \leqslant a$.

Change the order of integration in the following double integrals:

2136. $\int\limits_0^4 dx \int\limits_{3x^2}^{12x} f(x, y)\,dy$.

2137. $\int\limits_0^1 dx \int\limits_{2x}^{x} f(x, y)\,dy$.

2138. $\int\limits_0^a dx \int\limits_{\frac{a^2-x^2}{2a}}^{\sqrt{a^2-x^2}} f(x, y)\,dy.$

2139. $\int\limits_{\frac{a}{2}}^{a} dx \int\limits_0^{\sqrt{2ax-x^2}} f(x, y)\,dy.$

2140. $\int\limits_0^{2a} dx \int\limits_{\sqrt{2ax-x^2}}^{\sqrt{4ax}} f(x, y)\,dy.$

2141. $\int\limits_0^1 dy \int\limits_{-\sqrt{1-y^2}}^{1-y} f(x, y)\,dx.$

2142. $\int\limits_0^1 dy \int\limits_{\frac{y^2}{2}}^{\sqrt{3-y^2}} f(x, y)\,dx.$

2143. $\int\limits_0^{\frac{R\sqrt{2}}{2}} dx \int\limits_0^x f(x, y)\,dy + \int\limits_{\frac{R\sqrt{2}}{2}}^{R} dx \int\limits_0^{\sqrt{R^2-x^2}} f(x, y)\,dy.$

2144. $\int\limits_0^{\pi} dx \int\limits_0^{\sin x} f(x, y)\,dy.$

Evaluate the following double integrals:

2145. $\iint\limits_{(S)} x\,dx\,dy$, where S is a triangle with vertices $O(0, 0)$, $A(1, 1)$, and $B(0, 1)$.

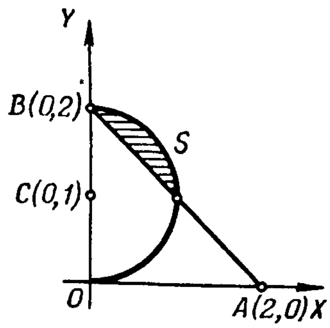

Fig. 90

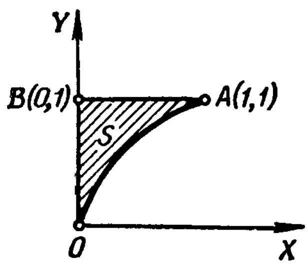

Fig. 91

2146. $\iint\limits_{(S)} x\,dx\,dy$, where the region of integration S is bounded by the straight line passing through the points $A(2, 0)$, $B(0, 2)$ and by the arc of a circle with centre at the point $C(0, 1)$, and radius 1 (Fig. 90).

2147. $\iint\limits_{(S)} \frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$, where S is a part of a circle of radius a with centre at $O(0, 0)$ lying in the first quadrant.

2148. $\iint\limits_{(S)} \sqrt{x^2-y^2}\,dx\,dy$, where S is a triangle with vertices $O(0, 0)$, $A(1, -1)$, and $B(1, 1)$.

2149. $\iint\limits_{(S)} \sqrt{xy-y^2}\,dx\,dy$, where S is a triangle with vertices $O(0, 0)$, $A(10, 1)$, and $B(1, 1)$.

2150. $\iint\limits_{(S)} e^{\frac{x}{y}}\,dx\,dy$, where S is a curvilinear triangle OAB bounded by the parabola $y^2=x$ and the straight lines $x=0$, $y=1$ (Fig. 91).

2151. $\iint\limits_{(S)} \frac{x\,dx\,dy}{x^2+y^2}$, where S is a parabolic segment bounded by the parabola $y=\frac{x^2}{2}$ and the straight line $y=x$.

2152. Compute the integrals and draw the regions over which they extend:

a) $\int\limits_0^{\pi} dx \int\limits_0^{1+\cos x} y^2 \sin x\,dy$;

b) $\int\limits_0^{\frac{\pi}{2}} dx \int\limits_{\cos x}^{1} y^4\,dy$;

c) $\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dy \int\limits_0^{3\cos y} x^2 \sin^2 y\,dx$.

When solving Problems 2153 to 2157 it is abvisable to make the drawings first.

2153. Evaluate the double integral

$$\iint\limits_{(S)} xy^2\,dx\,dy,$$

if S is a region bounded by the parabola $y^2=2px$ and the straight line $x=p$.

2154*. Evaluate the double integral

$$\iint\limits_{(S)} xy\,dx\,dy,$$

extended over the region S, which is bounded by the x-axis and an upper semicircle $(x-2)^2+y^2=1$.

2155. Evaluate the double integral

$$\iint_{(S)} \frac{dx\,dy}{\sqrt{2a-x}},$$

where S is the area of a circle of radius a, which circle is tangent to the coordinate axes and lies in the first quadrant.

2156*. Evaluate the double integral

$$\iint_{(S)} y\,dx\,dy,$$

where the region S is bounded by the axis of abscissas and an arc of the cycloid

$$x = R(t - \sin t),$$
$$y = R(1 - \cos t).$$

2157. Evaluate the double integral

$$\iint_{(S)} xy\,dx\,dy,$$

in which the region of integration S is bounded by the coordinate axes and an arc of the astroid

$$x = R\cos^3 t, \quad y = R\sin^3 t \left(0 \leqslant t \leqslant \frac{\pi}{2}\right).$$

2158. Find the mean value of the function $f(x, y) = xy^2$ in the region $S\{0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$.

Hint. The *mean value of a function* $f(x, y)$ in the region S is the number

$$\bar{f} = \frac{1}{S}\iint_{(S)} f(x, y)\,dx\,dy.$$

2159. Find the mean value of the square of the distance of a point $M(x, y)$ of the circle $(x-a)^2 + y^2 \leqslant R^2$ from the coordinate origin.

Sec. 2. Change of Variables in a Double Integral

1°. Double integral in polar coordinates. In a double integral, when passing from rectangular coordinates (x, y) to polar coordinates (r, φ), which are connected with rectangular coordinates by the relations

$$x = r\cos\varphi, \qquad y = r\sin\varphi,$$

we have the formula

$$\iint_{(S)} f(x, y)\,dx\,dy = \iint_{(S)} (r\cos\varphi,\ r\sin\varphi)\,r\,dr\,d\varphi, \tag{1}$$

If the region of integration (S) is bounded by the half-lines $r=\alpha$ and $r=\beta\ (\alpha<\beta)$ and the curves $r=r_1(\varphi)$ and $r=r_2(\varphi)$, where $r_1(\varphi)$ and $r_2(\varphi)\ [r_1(\varphi)\leqslant r_2(\varphi)]$ are single-valued functions on the interval $\alpha\leqslant\varphi\leqslant\beta$, then the double integral may be evaluated by the formula

$$\iint\limits_{(S)} F(\varphi, r)\, r\, dr\, d\varphi = \int\limits_{\alpha}^{\beta} d\varphi \int\limits_{r_1(\varphi)}^{r_2(\varphi)} F(\varphi, r)\, r\, dr,$$

where $F(\varphi, r)=f(r\cos\varphi, r\sin\varphi)$. In evaluating the integral $\int\limits_{r_1(\varphi)}^{r_2(\varphi)} F(\varphi, r)\, r\, dr$ we hold the quantity φ constant.

If the region of integration does not belong to one of the kinds that has been examined, it is broken up into parts, each of which is a region of a given type.

2°. **Double integral in curvilinear coordinates.** In the more general case, if in the double integral

$$\iint\limits_{(S)} f(x, y)\, dx\, dy$$

it is required to pass from the variables x, y to the variables u, v, which are connected with x, y by the continuous and differentiable relationships

$$x=\varphi(u, v), \qquad y=\psi(u, v)$$

that establish a one-to-one (and, in both directions, continuous) correspondence between the points of the region S of the xy-plane and the points of some region S' of the UV-plane, and if the *Jacobian*

$$I=\frac{D(x, y)}{D(u, v)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

retains a constant sign in the region S, then the formula

$$\iint\limits_{(S)} f(x, y)\, dx\, dy = \iint\limits_{(S')} f[\varphi(u, v), \psi(u, v)]\, |I|\, du\, dv$$

holds true.

The limits of the new integral are determined from general rules on the basis of the type of region S'.

Example 1. In passing to polar coordinates, evaluate

$$\iint\limits_{(S)} \sqrt{1-x^2-y^2}\, dx\, dy,$$

where the region S is a circle of radius $R=1$ with centre at the coordinate origin (Fig. 92).

Solution. Putting $x=r\cos\varphi$, $y=r\sin\varphi$, we obtain:

$$\sqrt{1-x^2-y^2}=\sqrt{1-(r\cos\varphi)^2-(r\sin\varphi)^2}=\sqrt{1-r^2}$$

Since the coordinate r in the region S varies from 0 to 1 for any φ, and φ varies from 0 to 2π, it follows that

$$\iint_{(S)} \sqrt{1-x^2-y^2}\,dx\,dy = \int_0^{2\pi} d\varphi \int_0^1 r\sqrt{1-r^2}\,dr = \frac{2}{3}\pi.$$

Pass to polar coordinates r and φ and set up the limits of integration with respect to the new variables in the following integrals:

2160. $\int_0^1 dx \int_0^1 f(x, y)\,dy.$ **2161.** $\int_0^2 dx \int_0^x f(\sqrt{x^2+y^2})\,dy.$

2162. $\iint_{(S)} f(x, y)\,dx\,dy,$

where S is a triangle bounded by the straight lines $y=x$, $y=-x$, $y=1$.

2163. $\int_{-1}^{1} dx \int_{x^2}^{1} f\left(\frac{y}{x}\right) dy.$

2164. $\iint_{(S)} f(x, y)\,dx\,dy$, where S is bounded by the lemniscate

$$(x^2+y^2)^2 = a^2(x^2-y^2).$$

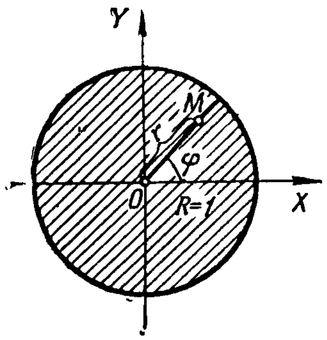

Fig. 92

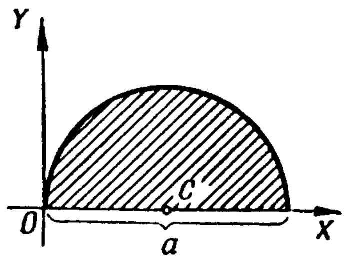

Fig. 93

2165. Passing to polar coordinates, calculate the double integral

$$\iint_{(S)} y\,dx\,dy,$$

where S is a semicircle of diameter a with centre at the point $C\left(\frac{a}{2}, 0\right)$ (Fig. 93).

2166. Passing to polar coordinates, evaluate the double integral

$$\iint_{(S)} (x^2+y^2)\,dx\,dy,$$

extended over a region bounded by the circle $x^2+y^2=2ax$.

2167. Passing to polar coordinates, evaluate the double integral

$$\iint_{(S)} \sqrt{a^2-x^2-y^2}\,dx\,dy,$$

where the region of integration S is a semicircle of radius a with centre at the coordinate origin and lying above the x-axis.

2168. Evaluate the double integral of a function $f(r, \varphi)=r$ over a region bounded by the cardioid $r=a(1+\cos\varphi)$ and the circle $r=a$. (This is a region that does not contain a pole.)

2169. Passing to polar coordinates, evaluate

$$\int_0^a dx \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dy.$$

2170. Passing to polar coordinates, evaluate

$$\iint_{(S)} \sqrt{a^2-x^2-y^2}\,dx\,dy,$$

where the region S is a loop of the lemniscate

$$(x^2+y^2)^2=a^2(x^2-y^2) \qquad (x\geqslant 0).$$

2171*. Evaluate the double integral

$$\iint_{(S)} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy,$$

extended over the region S bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ by passing to *generalized polar coordinates:*

$$\frac{x}{a}=r\cos\varphi,\ \frac{y}{b}=r\sin\varphi.$$

2172.** Transform

$$\int_0^c dx \int_{\alpha x}^{\beta x} f(x, y)\,dy$$

$(0<\alpha<\beta$ and $c>0)$ by introducing new variables $u=x+y$, $uv=y$.

2173*. Change the variables $u=x+y$, $v=x-y$ in the integral

$$\int_0^1 dx \int_0^1 f(x, y)\, dy.$$

2174**. Evaluate the double integral

$$\iint_{(S)} dx\, dy,$$

where S is a region bounded by the curve

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2}{h^2}-\frac{y^2}{k^2}.$$

Hint. Make the substitution

$$x=ar\cos\varphi, \quad y=br\sin\varphi.$$

Sec. 3. Computing Areas

1°. Area in rectangular coordinates. The *area of a plane region S* is

$$S=\iint_{(S)} dx\, dy.$$

If the region S is defined by the inequalities $a \leqslant x \leqslant b$, $\varphi(x) \leqslant y \leqslant \psi(x)$, then

$$S=\int_a^b dx \int_{\varphi(x)}^{\psi(x)} dy.$$

2°. Area in polar coordinates. If a region S in polar coordinates r and φ is defined by the inequalities $\alpha \leqslant \varphi \leqslant \beta$, $f(\varphi) \leqslant r \leqslant F(\varphi)$, then

$$S=\iint_{(S)} r\, d\varphi\, dr=\int_\alpha^\beta d\varphi \int_{f(\varphi)}^{F(\varphi)} r\, dr.$$

2175. Construct regions whose areas are expressed by the integrals

a) $\displaystyle\int_{-1}^{2} dx \int_{x^2}^{x+2} dy$; b) $\displaystyle\int_0^a dy \int_{a-y}^{\sqrt{a^2-y^2}} dx.$

Evaluate these areas and change the order of integration.

2176. Construct regions whose areas are expressed dy the integrals

a) $\displaystyle\int_{\frac{\pi}{4}}^{\arctan 2} d\varphi \int_0^{3\sec\varphi} r\, dr$; b) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_a^{a(1+\cos\varphi)} r\, dr.$

Compute these areas.

2177. Compute the area bounded by the straight lines $x=y$, $x=2y$, $x+y=a$, $x+3y=a\,(a>0)$.

2178. Compute the area lying above the x-axis and bounded by this axis, the parabola $y^2=4ax$, and the straight line $x+y=3a$.

2179*. Compute the area bounded by the ellipse

$$(y-x)^2+x^2=1.$$

2180. Find the area bounded by the parabolas

$$y^2=10x+25 \text{ and } y^2=-6x+9.$$

2181. Passing to polar coordinates, find the area bounded by the lines

$$x^2+y^2=2x, \quad x^2+y^2=4x, \quad y=x, \quad y=0.$$

2182. Find the area bounded by the straight line $r\cos\varphi=1$ and the circle $r=2$. (The area is not to contain a pole.)

2183. Find the area bounded by the curves

$$r=a(1+\cos\varphi) \text{ and } r=a\cos\varphi\,(a>0).$$

2184. Find the area bounded by the line

$$\left(\frac{x^2}{4}+\frac{y^2}{9}\right)^2=\frac{x^2}{4}-\frac{y^2}{9}.$$

2185*. Find the area bounded by the ellipse

$$(x-2y+3)^2+(3x+4y-1)^2=100.$$

2186. Find the area of a curvilinear quadrangle bounded by the arcs of the parabolas $x^2=ay$, $x^2=by$, $y^2=\alpha x$, $y^2=\beta x\,(0<a<b,\ 0<\alpha<\beta)$.

Hint. Introduce the new variables u and v, and put

$$x^2=uy, \quad y^2=vx.$$

2187. Find the area of a curvilinear quadrangle bounded by the arcs of the curves $y^2=ax$, $y^2=bx$, $xy=\alpha$, $xy=\beta\,(0<a<b,\ 0<\alpha<\beta)$.

Hint. Introduce the new variables u and v, and put

$$xy=u, \quad y^2=vx.$$

Sec. 4. Computing Volumes

The *volume V of a cylindroid* bounded above by a continuous surface $z=f(x, y)$, below by the plane $z=0$, and on the sides by a right cylindrical surface, which cuts out of the xy-plane a region S (Fig. 94), is equal to

$$V=\iint\limits_{(S)} f(x, y)\,dx\,dy.$$

2188. Use a double integral to express the volume of a pyramid with vertices $O(0, 0, 0)$, $A(1, 0, 0)$, $B(1, 1, 0)$ and $C(0, 0, 1)$ (Fig. 95). Set up the limits of integration.

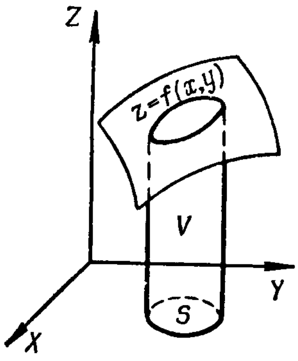

Fig. 94

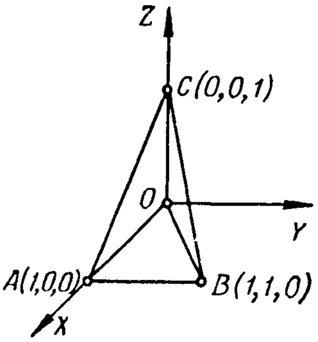

Fig. 95

In Problems 2189 to 2192 sketch the solid whose volume is expressed by the given double integral:

2189. $\int\limits_0^1 dx \int\limits_0^{1-x} (1-x-y)\,dy.$ 2191. $\int\limits_0^2 dx \int\limits_0^{\sqrt{1-x^2}} (1-x)\,dy.$

2190. $\int\limits_0^2 dx \int\limits_0^{2-x} (4-x-y)\,dy.$ 2192. $\int\limits_0^2 dx \int\limits_{2-x}^{2} (4-x-y)\,dy.$

2193. Sketch the solid whose volume is expressed by the integral $\int\limits_0^a dx \int\limits_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy$; reason geometrically to find the value of this integral.

2194. Find the volume of a solid bounded by the elliptical paraboloid $z=2x^2+y^2+1$, the plane $x+y=1$, and the coordinate planes.

2195. A solid is bounded by a hyperbolic paraboloid $z=x^2-y^2$ and the planes $y=0$, $z=0$, $x=1$. Compute its volume.

2196. A solid is bounded by the cylinder $x^2+z^2=a^2$ and the planes $y=0$, $z=0$, $y=x$. Compute its volume.

Find the volumes bounded by the following surfaces:

2197. $az=y^2$, $x^2+y^2=r^2$, $z=0$.

2198. $y=\sqrt{x}$, $y=2\sqrt{x}$, $x+z=6$, $z=0$.

2199. $z=x^2+y^2$, $y=x^2$, $y=1$, $z=0$.

2200. $x+y+z=a$, $3x+y=a$, $\frac{3}{2}x+y=a$, $y=0$, $z=0$.

2201. $\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$, $y=\frac{b}{a}x$, $y=0$, $z=0$.

2202. $x^2+y^2=2ax$, $z=\alpha x$, $z=\beta x$ $(\alpha>\beta)$.

In Problems 2203 to 2211 use polar and generalized polar coordinates.

2203. Find the entire volume enclosed between the cylinder $x^2+y^2=a^2$ and the hyperboloid $x^2+y^2-z^2=-a^2$.

2204. Find the entire volume contained between the cone $2(x^2+y^2)-z^2=0$ and the hyperboloid $x^2+y^2-z^2=-a^2$.

2205. Find the volume bounded by the surfaces $2az=x^2+y^2$, $x^2+y^2-z^2=a^2$, $z=0$.

2206. Determine the volume of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

2207. Find the volume of a solid bounded by the paraboloid $2az=x^2+y^2$ and the sphere $x^2+y^2+z^2=3a^2$. (The volume lying inside the paraboloid is meant.)

2208. Compute the volume of a solid bounded by the xy-plane, the cylinder $x^2+y^2=2ax$, and the cone $x^2+y^2=z^2$.

2209. Compute the volume of a solid bounded by the xy-plane, the surface $z=ae^{-(x^2+y^2)}$, and the cylinder $x^2+y^2=R^2$.

2210. Compute the volume of a solid bounded by the xy-plane, the paraboloid $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$, and the cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2\frac{x}{a}$.

2211. In what ratio does the hyperboloid $x^2+y^2-z^2=a^2$ divide the volume of the sphere $x^2+y^2+z^2\leqslant 3a^2$?

2212*. Find the volume of a solid bounded by the surfaces $z=x+y$, $xy=1$, $xy=2$, $y=x$, $y=2x$, $z=0\ (x>0,\ y>0)$.

Sec. 5. Computing the Areas of Surfaces

The *area* σ of a smooth single-valued surface $z=f(x, y)$, whose projection on the xy-plane is the region S, is equal to

$$\sigma=\iint\limits_{(S)}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy.$$

2213. Find the area of that part of the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ which lies between the coordinate planes.

2214. Find the area of that part of the surface of the cylinder $x^2+y^2=R^2$ $(z\geqslant 0)$ which lies between the planes $z=mx$ and $z=nx\,(m>n>0)$.

2215*. Compute the area of that part of the surface of the cone $x^2-y^2=z^2$ which is situated in the first octant and is bounded by the plane $y+z=a$.

2216. Compute the area of that part of the surface of the cylinder $x^2+y^2=ax$ which is cut out of it by the sphere $x^2+y^2+z^2=a^2$.

2217. Compute the area of that part of the surface of the sphere $x^2+y^2+z^2=a^2$ cut out by the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

2218. Compute the area of that part of the surface of the paraboloid $y^2+z^2=2ax$ which lies between the cylinder $y^2=ax$ and the plane $x=a$.

2219. Compute the area of that part of the surface of the cylinder $x^2+y^2=2ax$ which lies between the xy-plane and the cone $x^2+y^2=z^2$.

2220*. Compute the area of that part of the surface of the cone $x^2-y^2=z^2$ which lies inside the cylinder $x^2+y^2=2ax$.

2221*. Prove that the areas of the parts of the surfaces of the paraboloids $x^2+y^2=2az$ and $x^2-y^2=2az$ cut out by the cylinder $x^2+y^2=R^2$ are of equivalent size.

2222*. A sphere of radius a is cut by two circular cylinders whose base diameters are equal to the radius of the sphere and which are tangent to each other along one of the diameters of the sphere. Find the volume and the area of the surface of the remaining part of the sphere.

2223* An opening with square base whose side is equal to a is cut out of a sphere of radius a. The axis of the opening coincides with the diameter of the sphere. Find the area of the surface of the sphere cut out by the opening.

2224*. Compute the area of that part of the helicoid $z=c\arctan\frac{y}{x}$ which lies in the first octant between the cylinders $x^2+y^2=a^2$ and $x^2+y^2=b^2$.

Sec. 6. Applications of the Double Integral in Mechanics

1°. The mass and static moments of a lamina. If S is a region in an xy-plane occupied by a lamina, and $\varrho(x, y)$ is the surface density of the lamina at the point (x, y), then the mass M of the lamina and its static moments M_X and M_Y relative to the x- and y-axes are expressed by the double integrals

$$M=\iint\limits_{(S)}\varrho(x, y)\,dx\,dy,\quad M_X=\iint\limits_{(S)}y\varrho(x, y)\,dx\,dy,$$

$$M_Y=\iint\limits_{(S)}x\varrho(x, y)\,dx\,dy. \qquad (1)$$

If the lamina is homogeneous, then $\varrho(x, y)=\text{const}$.

2°. The coordinates of the centre of gravity of a lamina. If $C(\bar{x}, \bar{y})$ is the centre of gravity of a lamina, then

$$\bar{x}=\frac{M_Y}{M},\quad \bar{y}=\frac{M_X}{M},$$

where M is the mass of the lamina and M_X, M_Y are its static moments relative to the coordinate axes (see 1°). If the lamina is homogeneous, then in formulas (1) we can put $\varrho=1$.

3°. The moments of inertia of a lamina. The moments of inertia of a lamina relative to the x- and y-axes are, respectively, equal to

$$I_X=\iint\limits_{(S)}y^2\varrho(x, y)\,dx\,dy,\quad I_Y=\iint\limits_{(S)}x^2\varrho(x, y)\,dx\,dy. \qquad (2)$$

The moment of inertia of a lamina relative to the origin is

$$I_0=\iint\limits_{(S)}(x^2+y^2)\,\varrho(x, y)\,dx\,dy=I_X+I_Y. \qquad (3)$$

Putting $\varrho(x, y)=1$ in formulas (2) and (3), we get the geometric moments of inertia of a plane figure.

2225. Find the mass of a circular lamina of radius R if the density is proportional to the distance of a point from the centre and is equal to δ at the edge of the lamina.

2226. A lamina has the shape of a right triangle with legs $OB=a$ and $OA=b$, and its density at any point is equal to the distance of the point from the leg OA. Find the static moments of the lamina relative to the legs OA and OB.

2227. Compute the coordinates of the centre of gravity of the area $OmAnO$ (Fig. 96), which is bounded by the curve $y=\sin x$ and the straight line OA that passes through the coordinate origin and the vertex $A\left(\frac{\pi}{2}, 1\right)$ of a sine curve.

2228. Find the coordinates of the centre of gravity of an area bounded by the cardioid $r=a(1+\cos\varphi)$.

2229. Find the coordinates of the centre of gravity of a circular sector of radius a with angle at the vertex 2α (Fig. 97).

2230. Compute the coordinates of the centre of gravity of an area bounded by the parabolas $y^2=4x+4$ and $y^2=-2x+4$.

2231. Compute the moment of inertia of a triangle bounded by the straight lines $x+y=2$, $x=2$, $y=2$ relative to the x-axis.

2232. Find the moment of inertia of an annulus with diameters d and $D\,(d<D)$: a) relative to its centre, and b) relative to its diameter.

2233. Compute the moment of inertia of a square with side a relative to the axis passing through its vertex perpendicularly to the plane of the square.

2234*. Compute the moment of inertia of a segment cut off the parabola $y^2=ax$ by the straight line $x=a$ relative to the straight line $y=-a$.

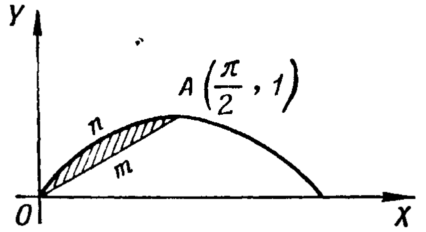

Fig. 96

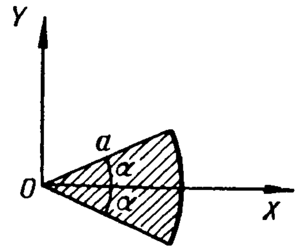

Fig. 97

2235*. Compute the moment of inertia of an area bounded by the hyperbola $xy=4$ and the straight line $x+y=5$ relative to the straight line $x=y$.

2236*. In a square lamina with side a, the density is proportional to the distance from one of its vertices. Compute the moment of inertia of the lamina relative to the side that passes through this vertex.

2237. Find the moment of inertia of the cardioid $r=a(1+\cos\varphi)$ relative to the pole.

2238. Compute the moment of inertia of the area of the lemniscate $r^2=2a^2\cos 2\varphi$ relative to the axis perpendicular to its plane in the pole.

2239*. Compute the moment of inertia of a homogeneous lamina bounded by one arc of the cycloid $x=a(t-\sin t)$, $y=a(1-\cos t)$ and the x-axis, relative to the x-axis.

Sec. 7. Triple Integrals

1°. Triple integrals in rectangular coordinates. The triple integral of the function $f(x, y, z)$ extended over the region V is the limit of the corresponding threefold iterated sum:

$$\iiint\limits_V f(x, y, z)\,dx\,dy\,dz = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_j \to 0 \\ \max \Delta z_k \to 0}} \sum_i \sum_j \sum_k f(x_i, y_j, z_k)\,\Delta x_i\,\Delta y_j\,\Delta z_k.$$

Evaluation of a triple integral reduces to the successive computation of the three ordinary (onefold iterated) integrals or to the computation of one double and one single integral.

Example 1. Compute

$$I=\int\int\limits_{V}\int x^3y^2z\,dx\,dy\,dz,$$

where the region V is defined by the inequalities

$$0\leqslant x\leqslant 1,\quad 0\leqslant y\leqslant x,\quad 0\leqslant z\leqslant xy.$$

Solution. We have

$$I=\int\limits_0^1 dx\int\limits_0^x dy\int\limits_0^{xy} x^3y^2z\,dz=\int\limits_0^1 dx\int\limits_0^x x^3y^2\frac{z^2}{2}\Bigg|_0^{xy}dy=$$

$$=\int\limits_0^1 dx\int\limits_0^x \frac{x^5y^4}{2}\,dy=\int\limits_0^1 \frac{x^5}{2}\frac{y^5}{5}\Bigg|_0^x dx=\int\limits_0^1 \frac{x^{10}}{10}\,dx=\frac{1}{110}.$$

Example 2. Evaluate

$$\int\int\limits_{(V)}\int x^2\,dx\,dy\,dz,$$

extended over the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Solution.

$$\int\int\limits_{(V)}\int x^2\,dx\,dy\,dz=\int\limits_{-a}^{a}x^2\,dx\int\limits_{(S_{yz})}\int dy\,dz=\int\limits_{-a}^{a}x^2S_{yz}\,dx,$$

where S_{yz} is the area of the ellipse $\frac{y^2}{b^2}+\frac{z^2}{c^2}=1-\frac{x^2}{a^2}$, $x=\text{const}$, and is equal to

$$S_{yz}=\pi b\sqrt{1-\frac{x^2}{a^2}}\cdot c\sqrt{1-\frac{x^2}{a^2}}=\pi bc\left(1-\frac{x^2}{a^2}\right).$$

We therefore finally get

$$\int\int\limits_{(V)}\int x^2\,dx\,dy\,dz=\pi bc\int\limits_{-a}^{a}x^2\left(1-\frac{x^2}{a^2}\right)dx=\frac{4}{15}\pi a^3bc.$$

2°. Change of variables in a triple integral. If in the triple integral

$$\int\int\limits_{(V)}\int f(x,\,y,\,z)\,dx\,dy\,dz$$

it is required to pass from the variables x, y, z to the variables u, v, w, which are connected with x, y, z by the relations $x=\varphi(u,\,v,\,w)$, $y=\psi(u,\,v,\,w)$, $z=\chi(u,\,v,\,w)$, where the functions φ, ψ, χ are:

1) continuous together with their partial first derivatives;

2) in one-to-one (and, in both directions, continuous) correspondence between the points of the region of integration V of xyz-space and the points of some region V' of UVW-space;

3) the functional determinant (Jacobian) of these functions

$$I=\frac{D(x, y, z)}{D(u, v, w)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

retains a constant sign in the region V, then we can make use of the formula

$$\iiint\limits_{(V)} f(x, y, z)\,dx\,dy\,dz=$$

$$=\iiint\limits_{(V')} f[\varphi(u, v, w),\ \psi(u, v, w),\ \chi(u, v, w)]\,|I|\,du\,dv\,dw.$$

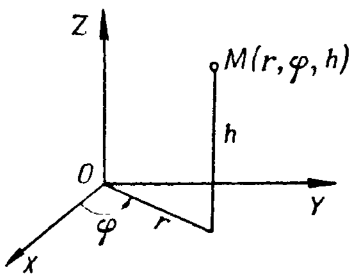

Fig. 98

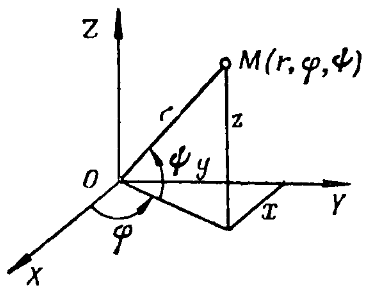

Fig. 99

In particular,

1) for cylindrical coordinates r, φ, h (Fig. 98), where

$$x=r\cos\varphi,\quad y=r\sin\varphi,\quad z=h,$$

we get $I=r$;

2) for spherical coordinates φ, ψ, r (φ is the longitude, ψ the latitude, r the radius vector) (Fig. 99), where

$$x=r\cos\psi\cos\varphi,\quad y=r\cos\psi\sin\varphi,\quad z=r\sin\psi,$$

we have $I=r^2\cos\psi$.

Example 3. Passing to spherical coordinates, compute

$$\iiint\limits_{(V)} \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz,$$

where V is a sphere of radius R.

Solution. For a sphere, the ranges of the spherical coordinates φ (longitude), ψ (latitude), and r (radius vector) will be

$$0\leqslant\varphi\leqslant 2\pi,\quad -\frac{\pi}{2}\leqslant\psi\leqslant\frac{\pi}{2},\quad 0\leqslant r\leqslant R.$$

We therefore have

$$\iiint_{(V)} \sqrt{x^2+y^2+z^2}\, dx\, dy\, dz = \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^R r\, r^2 \cos\psi\, dr = \pi R^4.$$

3°. **Applications of triple integrals.** The *volume* of a region of three-dimensional *xyz*-space is

$$V = \iiint_{(V)} dx\, dy\, dz.$$

The *mass* of a solid occupying the region V is

$$M = \iiint_{(V)} \gamma(x, y, z)\, dx\, dy\, dz,$$

where $\gamma(x, y, z)$ is the density of the body at the point (x, y, z).

The *static moments* of the body relative to the coordinate planes are

$$M_{XY} = \iiint_{(V)} \gamma(x, y, z)\, z\, dx\, dy\, dz;$$

$$M_{YZ} = \iiint_{(V)} \gamma(x, y, z)\, x\, dx\, dy\, dz;$$

$$M_{ZX} = \iiint_{(V)} \gamma(x, y, z)\, y\, dx\, dy\, dz.$$

The *coordinates of the centre of gravity* are

$$\bar{x} = \frac{M_{YZ}}{M}, \quad \bar{y} = \frac{M_{ZX}}{M}, \quad \bar{z} = \frac{M_{XY}}{M}.$$

If the solid is homogeneous, then we can put $\gamma(x, y, z) = 1$ in the formulas for the coordinates of the centre of gravity.

The *moments of inertia* relative to the coordinate axes are

$$I_X = \iiint_{(V)} (y^2+z^2)\, \gamma(x, y, z)\, dx\, dy\, dz;$$

$$I_Y = \iiint_{(V)} (z^2+x^2)\, \gamma(x, y, z)\, dx\, dy\, dz;$$

$$I_Z = \iiint_{(V)} (x^2+y^2)\, \gamma(x, y, z)\, dx\, dy\, dz.$$

Putting $\gamma(x, y, z) = 1$ in these formulas, we get the geometric moments of inertia of the body.

A. Evaluating triple integrals

Set up the limits of integration in the triple integral

$$\iiint_{(V)} f(x, y, z)\, dx\, dy\, dz$$

for the indicated regions V.

2240. V is a tetrahedron bounded by the planes

$$x+y+z=1, \quad x=0, \quad y=0, \quad z=0.$$

2241. V is a cylinder bounded by the surfaces

$$x^2+y^2=R^2, \quad z=0, \quad z=H.$$

2242*. V is a cone bounded by the surfaces

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}, \quad z=c.$$

2243. V is a volume bounded by the surfaces

$$z=1-x^2-y^2, \quad z=0.$$

Compute the following integrals:

2244. $\int\limits_0^1 dx \int\limits_0^1 dy \int\limits_0^1 \frac{dz}{\sqrt{x+y+z+1}}$.

2245. $\int\limits_0^2 dx \int\limits_0^{2\sqrt{x}} dy \int\limits_0^{\sqrt{\frac{4x-y^2}{2}}} x\,dz$.

2246. $\int\limits_0^a dx \int\limits_0^{\sqrt{a^2-x^2}} dy \int\limits_0^{\sqrt{a^2-x^2-y^2}} \frac{dz}{\sqrt{a^2-x^2-y^2-z^2}}$.

2247. $\int\limits_0^1 dx \int\limits_0^{1-x} dy \int\limits_0^{1-x-y} xyz\,dz$.

2248. Evaluate

$$\iiint\limits_{(V)} \frac{dx\,dy\,dz}{(x+y+z+1)^3},$$

where V is the region of integration bounded by the coordinate planes and the plane $x+y+z=1$.

2249. Evaluate

$$\iiint\limits_{(V)} (x+y+z)^2\,dx\,dy\,dz,$$

where V (the region of integration) is the common part of the paraboloid $2az \geqslant x^2+y^2$ and the sphere $x^2+y^2+z^2 \leqslant 3a^2$.

2250. Evaluate

$$\iiint\limits_{(V)} z^2\,dx\,dy\,dz.$$

where V (region of integration) is the common part of the spheres $x^2+y^2+z^2 \leqslant R^2$ and $x^2+y^2+z^2 \leqslant 2Rz$

2251. Evaluate

$$\iiint\limits_{(V)} z\,dx\,dy\,dz,$$

where V is a volume bounded by the plane $z=0$ and the upper half of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

2252. Evaluate

$$\iiint\limits_{(V)} \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) dx\,dy\,dz,$$

where V is the interior of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

2253. Evaluate

$$\iiint\limits_{(V)} z\,dx\,dy\,dz,$$

where V (the region of integration) is bounded by the cone $z^2=\frac{h^2}{R^2}(x^2+y^2)$ and the plane $z=h$.

2254. Passing to cylindrical coordinates, evaluate

$$\iiint\limits_{(V)} dx\,dy\,dz,$$

where V is a region bounded by the surfaces $x^2+y^2+z^2=2Rz$, $x^2+y^2=z^2$ and containing the point $(0,0,R)$.

2255. Evaluate

$$\int\limits_0^2 dx \int\limits_0^{\sqrt{2x-x^2}} dy \int\limits_0^a z\sqrt{x^2+y^2}\,dz,$$

first transforming it to cylindrical coordinates.

2256. Evaluate

$$\int\limits_0^{2r} dx \int\limits_{-\sqrt{2rx-x^2}}^{\sqrt{2rx-x^2}} dy \int\limits_0^{\sqrt{4r^2-x^2-y^2}} dz,$$

first transforming it to cylindrical coordinates.

2257. Evaluate

$$\int\limits_{-R}^{R} dx \int\limits_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int\limits_0^{\sqrt{R^2-x^2-y^2}} (x^2+y^2)\,dz,$$

first transforming it to spherical coordinates.

2258. Passing to spherical coordinates, evaluate the integral

$$\iiint\limits_{(V)} \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz,$$

where V is the interior of the sphere $x^2+y^2+z^2 \leqslant x$.

B. *Computing volumes by means of triple integrals*

2259. Use a triple integral to compute the volume of a solid bounded by the surfaces

$$y^2=4a^2-3ax, \quad y^2=ax, \quad z=\pm h.$$

2260.** Compute the volume of that part of the cylinder $x^2+y^2=2ax$ which is contained between the paraboloid $x^2+y^2=2az$ and the xy-plane.

2261*. Compute the volume of a solid bounded by the sphere $x^2+y^2+z^2=a^2$ and the cone $z^2=x^2+y^2$ (external to the cone).

2262*. Compute the volume of a solid bounded by the sphere $x^2+y^2+z^2=4$ and the paraboloid $x^2+y^2=3z$ (internal to the paraboloid).

2263. Compute the volume of a solid bounded by the xy-plane, the cylinder $x^2+y^2=ax$ and the sphere $x^2+y^2+z^2=a^2$ (internal to the cylinder).

2264. Compute the volume of a solid bounded by the paraboloid $\frac{y^2}{b^2}+\frac{z^2}{c^2}=2\frac{x}{a}$ and the plane $x=a$.

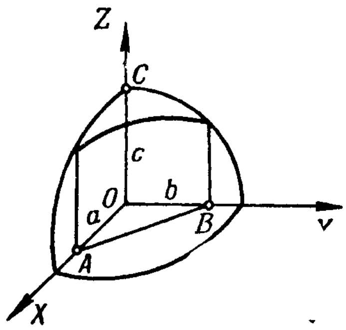

Fig. 100

C. *Applications of triple integrals to mechanics and physics*

2265. Find the mass M of a rectangular parallelepiped $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, $0 \leqslant z \leqslant c$, if the density at the point (x, y, z) is $\rho(x, y, z)=x+y+z$.

2266. Out of an octant of the sphere $x^2+y^2+z^2 \leqslant c^2$, $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ cut a solid $OABC$ bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}=1$ $(a \leqslant c,\ b \leqslant c)$ (Fig. 100). Find the mass of this body if the density at each point (x, y, z) is equal to the z-coordinate of the point.

2267*. In a solid which has the shape of a hemisphere $x^2+y^2+z^2 \leqslant a^2$, $z \geqslant 0$, the density varies in proportion to the distance of the point from the centre. Find the centre of gravity of the solid.

2268. Find the centre of gravity of a solid bounded by the paraboloid $y^2+2z^2=4x$ and the plane $x=2$.

2269*. Find the moment of inertia of a circular cylinder, whose altitude is h and the radius of the base is a, relative to the axis which serves as the diameter of the base of the cylinder.

2270*. Find the moment of inertia of a circular cone (altitude, h, radius of base, a, and density ϱ) relative to the diameter of the base.

2271.** Find the force of attraction exerted by a homogeneous cone of altitude h and vertex angle α (in axial cross-section) on a material point containing unit mass and located at its vertex.

2272.** Show that the force of attraction exerted by a homogeneous sphere on an external material point does not change if the entire mass of the sphere is concentrated at its centre.

Sec. 8. Improper Integrals Dependent on a Parameter. Improper Multiple Integrals

1°. Differentiation with respect to a parameter. In the case of certain restrictions imposed on the functions $f(x, \alpha)$, $f'_\alpha(x, \alpha)$ and on the corresponding improper integrals we have the *Leibniz rule*

$$\frac{d}{d\alpha}\int_a^\infty f(x, \alpha)\,dx=\int_a^\infty f'_\alpha(x, \alpha)\,dx.$$

Example 1. By differentiating with respect to a parameter, evaluate

$$\int_0^\infty \frac{e^{-\alpha x^2}-e^{-\beta x^2}}{x}\,dx \quad (\alpha>0,\ \beta>0).$$

Solution. Let

$$\int_0^\infty \frac{e^{-\alpha x^2}-e^{-\beta x^2}}{x}\,dx=F(\alpha, \beta).$$

Then

$$\frac{\partial F(\alpha, \beta)}{\partial\alpha}=-\int_0^\infty xe^{-\alpha x^2}\,dx=\frac{1}{2\alpha}e^{-\alpha x^2}\Big|_0^\infty=-\frac{1}{2\alpha}.$$

Whence $F(\alpha, \beta)=-\frac{1}{2}\ln\alpha+C(\beta)$. To find $C(\beta)$, we put $\alpha=\beta$ in the latter equation. We have $0=-\frac{1}{2}\ln\beta+C(\beta)$.

Whence $C(\beta)=\frac{1}{2}\ln\beta$. Hence,

$$F(\alpha, \beta)=-\frac{1}{2}\ln\alpha+\frac{1}{2}\ln\beta=\frac{1}{2}\ln\frac{\beta}{\alpha}.$$

2°. Improper double and triple integrals.

a) An infinite region. If a function $f(x, y)$ is continuous in an unbounded region S, then we put

$$\iint_{(S)} f(x, y)\,dx\,dy = \lim_{\sigma \to S} \iint_{(\sigma)} f(x, y)\,dx\,dy, \tag{1}$$

where σ is a finite region lying entirely within S, where $\sigma \to S$ signifies that we expand the region σ by an arbitrary law so that any point of S should enter it and remain in it. If there is a limit on the right and if it does not depend on the choice of the region σ, then the corresponding improper integral is called *convergent*, otherwise it is *divergent*.

If the integrand $f(x, y)$ is nonnegative $[f(x, y) \geqslant 0]$, then for the convergence of an improper integral it is necessary and sufficient for the limit on the right of (1) to exist at least for one system of regions σ that exhaust the region S.

b) A discontinuous function. If a function $f(x, y)$ is everywhere continuous in a bounded closed region S, except the point $P(a, b)$, then we put

$$\iint_{(S)} f(x, y)\,dx\,dy = \lim_{\varepsilon \to 0} \iint_{(S_\varepsilon)} f(x, y)\,dx\,dy, \tag{2}$$

where S_ε is a region obtained from S by eliminating a small region of diameter ε that contains the point P. If (2) has a limit that does not depend on the type of small regions eliminated from S, the improper integral under consideration is called *convergent*, otherwise it is *divergent*.

If $f(x, y) \gtrless 0$, then the limit on the right of (2) is not dependent on the type of regions eliminated from S; for instance, such regions may be circles of radius $\frac{\varepsilon}{2}$ with centre at P.

The concept of improper double integrals is readily extended to the case of triple integrals.

Example 2. Test for convergence

$$\iint_{(S)} \frac{dx\,dy}{(1+x^2+y^2)^p}, \tag{3}$$

where S is the entire xy-plane.

Solution. Let σ be a circle of radius ϱ with centre at the coordinate origin. Passing to polar coordinates for $p \neq 1$, we have

$$I(\sigma) = \iint_{(\sigma)} \frac{dx\,dy}{(1+x^2+y^2)^p} = \int_0^{2\pi} d\varphi \int_0^{\varrho} \frac{r\,dr}{(1+r^2)^p} =$$

$$= \int_0^{2\pi} \frac{1}{2}\frac{(1+r^2)^{1-p}}{1-p}\Big|_0^{\varrho}\,d\varphi = \frac{\pi}{1-p}\left[(1+\varrho^2)^{1-p} - 1\right].$$

If $p < 1$, then $\lim\limits_{\sigma \to S} I(\sigma) = \lim\limits_{\varrho \to \infty} I(\sigma) = \infty$ and the integral diverges. But if $p > 1$, then $\lim\limits_{\varrho \to \infty} I(\sigma) = \frac{\pi}{p-1}$ and the integral converges. For $p = 1$ we have

$I(\sigma)=\int\limits_0^{2\pi} d\varphi\int\limits_0^{\varrho}\frac{r\,dr}{1+r^2}=\pi\ln(1+\varrho^2)$; $\lim\limits_{\varrho\to\infty} I(\sigma)=\infty$, that is, the integral diverges.

Thus, the integral (3) converges for $p>1$.

2273. Find $f'(x)$, if

$$f(x)=\int\limits_x^{\infty} e^{-xy^2}\,dy\quad (x>0).$$

2274. Prove that the function

$$u=\int\limits_{-\infty}^{+\infty}\frac{xf(z)}{x^2+(y-z)^2}\,dz$$

satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

2275. The Laplace transformation $F(p)$ for the function $f(t)$ is defined by the formula

$$F(p)=\int\limits_0^{\infty} e^{-pt}f(t)\,dt.$$

Find $F(p)$, if: a) $f(t)=1$; b) $f(t)=e^{\alpha t}$; c) $f(t)=\sin\beta t$; d) $f(t)=\cos\beta t$.

2276. Taking advantage of the formula

$$\int\limits_0^1 x^{n-1}dx=\frac{1}{n}\ (n>0),$$

compute the integral

$$\int\limits_0^1 x^{n-1}\ln x\,dx.$$

2277*. Using the formula

$$\int\limits_0^{\infty} e^{-pt}\,dt=\frac{1}{p}\quad (p>0),$$

evaluate the integral

$$\int\limits_0^{\infty} t^2e^{-pt}\,dt.$$

Applying differentiation with respect to a parameter, evaluate the following integrals:

2278. $\int_0^\infty \frac{e^{-\alpha x}-e^{-\beta x}}{x}\,dx \quad (\alpha>0,\ \beta>0)$.

2279. $\int_0^\infty \frac{e^{-\alpha x}-e^{-\beta x}}{x}\sin mx\,dx \quad (\alpha>0,\ \beta>0)$.

2280. $\int_0^\infty \frac{\arctan \alpha x}{x(1+x^2)}\,dx$.

2281. $\int_0^1 \frac{\ln(1-\alpha^2x^2)}{x^2\sqrt{1-x^2}}\,dx \quad (|\alpha|<1)$.

2282. $\int_0^\infty e^{-\alpha x}\frac{\sin\beta x}{x}\,dx \quad (\alpha\geqslant 0)$.

Evaluate the following improper integrals:

2283. $\int_0^\infty dx\int_0^\infty e^{-(x+y)}\,dy$.

2284. $\int_0^1 dy\int_0^{y^2} e^{\frac{x}{y}}\,dx$.

2285. $\iint\limits_{(S)} \frac{dx\,dy}{x^4+y^2}$, where S is a region defined by the inequalities $x\geqslant 1,\ y\geqslant x^2$.

2286*. $\int_0^\infty dx\int_0^\infty \frac{dy}{(x^2+y^2+a^2)^2} \quad (a>0)$.

2287. The *Euler-Poisson integral* defined by the formula $I=\int_0^\infty e^{-x^2}dx$ may also be written in the form $I=\int_0^\infty e^{-y^2}dy$. Evaluate I by multiplying these formulas and then passing to polar coordinates.

2288. Evaluate

$$\int_0^\infty dx\int_0^\infty dy\int_0^\infty \frac{dz}{(x^2+y^2+z^2+1)^2}.$$

Test for convergence the improper double integrals:

2289**. $\iint\limits_{(S)} \ln \sqrt{x^2+y^2}\,dx\,dy$, where S is a circle $x^2+y^2 \leqslant 1$.

2290. $\iint\limits_{(S)} \frac{dx\,dy}{(x^2+y^2)^{\alpha}}$, where S is a region defined by the inequality $x^2+y^2 \geqslant 1$ ("exterior" of the circle).

2291*. $\iint\limits_{(S)} \frac{dx\,dy}{\sqrt[3]{(x-y)^2}}$, where S is a square $|x| \leqslant 1$, $|y| \leqslant 1$.

2292. $\iiint\limits_{(V)} \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{\alpha}}$, where V is a region defined by the inequality $x^2+y^2+z^2 \geqslant 1$ ("exterior" of a sphere).

Sec. 9. Line Integrals

1°. Line integrals of the first type. Let $f(x, y)$ be a continuous function and $y=\varphi(x)$ $[a \leqslant x \leqslant b]$ be the equation of some smooth curve C.

Let us construct a system of points $M_i(x_i, y_i)$ $(i=0, 1, 2, \ldots, n)$ that break up the curve C into elementary arcs $\overset{\frown}{M_{i-1}M_i}=\Delta s_i$ and let us form the integral sum $S_n=\sum\limits_{i=1}^{n} f(x_i, y_i)\,\Delta s_i$. The limit of this sum, when $n \to \infty$ and $\max \Delta s_i \to 0$, is called a *line integral of the first type*

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta s_i = \int\limits_C f(x, y)\,ds$$

(ds is the arc differential) and is evaluated from the formula

$$\int\limits_C f(x, y)\,ds = \int\limits_a^b f(x, \varphi(x))\sqrt{1+(\varphi'(x))^2}\,dx.$$

In the case of parametric representation of the curve C: $x=\varphi(t)$, $y=\psi(t)$ $[\alpha \leqslant t \leqslant \beta]$, we have

$$\int\limits_C f(x, y)\,ds = \int\limits_{\alpha}^{\beta} f(\varphi(t), \psi(t))\sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt.$$

Also considered are line integrals of the first type of functions of three variables $f(x, y, z)$ taken along a space curve. These integrals are evaluated in like fashion. A line integral of the first type *does not depend on the direction of the path of integration*; if the integrand f is interpreted as a linear density of the curve of integration C, then this integral represents the *mass of the curve* C.

Example 1. Evaluate the line integral

$$\int_C (x+y)\,ds,$$

where C is the contour of the triangle ABO with vertices $A(1, 0)$, $B(0, 1)$, and $O(0, 0)$ (Fig 101).

Solution. Here, the equation AB is $y=1-x$, the equation OB is $x=0$, and the equation OA is $y=0$. We therefore have

$$\int_C (x+y)\,ds=\int_{AB}(x+y)\,ds+\int_{BO}(x+y)\,ds+\int_{OA}(x+y)\,ds=$$

$$=\int_0^1 \sqrt{2}\,dx+\int_0^1 y\,dy+\int_0^1 x\,dx=\sqrt{2}+1.$$

2°. Line integrals of the second type. If $P(x, y)$ and $Q(x, y)$ are continuous functions and $y=\varphi(x)$ is a smooth curve C that runs from a to b as

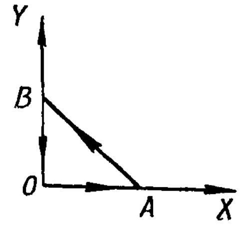

Fig. 101

x varies, then the corresponding *line integral of the second type* is expressed as follows:

$$\int_C P(x, y)\,dx+Q(x, y)\,dy=\int_a^b [P(x, \varphi(x))+\varphi'(x)\,Q(x, \varphi(x))]\,dx.$$

In the more general case when the curve C is represented parametrically: $x=\varphi(t)$, $y=\psi(t)$, where t varies from α to β, we have

$$\int_C P(x, y)\,dx+Q(x, y)\,dy+\int_\alpha^\beta [P(\varphi(t), \psi(t))\,\varphi'(t)+Q(\varphi(t), \psi(t))\,\psi'(t)]\,dt.$$

Similar formulas hold for a line integral of the second type taken over a space curve.

A line integral of the second type *changes sign when the direction of the path of integration is reversed*. This integral may be interpreted mechanically as the work of an appropriate variable force $\{P(x, y), Q(x, y)\}$ along the curve of integration C

Example 2. Evaluate the line integral

$$\int_C y^2\,dx+x^2\,dy,$$

where C is the upper half of the ellipse $x=a\cos t$, $y=b\sin t$ traversed clockwise.

Solution. We have

$$\int\limits_C y^2\,dx+x^2\,dy=\int\limits_{\pi}^{0}[b^2\sin^2 t\cdot(-a\sin t)+a^2\cos^2 t\cdot b\cos t]\,dt=$$
$$=-ab^2\int\limits_{\pi}^{0}\sin^3 t\,dt+a^2b\int\limits_{\pi}^{0}\cos^3 t\,dt=\frac{4}{3}ab^2.$$

3°. The case of a total differential. If the integrand of a line integral of the second type is a total differential of some single-valued function $U=U(x, y)$, that is, $P(x, y)\,dx+Q(x, y)\,dy=dU(x, y)$, then this line integral is not dependent on the path of integration and we have the Newton-Leibniz formula

$$\int\limits_{(x_1, y_1)}^{(x_2, y_2)} P(x, y)\,dx+Q(x, y)\,dy=U(x_2, y_2)-U(x_1, y_1), \tag{1}$$

where (x_1, y_1) is the initial and (x_2, y_2) is the terminal point of the path In particular, if the contour of integration C is closed, then

$$\oint\limits_C P(x, y)\,dx+Q(x, y)\,dy=0 \tag{2}$$

If 1) the contour of integration C is contained entirely within some simply-connected region S and 2) the functions $P(x, y)$ and $Q(x, y)$ together with their partial derivatives of the first order are continuous in S, then a necessary and sufficient condition for the existence of the function U is the identical fulfilment (in S) of the equality

$$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y} \tag{3}$$

(see integration of total differentials) If conditions one and two are not fulfilled, the presence of condition (3) does not guarantee the existence of a single-valued function U, and formulas (1) and (2) may prove wrong (see Problem 2332) We give a method of finding a function $U(x, y)$ from its total differential based on the use of line integrals (which is yet another method of integrating a total differential). For the contour of integration C let us take a broken line P_0P_1M (Fig 102), where $P_0(x_0, y_0)$ is a fixed point and $M(x, y)$ is a variable point. Then along P_0P_1 we have $y=y_0$ and $dy=0$, and along P_1M we have $dx=0$ We get:

$$U(x, y)-U(x_0, y_0)=\int\limits_{(x_0, y_0)}^{(x, y)} P(x, y)\,dx+Q(x, y)\,dy=$$
$$=\int\limits_{x_0}^{x} P(x, y_0)\,dx+\int\limits_{y_0}^{y} Q(x, y)\,dy.$$

Similarly, integrating with respect to P_0P_2M, we have

$$U(x, y)-U(x_0, y_0)=\int\limits_{y_0}^{y} Q(x_0, y)\,dy+\int\limits_{x_0}^{x} P(x, y)\,dx.$$

Example 3. $(4x+2y)\,dx+(2x-6y)\,dy=dU$. Find U.

Solution. Let $x_0=0$, $y_0=0$. Then

$$U(x, y)=\int_0^x 4x\,dx+\int_0^y (2x-6y)\,dy+C=2x^2+2xy-3y^2+C$$

or

$$U(x, y)=\int_0^y -6y\,dy+\int_0^x (4x+2y)\,dx+C=-3y^2+2x^2+3xy+C,$$

where $C=U(0, 0)$ is an arbitrary constant.

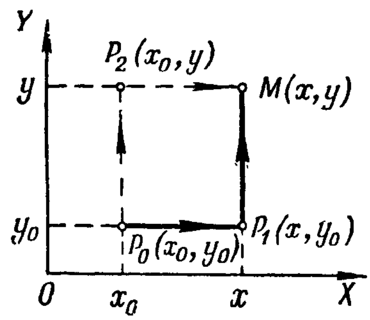

Fig. 102

4°. **Green's formula for a plane.** If C is the boundary of a region S and the functions $P(x, y)$ and $Q(x, y)$ are continuous together with their first-order partial derivatives in the closed region $S+C$, then *Green's formula* holds:

$$\oint_C P\,dx+Q\,dy=\iint_{(S)}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy,$$

here the circulation about the contour C is chosen so that the region S should remain to the left.

5°. **Applications of line integrals.**[1]) An *area* bounded by the closed contour C is

$$S=-\oint_C y\,dx=\oint_C x\,dy$$

(the direction of circulation of the contour is chosen counterclockwise).

The following formula for area is more convenient for application:

$$S=\frac{1}{2}\oint_C (x\,dy-y\,dx)=\frac{1}{2}\oint_C x^2\,d\left(\frac{y}{x}\right).$$

2) The *work of a force*, having projections $X=X(x, y, z)$, $Y=Y(x, y, z)$, $Z=Z(x, y, z)$ (or, accordingly, the work of a force field), along a path C is

expressed by the integral

$$A=\int_C X\,dx+Y\,dy+Z\,dz.$$

If the force has a potential, i.e., if there exists a function $U=U(x, y, z)$ (a potential function or a force function) such that

$$\frac{\partial U}{\partial x}=X,\ \frac{\partial U}{\partial y}=Y,\ \frac{\partial U}{\partial z}=Z,$$

then the work, irrespective of the shape of the path C, is equal to

$$A=\int_{(x_1,\,y_1,\,z_1)}^{(x_2,\,y_2,\,z_2)} X\,dx+Y\,dy+Z\,dz=\int_{(x_1,\,y_1,\,z_1)}^{(x_2,\,y_2,\,z_2)} dU=U(x_2,\ y_2,\ z_2)-U(x_1,\ y_1\ z_1),$$

where (x_1, y_1, z_1) is the initial and (x_2, y_2, z_2) is the terminal point of the path.

A. *Line Integrals of the First Type*

Evaluate the following line integrals:

2293. $\int_C xy\,ds$, where C is the contour of the square $|x|+|y|=a$ $(a>0)$.

2294. $\int_C \frac{ds}{\sqrt{x^2+y^2+4}}$, where C is a segment of the straight line connecting the points $O(0,\ 0)$ and $A(1,\ 2)$.

2295. $\int_C xy\,ds$, where C is a quarter of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ lying in the first quadrant.

2296. $\int_C y^2ds$, where C is the first arc of the cycloid $x=a(t-\sin t)$, $y=a(1-\cos t)$.

2297. $\int_C \sqrt{x^2+y^2}\,ds$, where C is an arc of the involute of the circle $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$ $[0\leqslant t\leqslant 2\pi]$.

2298. $\int_C (x^2+y^2)^2\,ds$, where C is an arc of the logarithmic spiral $r=ae^{m\varphi}$ $(m>0)$ from the point $A(0,\ a)$ to the point $O(-\infty,\ 0)$.

2299. $\int_C (x+y)\,ds$, where C is the right-hand loop of the lemniscate $r^2=a^2\cos 2\varphi$.

2300. $\int_C (x+y)\,ds$, where C is an arc of the curve $x=t$, $y=\frac{3t^2}{\sqrt{2}}$, $z=t^3$ $(0\leqslant t\leqslant 1]$.

2301. $\int\limits_C \frac{ds}{x^2+y^2+z^2}$, where C is the first turn of the screw-line $x=a\cos t$, $y=a\sin t$, $z=bt$.

2302. $\int\limits_C \sqrt{2y^2+z^2}\,ds$, where C is the circle $x^2+y^2+z^2=a^2$, $x=y$.

2303*. Find the area of the lateral surface of the parabolic cylinder $y=\frac{3}{8}x^2$ bounded by the planes $z=0$, $x=0$, $z=x$, $y=6$.

2304. Find the arc length of the conic screw-line C $x=ae^t\cos t$, $y=ae^t\sin t$, $z=ae^t$ from the point $O(0, 0, 0)$ to the point $A(a, 0, a)$.

2305. Determine the mass of the contour of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, if the linear density of it at each point $M(x, y)$ is equal to $|y|$.

2306. Find the mass of the first turn of the screw-line $x=a\cos t$, $y=a\sin t$, $z=bt$, if the density at each point is equal to the radius vector of this point.

2307. Determine the coordinates of the centre of gravity of a half-arc of the cycloid

$$x=a(t-\sin t),\ y=a(1-\cos t)\quad [0\leqslant t\leqslant\pi].$$

2308. Find the moment of inertia, about the z-axis, of the first turn of the screw-line $x=a\cos t$, $y=a\sin t$, $z=bt$.

2309. With what force will a mass M distributed with uniform density over the circle $x^2+y^2=a^2$, $z=0$, act on a mass m located at the point $A(0, 0, b)$?

B. *Line Integrals of the Second Type*

Evaluate the following line integrals:

2310. $\int\limits_{AB}(x^2-2xy)\,dx+(2xy+y^2)\,dy$, where AB is an arc of the parabola $y=x^2$ from the point $A(1, 1)$ to the point $B(2, 4)$.

2311. $\int\limits_C (2a-y)\,dx+x\,dy$, where C is an arc of the first arch of the cycloid

$$x=a(t-\sin t),\ y=a(1-\cos t)$$

which arc runs in the direction of increasing parameter t.

2312. $\int\limits_{OA} 2xy\,dx-x^2\,dy$ taken along different paths emanating from the coordinate origin $O(0, 0)$ and terminating at the point $A(2, 1)$ (Fig. 103):

a) the straight line OmA;

b) the parabola OnA, the axis of symmetry of which is the y-axis;
c) the parabola OpA, the axis of symmetry of which is the x-axis;
d) the broken line OBA;
e) the broken line OCA.

2313. $\int\limits_{OA} 2xy\,dx + x^2\,dy$ as in Problem 2312.

2314*. $\oint \frac{(x+y)\,dx-(x-y)\,dy}{x^2+y^2}$ taken along the circle $x^2+y^2=a^2$ counterclockwise.

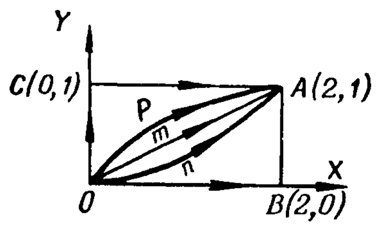

Fig. 103

2315. $\int\limits_{C} y^2\,dx + x^2\,dy$, where C is the upper half of the ellipse $x=a\cos t$, $y=b\sin t$ traced clockwise.

2316. $\int\limits_{AB} \cos y\,dx - \sin x\,dy$ taken along the segment AB of the bisector of the second quadrantal angle, if the abscissa of the point A is 2 and the ordinate of B is 2.

2317. $\oint\limits_{C} \frac{xy(y\,dx - x\,dy)}{x^2+y^2}$, where C is the right-hand loop of the lemniscate $r^2=a^2\cos 2\varphi$ traced counterclockwise.

2318. Evaluate the line integrals with respect to expressions which are total differentials:

a) $\int\limits_{(-1,\,2)}^{(2,\,3)} x\,dy + y\,dx$, b) $\int\limits_{(0,\,1)}^{(3,\,4)} x\,dx + y\,dy$, c) $\int\limits_{(0,\,0)}^{(1,\,1)} (x+y)(dx+dy)$,

d) $\int\limits_{(1,\,2)}^{(2,\,1)} \frac{y\,dx - x\,dy}{y^2}$ (along a path that does not intersect the x-axis),

e) $\int\limits_{\left(\frac{1}{2},\,\frac{1}{2}\right)}^{(x,\,y)} \frac{dx+dy}{x+y}$ (along a path that does not intersect the straight line $x+y=0$),

f) $\int\limits_{(x_1,\,y_1)}^{(x_2,\,y_2)} \varphi(x)\,dx+\psi(y)\,dy$.

2319. Find the antiderivative functions of the integrands and evaluate the integrals:

a) $\int\limits_{(-2,\,-1)}^{(3,\,0)} (x^4+4xy^3)\,dx+(6x^2y^2-5y^4)\,dy$,

b) $\int\limits_{(0,\,-1)}^{(1,\,0)} \frac{x\,dy-y\,dx}{(x-y)^2}$ (the integration path does not intersect the straight line $y=x$),

c) $\int\limits_{(1,\,1)}^{(3,\,1)} \frac{(x+2y)\,dx+y\,dy}{(x+y)^2}$ (the integration path does not intersect the straight line $y=-x$),

d) $\int\limits_{(0,\,0)}^{(1,\,1)} \left(\frac{x}{\sqrt{x^2+y^2}}+y\right)dx+\left(\frac{y}{\sqrt{x^2+y^2}}+x\right)dy$.

2320. Compute

$$I=\int \frac{x\,dx+y\,dy}{\sqrt{1+x^2+y^2}},$$

taken clockwise along the quarter of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ that lies in the first quadrant.

2321. Show that if $f(u)$ is a continuous function and C is a closed piecewise-smooth contour, then

$$\oint_C f(x^2+y^2)(x\,dx+y\,dy)=0.$$

2322. Find the antiderivative function U if:

a) $du=(2x+3y)\,dx+(3x-4y)\,dy$;

b) $du=(3x^2-2xy+y^2)\,dx-(x^2-2xy+3y^2)\,dy$;

c) $du=e^{x-y}[(1+x+y)\,dx+(1-x-y)\,dy]$;

d) $du=\frac{dx}{x+y}+\frac{dy}{x+y}$.

Evaluate the line integrals taken along the following space curves:

2323. $\int\limits_C (y-z)\,dx+(z-x)\,dy+(x-y)\,dz$, where C is a turn of the screw-line

$$\begin{cases} x=a\cos t, \\ y=a\sin t, \\ z=bt, \end{cases}$$

corresponding to the variation of the parameter t from 0 to 2π.

2324. $\oint\limits_C y\,dx+z\,dy+x\,dz$, where C is the circle

$$\begin{cases} x=R\cos\alpha\cos t, \\ y=R\cos\alpha\sin t, \\ z=R\sin\alpha\ (\alpha=\text{const}), \end{cases}$$

traced in the direction of increasing parameter.

2325. $\int\limits_{OA} xy\,dx+yz\,dy+zx\,dz$, where OA is an arc of the circle

$$x^2+y^2+z^2=2Rx,\ z=x,$$

situated on the side of the xz-plane where $y>0$.

2326. Evaluate the line integrals of the total differentials:

a) $\int\limits_{(1,\ 0,\ -3)}^{(6,\ 4,\ 8)} x\,dx+y\,dy-z\,dz$,

b) $\int\limits_{(1,\ 1,\ 1)}^{(a,\ b,\ c)} yz\,dx+zx\,dy+xy\,dz$,

c) $\int\limits_{(0,\ 0,\ 0)}^{(3,\ 4,\ 5)} \dfrac{x\,dx+y\,dy+z\,dz}{\sqrt{x^2+y^2+z^2}}$,

d) $\int\limits_{(1,\ 1,\ 1)}^{\left(x,\ y,\ \frac{1}{xy}\right)} \dfrac{yz\,dx+zx\,dy+xy\,dz}{xyz}$ (the integration path is situated in the first octant).

C. *Green's Formula*

2327. Using Green's formula, transform the line integral

$$I=\oint\limits_C \sqrt{x^2+y^2}\,dx+y\left[xy+\ln\left(x+\sqrt{x^2+y^2}\right)\right]dy,$$

where the contour C bounds the region S.

2328. Applying Green's formula, evaluate

$$I=\oint_C 2(x^2+y^2)\,dx+(x+y)^2\,dy,$$

where C is the contour of a triangle (traced in the positive direction) with vertices at the points $A(1, 1)$, $B(2, 2)$ and $C(1, 3)$. Verify the result obtained by computing the integral directly.

2329. Applying Green's formula, evaluate the integral

$$\oint_C -x^2y\,dx+xy^2\,dy,$$

where C is the circle $x^2+y^2=R^2$ traced counterclockwise.

2330. A parabola AmB, whose axis is the y-axis and whose chord is AnB, is drawn through the points $A(1, 0)$ and $B(2, 3)$. Find $\oint\limits_{AmBnA}(x+y)\,dx-(x-y)\,dy$ directly and by applying Green's formula.

2331. Find $\int\limits_{AmB} e^{xy}[y^2\,dx+(1+xy)\,dy]$, if the points A and B lie on the x-axis, while the area, bounded by the integration path AmB and the segment AB, is equal to S.

2332*. Evaluate $\oint\limits_C \frac{x\,dy-y\,dx}{x^2+y^2}$. Consider two cases:

a) when the origin is outside the contour C,
b) when the contour encircles the origin n times.

2333.** Show that if C is a closed curve, then

$$\oint_C \cos(X,\ n)\,ds=0,$$

where s is the arc length and n is the outer normal.

2334. Applying Green's formula, find the value of the integral

$$I=\oint_C [x\cos(X,\ n)+y\sin(X,\ n)]\,ds,$$

where ds is the differential of the arc and n is the outer normal to the contour C.

2335*. Evaluate the integral

$$\oint_C \frac{dx-dy}{x+y},$$

taken along the contour of a square with vertices at the points $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$, provided the contour is traced counterclockwise.

D. *Applications of the Line Integral*

Evaluate the areas of figures bounded by the following curves:

2336. The ellipse $x = a\cos t$, $y = b\sin t$.

2337. The astroid $x = a\cos^3 t$, $y = a\sin^3 t$.

2338. The cardioid $x = a(2\cos t - \cos 2t)$, $y = a(2\sin t - \sin 2t)$.

2339*. A loop of the folium of Descartes $x^3 + y^3 - 3axy = 0$ $(a > 0)$.

2340. The curve $(x + y)^3 = axy$.

2341*. A circle of radius r is rolling without sliding along a fixed circle of radius R and outside it. Assuming that $\frac{R}{r}$ is an integer, find the area bounded by the curve (epicycloid) described by some point of the moving circle. Analyze the particular case of $r = R$ (cardioid).

2342*. A circle of radius r is rolling without sliding along a fixed circle of radius R and inside it. Assuming that $\frac{R}{r}$ is an integer, find the area bounded by the curve (hypocycloid) described by some point of the moving circle. Analyze the particular case when $r = \frac{R}{4}$ (astroid).

2343. A field is generated by a force of constant magnitude F in the positive x-direction Find the work that the field does when a material point traces clockwise a quarter of the circle $x^2 + y^2 = R^2$ lying in the first quadrant.

2344. Find the work done by the force of gravity when a material point of mass m is moved from position $A(x_1, y_1, z_1)$ to position $B(x_2, y_2, z_2)$ (the z-axis is directed vertically upwards).

2345. Find the work done by an elastic force directed towards the coordinate origin if the magnitude of the force is proportional to the distance of the point from the origin and if the point of application of the force traces counterclockwise a quarter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ lying in the first quadrant.

2346. Find the potential function of a force $R\{X, Y, Z\}$ and determine the work done by the force over a given path if:

a) $X = 0$, $Y = 0$, $Z = -mg$ (force of gravity) and the material point is moved from position $A(x_1, y_1, z_1)$ to position $B(x_2, y_2, z_2)$;

b) $X = -\frac{\mu x}{r^3}$, $Y = -\frac{\mu y}{r^3}$, $Z = -\frac{\mu z}{r^3}$, where $\mu = \text{const}$ and $r = \sqrt{x^2 + y^2 + z^2}$ (Newton attractive force) and the material point moves from position $A(a, b, c)$ to infinity;

c) $X=-k^2x$, $Y=-k^2y$, $Z=-k^2z$, where $k=\text{const}$ (elastic force), and the initial point of the path is located on the sphere $x^2+y^2+z^2=R^2$, while the terminal point is located on the sphere $x^2+y^2+z^2=r^2$ $(R>r)$.

Sec. 10. Surface Integrals

1°. Surface integral of the first type. Let $f(x, y, z)$ be a continuous function and $z=\varphi(x, y)$ a smooth surface S.

The *surface integral of the first type* is the limit of the integral sum

$$\iint\limits_S f(x, y, z)\,dS=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta S_i,$$

where ΔS_i is the area of the ith element of the surface S, the point (x_i, y_i, z_i) belongs to this element, and the maximum diameter of elements of partition tends to zero.

The value of this integral is not dependent on the choice of side of the surface S over which the integration is performed.

If a projection σ of the surface S on the xy-plane is single-valued, that is, every straight line parallel to the z-axis intersects the surface S at only one point, then the appropriate surface integral of the first type may be calculated from the formula

$$\iint\limits_S f(x, y, z)\,dS=\iint\limits_{(\sigma)} f[x, y, \varphi(x, y)]\sqrt{1+\varphi_x'^2(x, y)+\varphi_y'^2(x, y)}\,dx\,dy.$$

Example 1. Compute the surface integral

$$\iint\limits_S (x+y+z)\,dS,$$

where S is the surface of the cube $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$, $0\leqslant z\leqslant 1$.

Let us compute the sum of the surface integrals over the upper edge of the cube $(z=1)$ and over the lower edge of the cube $(z=0)$:

$$\int_0^1\!\!\int_0^1 (x+y+1)\,dx\,dy+\int_0^1\!\!\int_0^1 (x+y)\,dx\,dy=\int_0^1\!\!\int_0^1 (2x+2y+1)\,dx\,dy=3.$$

The desired surface integral is obviously three times greater and equal to

$$\iint\limits_S (x+y+z)\,dS=9.$$

2°. Surface integral of the second type. If $P=P(x, y, z)$, $Q=Q(x, y, z)$, $R=R(x, y, z)$ are continuous functions and S^+ is a side of the smooth surface S characterized by the direction of the normal $n\{\cos\alpha, \cos\beta, \cos\gamma\}$, then the corresponding *surface integral of the second type* is expressed as follows:

$$\iint\limits_{S+} P\,dy\,dz+Q\,dz\,dx+R\,dx\,dy=\iint\limits_S (P\cos\alpha+Q\cos\beta+R\cos\gamma)\,dS.$$

When we pass to the other side, S^{-}, of the surface, this integral reverses sign.

If the surface S is represented implicitly, $F(x, y, z)=0$, then the direction cosines of the normal of this surface are determined from the formulas

$$\cos\alpha=\frac{1}{D}\frac{\partial F}{\partial x}, \quad \cos\beta=\frac{1}{D}\frac{\partial F}{\partial y}, \quad \cos\gamma=\frac{1}{D}\frac{\partial F}{\partial z},$$

where

$$D=\pm\sqrt{\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+\left(\frac{\partial F}{\partial z}\right)^2},$$

and the choice of sign before the radical should be brought into agreement with the side of the surface S.

3°. **Stokes' formula.** If the functions $P=P(x, y, z)$, $Q=Q(x, y, z)$, $R=R(x, y, z)$ are continuously differentiable and C is a closed contour bounding a two-sided surface S, we then have the *Stokes' formula*

$$\oint_C P\,dx+Q\,dy+R\,dz=$$

$$=\iint_S\left[\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\cos\alpha+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\cos\beta+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\cos\gamma\right]dS,$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the normal to the surface S, and the direction of the normal is defined so that on the side of the normal the contour S is traced counterclockwise (in a right-handed coordinate system).

Evaluate the following surface integrals of the first type:

2347. $\iint_S (x^2+y^2)\,dS$, where S is the sphere $x^2+y^2+z^2=a^2$.

2348. $\iint_S \sqrt{x^2+y^2}\,dS$ where S is the lateral surface of the cone $\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{b^2}=0$ $[0\leqslant z\leqslant b]$.

Evaluate the following surface integrals of the second type:

2349. $\iint_S yz\,dy\,dz+xz\,dz\,dx+xy\,dx\,dy$, where S is the external side of the surface of a tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$, $x+y+z=a$.

2350. $\iint_S z\,dx\,dy$, where S is the external side of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

2351. $\iint_S x^2\,dy\,dz+y^2\,dz\,dx+z^2\,dx\,dy$, where S is the external side of the surface of the hemisphere $x^2+y^2+z^2=a^2$ $(z\geqslant 0)$.

2352. Find the mass of the surface of the cube $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$, $0\leqslant z\leqslant 1$, if the surface density at the point $M(x, y, z)$ is equal to xyz.

2353. Determine the coordinates of the centre of gravity of a homogeneous parabolic envelope $az = x^2 + y^2$ $(0 \leqslant z \leqslant a)$.

2354. Find the moment of inertia of a part of the lateral surface of the cone $z = \sqrt{x^2 + y^2}$ $[0 \leqslant z \leqslant h]$ about the z-axis.

2355. Applying Stokes' formula, transform the integrals:

a) $\oint\limits_C (x^2 - yz)\,dx + (y^2 - zx)\,dy + (z^2 - xy)\,dz$;

b) $\oint\limits_C y\,dx + z\,dy + x\,dz$.

Applying Stokes' formula, find the given integrals and verify the results by direct calculations:

2356. $\oint\limits_C (y + z)\,dx + (z + x)\,dy + (x + y)\,dz$, where C is the circle

$$x^2 + y^2 + z^2 = a^2, \quad x + y + z = 0.$$

2357. $\oint\limits_C (y - z)\,dx + (z - x)\,dy + (x - y)\,dz$, where C is the ellipse

$$x^2 + y^2 = 1, \quad x + z = 1.$$

2358. $\oint\limits_C x\,dx + (x + y)\,dy + (x + y + z)\,dz$, where C is the curve $x = a\sin t$, $y = a\cos t$, $z = a(\sin t + \cos t)$ $[0 \leqslant t \leqslant 2\pi]$.

2359. $\oint\limits_{ABCA} y^2\,dx + z^2\,dy + x^2\,dz$, where $ABCA$ is the contour of $\triangle ABC$ with vertices $A(a, 0, 0)$, $B(0, a, 0)$, $C(0, 0, a)$.

2360. In what case is the line integral

$$I = \oint\limits_C P\,dx + Q\,dy + R\,dz$$

over any closed contour C equal to zero?

Sec. 11. The Ostrogradsky-Gauss Formula

If S is a closed smooth surface bounding the volume V, and $P = P(x, y, z)$, $Q = Q(x, y, z)$, $R = R(x, y, z)$ are functions that are continuous together with their first partial derivatives in the closed region V, then we have the *Ostrogradsky-Gauss formula*

$$\iint\limits_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,dS = \iiint\limits_{(V)} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz,$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the outer normal to the surface S

Applying the Ostrogradsky-Gauss formula, transform the following surface integrals over the closed surfaces S bounding the volume V ($\cos\alpha$, $\cos\beta$, $\cos\gamma$ are direction cosines of the outer normal to the surface S).

2361. $\iint\limits_S xy\,dx\,dy + yz\,dy\,dz + zx\,dz\,dx.$

2362. $\iint\limits_S x^2\,dy\,dz + y^2\,dz\,dx + z^2\,dx\,dy.$

2363. $\iint\limits_S \frac{x\cos\alpha + y\cos\beta + z\cos\gamma}{\sqrt{x^2+y^2+z^2}}\,dS.$

2364. $\iint\limits_S \left(\frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma\right) dS.$

Using the Ostrogradsky-Gauss formula, compute the following surface integrals:

2365. $\iint\limits_S x^2\,dy\,dz + y^2\,dz\,dx + z^2\,dx\,dy$, where S is the external side of the surface of the cube $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant a$, $0 \leqslant z \leqslant a$.

2366. $\iint\limits_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$, where S is the external side of a pyramid bounded by the surfaces $x+y+z=a$, $x=0$, $y=0$, $z=0$.

2367. $\iint\limits_S x^3\,dy\,dz + y^3\,dz\,dx = z^3\,dx\,dy$, where S is the external side of the sphere $x^2+y^2+z^2=a^2$.

2368 $\iint\limits_S (x^2\cos\alpha + y^2\cos\beta + z^2\cos\gamma)\,dS$, where S is the external total surface of the cone

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 0 \qquad [0 \leqslant z \leqslant b].$$

2369. Prove that if S is a closed surface and l is any fixed direction, then

$$\iint\limits_S \cos(\boldsymbol{n}, \boldsymbol{l})\,dS = 0,$$

where $\boldsymbol{n}$ is the outer normal to the surface S.

2370. Prove that the volume of the solid V bounded by the surface S is equal to

$$V = \frac{1}{3}\iint\limits_S (x\cos\alpha + y\cos\beta + z\cos\gamma)\,dS,$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the outer normal to the surface S.

Sec. 12. Fundamentals of Field Theory

1°. Scalar and vector fields. A *scalar field* is defined by the scalar function of the point $u=f(P)=f(x, y, z)$, where $P(x, y, z)$ is a point of space. The surfaces $f(x, y, z)=C$, where $C=\text{const}$, are called *level surfaces* of the scalar field.

A *vector field* is defined by the vector function of the point $\boldsymbol{a}=\boldsymbol{a}(P)=\boldsymbol{a}(\boldsymbol{r})$, where P is a point of space and $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ is the radius vector of the point P. In coordinate form, $\boldsymbol{a}=a_x\boldsymbol{i}+a_y\boldsymbol{j}+a_z\boldsymbol{k}$, where $a_x=a_x(x, y, z)$, $a_y=a_y(x, y, z)$, and $a_z=a_z(x, y, z)$ are projections of the vector $\boldsymbol{a}$ on the coordinate axes. The *vector lines* (*force lines*, *flow lines*) of a vector field are found from the following system of differential equations

$$\frac{dx}{a_x}=\frac{dy}{a_y}=\frac{dz}{a_z}.$$

A scalar or vector field that does not depend on the time t is called *stationary*; if it depends on the time, it is called *nonstationary*.

2°. Gradient. The vector

$$\operatorname{grad} U(P)=\frac{\partial U}{\partial x}\boldsymbol{i}+\frac{\partial U}{\partial y}\boldsymbol{j}+\frac{\partial U}{\partial z}\boldsymbol{k}\equiv\nabla U,$$

where $\nabla=\boldsymbol{i}\frac{\partial}{\partial x}+\boldsymbol{j}\frac{\partial}{\partial y}+\boldsymbol{k}\frac{\partial}{\partial z}$ is the Hamiltonian operator (del, or nabla), is called the *gradient* of the field $U=f(P)$ at the given point P (cf. Ch. VI, Sec. 6). The gradient is in the direction of the normal n to the level surface at the point P and in the direction of increasing function U, and has length equal to

$$\frac{\partial U}{\partial n}=\sqrt{\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2+\left(\frac{\partial U}{\partial z}\right)^2}.$$

If the direction is given by the unit vector $\boldsymbol{l}\{\cos\alpha, \cos\beta, \cos\gamma\}$, then

$$\frac{\partial U}{\partial l}=\operatorname{grad}\boldsymbol{U}\cdot\boldsymbol{l}=\operatorname{grad}_l\boldsymbol{U}=\frac{\partial U}{\partial x}\cos\alpha+\frac{\partial U}{dy}\cos\beta+\frac{\partial U}{\partial z}\cos\gamma$$

(the derivative of the function U in the direction l).

3°. Divergence and rotation. The *divergence* of a vector field $\boldsymbol{a}(P)=a_x\boldsymbol{i}+a_y\boldsymbol{j}+a_z\boldsymbol{k}$ is the scalar $\operatorname{div}\boldsymbol{a}=\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\equiv\nabla\boldsymbol{a}$.

The *rotation* (curl) of a vector field $\boldsymbol{a}(P)=a_x\boldsymbol{i}+a_y\boldsymbol{j}+a_z\boldsymbol{k}$ is the vector

$$\operatorname{rot}\boldsymbol{a}=\left(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x}\right)\boldsymbol{j}+\left(\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y}\right)\boldsymbol{k}\equiv\nabla\times\boldsymbol{a}.$$

4°. Flux of a vector. The *flux* of a vector field $\boldsymbol{a}(P)$ through a surface S in a direction defined by the unit vector of the normal $n\{\cos\alpha, \cos\beta, \cos\gamma\}$ to the surface S is the integral

$$\iint\limits_S \boldsymbol{an}\,dS=\iint\limits_S a_n\,dS=\iint\limits_S (a_x\cos\alpha+a_y\cos\beta+a_z\cos\gamma)\,dS.$$

If S is a closed surface bounding a volume V, and n is a unit vector of the outer normal to the surface S, then the *Ostrogradsky-Gauss formula* holds, which in vector form is

$$\iint_S a_n \, dS = \iiint_{(V)} \operatorname{div} \boldsymbol{a} \, dx \, dy \, dz.$$

5°. Circulation of a vector, the work of a field. The *line integral* of the vector $\boldsymbol{a}$ along the curve C is defined by the formula

$$\int_C \boldsymbol{a} \, d\boldsymbol{r} = \int_C a_s \, ds = \int_C a_x \, dx + a_y dy + a_z \, dz \tag{1}$$

and represents the *work* done by the field $\boldsymbol{a}$ along the curve C (a_s is the projection of the vector $\boldsymbol{a}$ on the tangent to C).

If C is closed, then the line integral (1) is called the *circulation* of the vector field $\boldsymbol{a}$ around the contour C.

If the closed curve C bounds a two-sided surface S, then *Stokes' formula* holds, which in vector form has the form

$$\oint_C \boldsymbol{a} \, d\boldsymbol{r} = \iint_S \boldsymbol{n} \operatorname{rot} \boldsymbol{a} \, dS,$$

where $\boldsymbol{n}$ is the vector of the normal to the surface S; the direction of the vector should be chosen so that for an observer looking in the direction of $\boldsymbol{n}$ the circulation of the contour C should be counterclockwise in a right-handed coordinate system.

6°. Potential and solenoidal fields. The vector field $\boldsymbol{a}(\boldsymbol{r})$ is called *potential* if

$$\boldsymbol{a} = \operatorname{grad} U,$$

where $U = f(\boldsymbol{r})$ is a scalar function (the *potential* of the field).

For the potentiality of a field $\boldsymbol{a}$, given in a simply-connected domain, it is necessary and sufficient that it be *nonrotational*, that is, $\operatorname{rot} \boldsymbol{a} = 0$. In that case there exists a potential U defined by the equation

$$dU = a_x \, dx + a_y \, dy + a_z \, dz.$$

If the potential U is a single-valued function, then $\int_{AB} \boldsymbol{a} \, d\boldsymbol{r} = U(B) - U(A)$; in particular, the circulation of the vector $\boldsymbol{a}$ is equal to zero: $\oint_C \boldsymbol{a} \, d\boldsymbol{r} = 0$.

A vector field $\boldsymbol{a}(\boldsymbol{r})$ is called *solenoidal* if at each point of the field $\operatorname{div} \boldsymbol{a} = 0$; in this case the flux of the vector through any closed surface is zero.

If the field is at the same time potential and solenoidal, then $\operatorname{div}(\operatorname{grad} U) = 0$ and the potential function U is harmonic; that is, it satisfies the Laplace equation $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$, or $\Delta U = 0$, where $\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator

2371. Determine the level surfaces of the scalar field $U = f(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$. What will the level surfaces be of a field $U = F(\varrho)$, where $\varrho = \sqrt{x^2 + y^2}$?

2372. Determine the level surfaces ot the scalar field

$$U = \arcsin \frac{z}{\sqrt{x^2+y^2}}.$$

2373. Show that straight lines parallel to a vector $\boldsymbol{c}$ are the vector lines of a vector field $\boldsymbol{a}(P) = \boldsymbol{c}$, where $\boldsymbol{c}$ is a constant vector.

2374. Find the vector lines of the field $\boldsymbol{a} = -\omega y\boldsymbol{i} + \omega x\boldsymbol{j}$, where ω is a constant.

2375. Derive the formulas:

a) $\operatorname{grad}(C_1U + C_2V) = C_1 \operatorname{grad} U + C_2 \operatorname{grad} V$, where C_1 and C_2 are constants;

b) $\operatorname{grad}(UV) = U \operatorname{grad} V + V \operatorname{grad} U$;

c) $\operatorname{grad}(U^2) = 2U \operatorname{grad} U$;

d) $\operatorname{grad}\left(\frac{U}{V}\right) = \frac{V \operatorname{grad} U - U \operatorname{grad} V}{V^2}$;

e) $\operatorname{grad} \varphi(U) = \varphi'(U) \operatorname{grad} U$.

2376. Find the magnitude and the direction of the gradient of the field $U = x^3 + y^3 + z^3 - 3xyz$ at the point $A(2, 1, 1)$. Determine at what points the gradient of the field is perpendicular to the z-axis and at what points it is equal to zero.

2377. Evaluate grad U, if U is equal, respectively, to: a) r, b) r^2, c) $\frac{1}{r}$, d) $f(r)$ $(r = \sqrt{x^2+y^2+z^2})$.

2378. Find the gradient of the scalar field $U = \boldsymbol{cr}$, where $\boldsymbol{c}$ is a constant vector. What will the level surfaces be of this field, and what will their position be relative to the vector $\boldsymbol{c}$?

2379. Find the derivative of the function $U = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ at a given point $P(x, y, z)$ in the direction of the radius vector $\boldsymbol{r}$ of this point. In what case will this derivative be equal to the magnitude of the gradient?

2380. Find the derivative of the function $U = \frac{1}{r}$ in the direction of $\boldsymbol{l}\{\cos\alpha, \cos\beta, \cos\gamma\}$. In what case will this derivative be equal to zero?

2381. Derive the formulas:

a) $\operatorname{div}(C_1\boldsymbol{a}_1 + C_2\boldsymbol{a}_2) = C_1 \operatorname{div} \boldsymbol{a}_1 + C_2 \operatorname{div} \boldsymbol{a}_2$, where C_1 and C_2 are constants;

b) $\operatorname{div}(U\boldsymbol{c}) = \operatorname{grad} U \cdot \boldsymbol{c}$, where $\boldsymbol{c}$ is a constant vector;

c) $\operatorname{div}(U\boldsymbol{a}) = \operatorname{grad} U \cdot \boldsymbol{a} + U \operatorname{div} \boldsymbol{a}$.

2382. Evaluate $\operatorname{div}\left(\frac{\boldsymbol{r}}{r}\right)$.

2383. Find div $\boldsymbol{a}$ for the central vector field $\boldsymbol{a}(P) = f(r)\frac{\boldsymbol{r}}{r}$, where $r = \sqrt{x^2+y^2+z^2}$.

2384. Derive the formulas:

a) $\text{rot}(C_1\boldsymbol{a}_1 + C_2\boldsymbol{a}_2) = C_1 \text{ rot } \boldsymbol{a}_1 + C_2 \text{ rot } \boldsymbol{a}_2$, where C_1 and C_2 are constants;

b) $\text{rot}(U\boldsymbol{c}) = \text{grad } U \cdot \boldsymbol{c}$, where $\boldsymbol{c}$ is a constant vector;

c) $\text{rot}(U\boldsymbol{a}) = \text{grad } U \cdot \boldsymbol{a} + U \text{ rot } \boldsymbol{a}$.

2385. Evaluate the divergence and the rotation of the vector $\boldsymbol{a}$ if $\boldsymbol{a}$ is, respectively, equal to: a) $\boldsymbol{r}$; b) $r\boldsymbol{c}$ and c) $f(r)\boldsymbol{c}$, where $\boldsymbol{c}$ is a constant vector.

2386. Find the divergence and rotation of the field of linear velocities of the points of a solid rotating counterclockwise with constant angular velocity ω about the z-axis.

2387. Evaluate the rotation of a field of linear velocities $\boldsymbol{v} = \boldsymbol{\omega} \cdot \boldsymbol{r}$ of the points of a body rotating with constant angular velocity $\boldsymbol{\omega}$ about some axis passing through the coordinate origin.

2388. Evaluate the divergence and rotation of the gradient of the scalar field U.

2389. Prove that $\text{div}(\text{rot } \boldsymbol{a}) = 0$.

2390. Using the Ostrogradsky-Gauss theorem, prove that the flux of the vector $\boldsymbol{a} = \boldsymbol{r}$ through a closed surface bounding an arbitrary volume v is equal to three times the volume.

2391. Find the flux of the vector $\boldsymbol{r}$ through the total surface of the cylinder $x^2 + y^2 \leqslant R^2$, $0 \leqslant z \leqslant H$.

2392. Find the flux of the vector $\boldsymbol{a} = x^3\boldsymbol{i} + y^3\boldsymbol{j} + z^3\boldsymbol{k}$ through: a) the lateral surface of the cone $\frac{x^2 + y^2}{R^2} \leqslant \frac{z^2}{H^2}$, $0 \leqslant z \leqslant H$; b) the total surface of the cone.

2393*. Evaluate the divergence and the flux of an attractive force $F = -\frac{m\boldsymbol{r}}{r^3}$ of a point of mass m, located at the coordinate origin, through an arbitrary closed surface surrounding this point.

2394. Evaluate the line integral of a vector $\boldsymbol{r}$ around one turn of the screw-line $x = R\cos t$; $y = R\sin t$; $z = ht$ from $t = 0$ to $t = 2\pi$.

2395. Using Stokes' theorem, evaluate the circulation of the vector $\boldsymbol{a} = x^2y^3\boldsymbol{i} + \boldsymbol{j} + z\boldsymbol{k}$ along the circumference $x^2 + y^2 = R^2$; $z = 0$, taking the hemisphere $z = \sqrt{R^2 - x^2 - y^2}$ for the surface.

2396. Show that if a force $\boldsymbol{F}$ is central, that is, it is directed towards a fixed point 0 and depends only on the distance r from this point: $\boldsymbol{F} = f(r)\boldsymbol{r}$, where $f(r)$ is a single-valued continuous function, then the field is a potential field. Find the potential U of the field.

2397. Find the potential U of a gravitational field generated by a material point of mass m located at the origin of coordinates: $\boldsymbol{a} = -\frac{m}{r^3}\boldsymbol{r}$. Show that the potential U satisfies the Laplace equation $\Delta U = 0$.

2398. Find out whether the given vector field has a potential U, and find U if the potential exists:

a) $\boldsymbol{a}=(5x^2y-4xy)\,\boldsymbol{i}+(3x^2-2y)\,\boldsymbol{j}$;
b) $\boldsymbol{a}=yz\boldsymbol{i}+zx\boldsymbol{j}+xy\boldsymbol{k}$;
c) $\boldsymbol{a}=(y+z)\,\boldsymbol{i}+(x+z)\,\boldsymbol{j}+(x+y)\,\boldsymbol{k}$.

2399. Prove that the central space field $\boldsymbol{a}=f(r)\,\boldsymbol{r}$ will be solenoidal only when $f(r)=\frac{k}{r^3}$, where k is constant.

2400. Will the vector field $\boldsymbol{a}=r\,(\boldsymbol{c}\times\boldsymbol{r})$ be solenoidal (where $\boldsymbol{c}$ is a constant vector)?

Chapter VIII

SERIES

Sec. 1. Number Series

1°. **Fundamental concepts.** A number series

$$a_1+a_2+\ldots+a_n+\ldots=\sum_{n=1}^{\infty} a_n \qquad (1)$$

is called *convergent* if its *partial sum*

$$S_n=a_1+a_2+\ldots+a_n$$

has a finite limit as $n \longrightarrow \infty$. The quantity $S=\lim_{n\to\infty} S_n$ is then called the *sum* of the series, while the number

$$R_n=S-S_n=a_{n+1}+a_{n+2}+\ldots$$

is called the *remainder* of the series. If the limit $\lim_{n\to\infty} S_n$ does not exist (or is infinite), the series is then called *divergent*.

If a series converges, then $\lim_{n\to\infty} a_n=0$ (*necessary condition for convergence*). The converse is not true.

For convergence of the series (1) it is necessary and sufficient that for any positive number ε it be possible to choose an N such that for $n>N$ and for any positive p the following inequality is fulfilled:

$$|a_{n+1}+a_{n+2}+\ldots+a_{n+p}|<\varepsilon$$

(*Cauchy's test*).

The convergence or divergence of a series is not violated if we add or subtract a finite number of its terms.

2°. **Tests of convergence and divergence of positive series.**

a) **Comparison test I.** If $0 \leqslant a_n \leqslant b_n$ after a certain $n=n_0$, and the series

$$b_1+b_2+\ldots+b_n+\ldots=\sum_{n=1}^{\infty} b_n \qquad (2)$$

converges, then the series (1) also converges. If the series (1) diverges, then (2) diverges as well.

It is convenient, for purposes of comparing series, to take a *geometric progression:*

$$\sum_{n=0}^{\infty} aq^n \qquad (a \neq 0),$$

which converges for $|q|<1$ and diverges for $|q|\geqslant 1$, and the *harmonic series*

$$\sum_{n=1}^{\infty}\frac{1}{n},$$

which is a divergent series.

Example 1. The series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\ldots+\frac{1}{n\cdot 2^n}+\ldots$$

converges, since here

$$a_n=\frac{1}{n\cdot 2^n}<\frac{1}{2^n},$$

while the geometric progression

$$\sum_{n=1}^{\infty}\frac{1}{2^n},$$

whose ratio is $q=\frac{1}{2}$, converges.

Example 2. The series

$$\frac{\ln 2}{2}+\frac{\ln 3}{3}+\ldots+\frac{\ln n}{n}+\ldots$$

diverges, since its general term $\frac{\ln n}{n}$ is greater than the corresponding term $\frac{1}{n}$ of the harmonic series (which diverges).

b) **Comparison test II.** If there exists a finite and nonzero limit $\lim\limits_{n\to\infty}\frac{a_n}{b_n}$ (in particular, if $a_n \sim b_n$), then the series (1) and (2) converge or diverge at the same time.

Example 3. The series

$$1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}+\ldots$$

diverges, since

$$\lim_{n\to\infty}\left(\frac{1}{2n-1}:\frac{1}{n}\right)=\frac{1}{2}\neq 0,$$

whereas a series with general term $\frac{1}{n}$ diverges.

Example 4. The series

$$\frac{1}{2-1}+\frac{1}{2^2-2}+\frac{1}{2^3-3}+\ldots+\frac{1}{2^n-n}+\ldots$$

converges, since

$$\lim_{n\to\infty}\left(\frac{1}{2^n-n}:\frac{1}{2^n}\right)=1,\quad \text{i.e.,}\quad \frac{1}{2^n-n}\sim\frac{1}{2^n},$$

while a series with general term $\frac{1}{2^n}$ converges.

c) **D'Alembert's test.** Let $a_n > 0$ (after a certain n) and let there be a limit

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = q.$$

Then the series (1) converges if $q < 1$, and diverges if $q > 1$. If $q = 1$, then it is not known whether the series is convergent or not.

Example 5. Test the convergence of the series

$$\frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \ldots + \frac{2n-1}{2^n} + \ldots$$

Solution. Here,

$$a_n = \frac{2n-1}{2^n}, \quad a_{n+1} = \frac{2n+1}{2^{n+1}}$$

and

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(2n+1)\, 2^n}{2^{n+1}(2n-1)} = \frac{1}{2} \lim_{n \to \infty} \frac{1 + \frac{1}{2n}}{1 - \frac{1}{2n}} = \frac{1}{2}.$$

Hence, the given series converges.

d) **Cauchy's test.** Let $a_n \geqslant 0$ (after a certain n) and let there be a limit

$$\lim_{n \to \infty} \sqrt[n]{a_n} = q.$$

Then (1) converges if $q < 1$, and diverges if $q > 1$. When $q = 1$, the question of the convergence of the series remains open.

e) **Cauchy's integral test.** If $a_n = f(n)$, where the function $f(x)$ is positive, monotonically decreasing and continuous for $x \geqslant a \geqslant 1$, the series (1) and the integral

$$\int_a^{\infty} f(x)\, dx$$

converge or diverge at the same time.

By means of the integral test it may be proved that the *Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad (3)$$

converges if $p > 1$, and diverges if $p \leqslant 1$. The convergence of a large number of series may be tested by comparing with the corresponding Dirichlet series (3)

Example 6. Test the following series for convergence

$$\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \ldots + \frac{1}{(2n-1)\, 2n} + \ldots$$

Solution. We have

$$a_n = \frac{1}{(2n-1)\, 2n} = \frac{1}{4n^2} \frac{1}{1 - \frac{1}{2n}} \sim \frac{1}{4n^2}.$$

Since the Dirichlet series converges for $p=2$, it follows that on the basis of comparison test II we can say that the given series likewise converges.

3°. Tests for convergence of alternating series. If a series

$$|a_1|+|a_2|+\ldots+|a_n|+\ldots, \tag{4}$$

composed of the absolute values of the terms of the series (1), converges, then (1) also converges and is called *absolutely convergent*. But if (1) converges and (4) diverges, then the series (1) is called *conditionally (not absolutely) convergent*.

For investigating the absolute convergence of the series (1), we can make use [for the series (4)] of the familiar convergence tests of positive series. For instance, (1) converges absolutely if

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1 \quad \text{or} \quad \lim_{n\to\infty}\sqrt[n]{|a_n|}<1.$$

In the general case, the divergence of (1) does not follow from the divergence of (4). But if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$ or $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}>1$, then not only does (4) diverge but the series (1) does also.

Leibniz test If for the alternating series

$$b_1-b_2+b_3-b_4+\ldots \quad (b_n\geqslant 0) \tag{5}$$

the following conditions are fulfilled: 1) $b_1\geqslant b_2\geqslant b_3\geqslant\ldots$; 2) $\lim\limits_{n\to\infty} b_n=0$, then (5) converges.

In this case, for the remainder of the series R_n the evaluation

$$|R_n|\leqslant b_{n+1}$$

holds.

Example 7. Test for convergence the series

$$1-\left(\frac{2}{3}\right)^2-\left(\frac{3}{5}\right)^3+\left(\frac{4}{7}\right)^4+\ldots+(-1)^{\frac{n(n-1)}{2}}\left(\frac{n}{2n-1}\right)^n+\ldots$$

Solution. Let us form a series of the absolute values of the terms of this series:

$$1+\left(\frac{2}{3}\right)^2+\left(\frac{3}{5}\right)^3+\left(\frac{4}{7}\right)^4+\ldots+\left(\frac{n}{2n-1}\right)^n+\ldots$$

Since

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{2n-1}\right)^n}=\lim_{n\to\infty}\frac{n}{2n-1}=\lim_{n\to\infty}\frac{1}{2-\frac{1}{n}}=\frac{1}{2},$$

the series converges absolutely.

Example 8. The series

$$1-\frac{1}{2}+\frac{1}{3}-\ldots+(-1)^{n+1}\cdot\frac{1}{n}+\ldots$$

converges, since the conditions of the Leibniz test are fulfilled. This series converges conditionally, since the series

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$$

diverges (harmonic series).

Note. For the convergence of an alternating series it is not sufficient that its general term should tend to zero. The Leibniz test only states that an alternating series converges if the absolute value of its general term tends to zero *monotonically*. Thus, for example, the series

$$1-\frac{1}{5}+\frac{1}{2}-\frac{1}{5^2}+\frac{1}{3}-\ldots+\frac{1}{k}-\frac{1}{5^k}+\ldots$$

diverges despite the fact that its general term tends to zero (here, of course, the monotonic variation of the absolute value of the general term has been violated). Indeed, here, $S_{2k}=S'_k+S''_k$, where

$$S'_k=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k},\quad S''_k=-\left(\frac{1}{5}+\frac{1}{5^2}+\ldots+\frac{1}{5^k}\right),$$

and $\lim\limits_{k\to\infty} S'_k=\infty$ (S'_k is a partial sum of the harmonic series), whereas the limit $\lim\limits_{k\to\infty} S''_k$ exists and is finite (S''_k is a partial sum of the convergent geometric progression), hence, $\lim\limits_{k\to\infty} S_{2k}=\infty$.

On the other hand, the Leibniz test is not necessary for the convergence of an alternating series: an alternating series may converge if the absolute value of its general term tends to zero in nonmonotonic fashion

Thus, the series

$$1-\frac{1}{2^2}+\frac{1}{3^3}-\frac{1}{4^2}+\ldots+\frac{1}{(2n-1)^3}-\frac{1}{(2n)^2}+\ldots$$

converges (and it converges absolutely), although the Leibniz test is not fulfilled: though the absolute value of the general term of the series tends to zero, it does not do so monotonically.

4°. **Series with complex terms** A series with the general term $c_n=a_n+ib_n$ ($i^2=-1$) converges if, and only if, the series with real terms $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ converge at the same time; in this case

$$\sum_{n=1}^{\infty} c_n=\sum_{n=1}^{\infty} a_n+i\sum_{n=1}^{\infty} b_n. \qquad (6)$$

The series (6) definitely converges and is called *absolutely convergent*, if the series

$$\sum_{n=1}^{\infty} |c_n|=\sum_{n=1}^{\infty}\sqrt{a_n^2+b_n^2},$$

whose terms are the moduli of the terms of the series (6), converges.

5°. **Operations on series.**

a) A convergent series may be multiplied termwise by any number k; that is, if

$$a_1+a_2+\ldots+a_n+\ldots=S,$$

then

$$ka_1+ka_2+\ldots+ka_n+\ldots=kS.$$

b) By the *sum (difference)* of two convergent series

$$a_1+a_2+\ldots+a_n+\ldots=S_1, \quad (7)$$

$$b_1+b_2+\ldots+b_n+\ldots=S_2 \quad (8)$$

we mean a series

$$(a_1\pm b_1)+(a_2\pm b_2)+\ldots+(a_n\pm b_n)+\ldots=S_1\pm S_2.$$

c) The *product* of the series (7) and (8) is the series

$$c_1+c_2+\ldots+c_n+\ldots, \quad (9)$$

where $c_n=a_1b_n+a_2b_{n-1}+\ldots+a_nb_1 (n=1, 2, \ldots)$.

If the series (7) and (8) converge absolutely, then the series (9) also converges absolutely and has a sum equal to S_1S_2.

d) If a series converges absolutely, its sum remains unchanged when the terms of the series are rearranged. This property is absent if the series converges conditionally.

Write the simplest formula of the nth term of the series using the indicated terms:

2401. $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots$

2402. $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\ldots$

2403. $1+\frac{2}{2}+\frac{3}{4}+\frac{4}{8}+\ldots$

2404. $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots$

2405. $\frac{3}{4}+\frac{4}{9}+\frac{5}{16}+\frac{6}{25}+\ldots$

2406. $\frac{2}{5}+\frac{4}{8}+\frac{6}{11}+\frac{8}{14}+\ldots$

2407. $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\ldots$

2408. $1+\frac{1\cdot 3}{1\cdot 4}+\frac{1\cdot 3\cdot 5}{1\cdot 4\cdot 7}+\frac{1\cdot 3\cdot 5\cdot 7}{1\cdot 4\cdot 7\cdot 10}+\ldots$

2409. $1-1+1-1+1-1+\ldots$

2410. $1+\frac{1}{2}+3+\frac{1}{4}+5+\frac{1}{6}+\ldots$

In Problems 2411-2415 it is required to write the first 4 or 5 terms of the series on the basis of the known general term a_n.

2411. $a_n=\frac{3n-2}{n^2+1}$.

2412. $\frac{(-1)^n n}{2^n}$.

2413 $a_n=\frac{2+(-1)^n}{n^2}$.

2414. $a_n=\frac{1}{[3+(-1)^n]^n}$.

2415. $a_n=\frac{\left(2+\sin\frac{n\pi}{2}\right)\cos n\pi}{n!}$.

Test the following series for convergence by applying the comparison tests (or the necessary condition):

2416. $1-1+1-1+\ldots+(-1)^{n-1}+\ldots$

2417. $\frac{2}{5}+\frac{1}{2}\left(\frac{2}{5}\right)^2+\frac{1}{3}\left(\frac{2}{5}\right)^3+\ldots+\frac{1}{n}\left(\frac{2}{5}\right)^n+\ldots$

2418. $\frac{2}{3}+\frac{3}{5}+\frac{4}{7}+\ldots+\frac{n+1}{2n+1}+\ldots$

2419. $\frac{1}{\sqrt{10}}-\frac{1}{\sqrt[3]{10}}+\frac{1}{\sqrt[4]{10}}-\ldots+\frac{(-1)^{n+1}}{\sqrt[n+1]{10}}+\ldots$

2420. $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}+\ldots$

2421. $\frac{1}{11}+\frac{1}{21}+\frac{1}{31}+\ldots+\frac{1}{10n+1}+\ldots$

2422. $\frac{1}{\sqrt{1\cdot 2}}+\frac{1}{\sqrt{2\cdot 3}}+\frac{1}{\sqrt{3\cdot 4}}+\ldots+\frac{1}{\sqrt{n(n+1)}}+\ldots$

2423. $2+\frac{2^2}{2}+\frac{2^3}{3}+\ldots+\frac{2^n}{n}+\ldots$

2424. $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}+\ldots$

2425. $\frac{1}{2^2}+\frac{1}{5^2}+\frac{1}{8^2}+\ldots+\frac{1}{(3n-1)^2}+\ldots$

2426. $\frac{1}{2}+\frac{\sqrt[3]{2}}{3\sqrt{2}}+\frac{\sqrt[3]{3}}{4\sqrt{3}}+\ldots+\frac{\sqrt[3]{n}}{(n+1)\sqrt{n}}+\ldots$

Using d'Alembert's test, test the following series for convergence:

2427. $\frac{1}{\sqrt{2}}+\frac{3}{2}+\frac{5}{2\sqrt{2}}+\ldots+\frac{2n-1}{(\sqrt{2})^n}+\ldots$

2428. $\frac{2}{1}+\frac{2\cdot 5}{1\cdot 5}+\frac{2\cdot 5\cdot 8}{1\cdot 5\cdot 9}+\ldots+\frac{2\cdot 5\cdot 8\ldots(3n-1)}{1\cdot 5\cdot 9\ldots(4n-3)}+\ldots$

Test for convergence, using Cauchy's test:

2429. $\frac{2}{1}+\left(\frac{3}{3}\right)^2+\left(\frac{4}{5}\right)^3+\ldots+\left(\frac{n+1}{2n-1}\right)^n+\ldots$

2430. $\frac{1}{2}+\left(\frac{2}{5}\right)^3+\left(\frac{3}{8}\right)^5+\ldots+\left(\frac{n}{3n-1}\right)^{2n-1}+\ldots$

Test for convergence the positive series:

2431. $1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\ldots$

2432. $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\ldots+\frac{1}{(n+1)^2-1}+\ldots$

2433. $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\ldots+\frac{1}{(3n-2)(3n+1)}+\ldots$

2434. $\frac{1}{3}+\frac{4}{9}+\frac{9}{19}+\ldots+\frac{n^2}{2n^2+1}+\ldots$

2435. $\frac{1}{2}+\frac{2}{5}+\frac{3}{10}+\ldots+\frac{n}{n^2+1}+\ldots$

2436. $\frac{3}{2^2\cdot 3^2}+\frac{5}{3^2\cdot 4^2}+\frac{7}{4^2\cdot 5^2}+\ldots+\frac{2n+1}{(n+1)^2(n+2)^2}+\ldots$

2437. $\frac{3}{4}+\left(\frac{6}{7}\right)^2+\left(\frac{9}{10}\right)^3+\ldots+\left(\frac{3n}{3n+1}\right)^n+\ldots$

2438. $\left(\frac{3}{4}\right)^{\frac{1}{2}}+\frac{5}{7}+\left(\frac{7}{10}\right)^{\frac{3}{2}}+\ldots+\left(\frac{2n+1}{3n+1}\right)^{\frac{n}{2}}+\ldots$

2439. $\frac{1}{e}+\frac{8}{e^2}+\frac{27}{e^3}+\ldots+\frac{n^3}{e^n}+\ldots$

2440. $1+\frac{2}{2^2}+\frac{4}{3^3}+\ldots+\frac{2^{n-1}}{n^n}+\ldots$

2441. $\frac{1!}{2+1}+\frac{2!}{2^2+1}+\frac{3!}{2^3+1}+\ldots+\frac{n!}{2^n+1}+\ldots$

2442. $1+\frac{2}{1!}+\frac{4}{2!}+\ldots+\frac{2^{n-1}}{(n-1)!}+\ldots$

2443. $\frac{1}{4}+\frac{1\cdot 3}{4\cdot 8}+\frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}+\ldots+\frac{1\cdot 3\cdot 5\ldots(2n-1)}{4\cdot 8\cdot 12\ldots 4n}+\ldots$

2444. $\frac{(1!)^2}{2!}+\frac{(2!)^2}{4!}+\frac{(3!)^2}{6!}+\ldots+\frac{(n!)^2}{(2n)!}+\ldots$

2445. $1000+\frac{1000\cdot 1002}{1\cdot 4}+\frac{1000\cdot 1002\cdot 1004}{1\cdot 4\cdot 7}+\ldots$

$\ldots+\frac{1000\cdot 1002\cdot 1004\ldots(998+2n)}{1\cdot 4\cdot 7\ldots(3n-2)}+\ldots$

2446. $\frac{2}{1}+\frac{2\cdot 5\cdot 8}{1\cdot 5\cdot 9}+\ldots+\frac{2\cdot 5\cdot 8\ldots(6n-7)(6n-4)}{1\cdot 5\cdot 9\ldots(8n-11)(8n-7)}+\ldots$

2447. $\frac{1}{2}+\frac{1\cdot 5}{2\cdot 4\cdot 6}+\ldots+\frac{1\cdot 5\ldots(4n-3)}{2\cdot 4\cdot 6\ldots(4n-4)(4n-2)}+\ldots$

2448. $\frac{1}{1!}+\frac{1\cdot 11}{3!}+\frac{1\cdot 11\cdot 21}{5!}+\ldots+\frac{1\cdot 11\cdot 21\ldots(10n-9)}{(2n-1)!}+\ldots$

2449. $1+\frac{1\cdot 4}{1\cdot 3\cdot 5}+\frac{1\cdot 4\cdot 9}{1\cdot 3\cdot 5\cdot 7\cdot 9}+\ldots+\frac{1\cdot 4\cdot 9\ldots n^2}{1\cdot 3\cdot 5\cdot 7\cdot 9\ldots(4n-3)}+\ldots$

2450. $\sum_{n=1}^{\infty}\arcsin\frac{1}{\sqrt{n}}$.

2451. $\sum_{n=1}^{\infty}\sin\frac{1}{n^2}$.

2452. $\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right)$.

2453. $\sum_{n=1}^{\infty}\ln\frac{n^2+1}{n^2}$.

2454. $\sum_{n=2}^{\infty}\frac{1}{\ln n}$.

2455. $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$.

2456. $\sum_{n=2}^{\infty}\frac{1}{n\ln^2 n}$.

2457. $\sum_{n=2}^{\infty}\frac{1}{n\cdot\ln n\cdot\ln\ln n}$.

2458. $\sum_{n=2}^{\infty}\frac{1}{n^2-n}$.

2459. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}}$.

2460. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)(n+2)}}$.

2461. $\sum_{n=2}^{\infty}\frac{1}{n\ln n+\sqrt{\ln^3 n}}$.

2462. $\sum_{n=2}^{\infty}\frac{1}{n\sqrt[3]{n}-\sqrt{n}}$.

2463. $\sum_{n=1}^{\infty}\frac{\sqrt[3]{n}}{(2n-1)(5\sqrt[3]{n}-1)}$.

2464. $\sum_{n=1}^{\infty}\left(1-\cos\frac{\pi}{n}\right)$.

2465. $\sum_{n=1}^{\infty}\frac{n!}{n^n}$.

2466. $\sum_{n=1}^{\infty}\frac{2^n n!}{n^n}$.

2467. $\sum_{n=1}^{\infty}\frac{3^n n!}{n^n}$.

2468*. $\sum_{n=1}^{\infty}\frac{e^n n!}{n^n}$.

2469. Prove that the series $\sum_{n=2}^{\infty}\frac{1}{n^p\ln^q n}$:

1) converges for arbitrary q, if $p>1$, and for $q>1$, if $p=1$;
2) diverges for arbitrary q, if $p<1$, and for $q\leqslant 1$, if $p=1$.

Test for convergence the following alternating series. For convergent series, test for absolute and conditional convergence.

2470. $1-\frac{1}{3}+\frac{1}{5}-\ldots+\frac{(-1)^{n-1}}{2n-1}+\ldots$

2471. $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\ldots+\frac{(-1)^{n-1}}{\sqrt{n}}+\ldots$

2472. $1-\frac{1}{4}+\frac{1}{9}-\ldots+\frac{(-1)^{n-1}}{n^2}+\ldots$

2473. $1-\frac{2}{7}+\frac{3}{13}-\ldots+\frac{(-1)^{n-1}n}{6n-5}+\ldots$

2474. $\frac{3}{1\cdot 2}-\frac{5}{2\cdot 3}+\frac{7}{3\cdot 4}-\ldots+(-1)^{n-1}\frac{2n+1}{n(n+1)}+\ldots$

2475. $-\frac{1}{2}-\frac{2}{4}+\frac{3}{8}+\frac{4}{16}-\ldots+(-1)^{\frac{n^2+n}{2}}\cdot\frac{n}{2^n}+\ldots$

2476. $-\frac{2}{2\sqrt{2}-1}+\frac{3}{3\sqrt{3}-1}-\frac{4}{4\sqrt{4}-1}+\ldots+$
$+(-1)^n\frac{n+1}{(n+1)\sqrt{n+1}-1}+\ldots$

2477. $-\frac{3}{4}+\left(\frac{5}{7}\right)^2-\left(\frac{7}{10}\right)^3+\ldots+(-1)^n\left(\frac{2n+1}{3n+1}\right)^n+\ldots$

2478. $\frac{3}{2}-\frac{3\cdot 5}{2\cdot 5}+\frac{3\cdot 5\cdot 7}{2\cdot 5\cdot 8}-\ldots+(-1)^{n-1}\frac{3\cdot 5\cdot 7\ldots(2n+1)}{2\cdot 5\cdot 8\ldots(3n-1)}+\ldots$

2479. $\frac{1}{7}-\frac{1\cdot 4}{7\cdot 9}+\frac{1\cdot 4\cdot 7}{7\cdot 9\cdot 11}-\ldots+(-1)^{n-1}\frac{1\cdot 4\cdot 7\ldots(3n-2)}{7\cdot 9\cdot 11\ldots(2n+5)}+\ldots$

2480. $\frac{\sin\alpha}{\ln 10}+\frac{\sin 2\alpha}{(\ln 10)^2}+\ldots+\frac{\sin n\alpha}{(\ln 10)^n}+\ldots$

2481. $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n}$.

2482. $\sum_{n=1}^{\infty}(-1)^{n-1}\tan\frac{1}{n\sqrt{n}}$.

2483. Convince yourself that the d'Alembert test for convergence does not decide the question of the convergence of the series $\sum_{n=1}^{\infty} a_n$, where

$$a_{2k-1}=\frac{2^{k-1}}{3^{k-1}},\quad a_{2k}=\frac{2^{k-1}}{3^k}\quad (k=1,\ 2,\ \ldots),$$

whereas by means of the Cauchy test it is possible to establish that this series converges.

2484*. Convince yourself that the Leibniz test cannot be applied to the alternating series a) to d). Find out which of these series diverge, which converge conditionally and which converge absolutely:

a) $\frac{1}{\sqrt{2}-1}-\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}-1}-\frac{1}{\sqrt{3}+1}+\frac{1}{\sqrt{4}-1}-\frac{1}{\sqrt{4}+1}+\ldots$

$$\left(a_{2k-1}=\frac{1}{\sqrt{k+1}-1},\quad a_{2k}=-\frac{1}{\sqrt{k+1}+1}\right);$$

b) $1-\frac{1}{3}+\frac{1}{2}-\frac{1}{3^3}+\frac{1}{2^2}-\frac{1}{3^5}+\ldots$

$$\left(a_{2k-1}=\frac{1}{2^{k-1}},\quad a_{2k}=-\frac{1}{3^{2k-1}}\right);$$

c) $1-\frac{1}{3}+\frac{1}{3}-\frac{1}{3^2}+\frac{1}{5}-\frac{1}{3^3}+\ldots$

$$\left(a_{2k-1}=\frac{1}{2k-1},\quad a_{2k}=-\frac{1}{3^k}\right);$$

d) $\frac{1}{3}-1+\frac{1}{7}-\frac{1}{5}+\frac{1}{11}-\frac{1}{9}+\ldots$

$$\left(a_{2k-1}=\frac{1}{4k-1},\quad a_{2k}=-\frac{1}{4k-3}\right).$$

Test the following series with complex terms for convergence:

2485. $\sum_{n=1}^{\infty}\frac{n(2+i)^n}{2^n}$.

2486. $\sum_{n=1}^{\infty}\frac{n(2i-1)^n}{3^n}$.

2487. $\sum_{n=1}^{\infty}\frac{1}{n(3+i)^n}$.

2488. $\sum_{n=1}^{\infty}\frac{i^n}{n}$.

2489. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+i}$.

2490. $\sum_{n=1}^{\infty}\frac{1}{(n+i)\sqrt{n}}$.

2491. $\sum_{n=1}^{\infty} \frac{1}{[n+(2n-1)i]^2}$.   2492. $\sum_{n=1}^{\infty} \left[\frac{n(2-i)+1}{n(3-2i)-3i}\right]^n$.

2493. Between the curves $y=\frac{1}{x^3}$ and $y=\frac{1}{x^2}$ and to the right of their point of intersection are constructed segments parallel to the y-axis at an equal distance from each other. Will the sum of the lengths of these segments be finite?

2494. Will the sum of the lengths of the segments mentioned in Problem 2493 be finite if the curve $y=\frac{1}{x^2}$ is replaced by the curve $y=\frac{1}{x}$?

2495. Form the sum of the series $\sum_{n=1}^{\infty} \frac{1+n}{3^n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n-n}{3^n}$. Does this sum converge?

2496. Form the difference of the divergent series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ and $\sum_{n=1}^{\infty} \frac{1}{2n}$ and test it for convergence.

2497. Does the series formed by subtracting the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ from the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

2498. Choose two series such that their sum converges while their difference diverges.

2499. Form the product of the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$. Does this product converge?

2500. Form the series $\left(1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^{n-1}}+\ldots\right)^2$. Does this series converge?

2501. Given the series $1+\frac{1}{2!}-\frac{1}{3!}+\ldots+\frac{(-1)^n}{n!}+\ldots$ Estimate the error committed when replacing the sum of this series with the sum of the first four terms, the sum of the first five terms. What can you say about the signs of these errors?

2502*. Estimate the error due to replacing the sum of the series

$$\frac{1}{2}+\frac{1}{2!}\left(\frac{1}{2}\right)^2+\frac{1}{3!}\left(\frac{1}{2}\right)^3+\ldots+\frac{1}{n!}\left(\frac{1}{2}\right)^n+\ldots$$

by the sum of its first n terms.

2503. Estimate the error due to replacing the sum of the series

$$1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\ldots$$

by the sum of its first n terms. In particular, estimate the accuracy of such an approximation for $n=10$.

2504.** Estimate the error due to replacing the sum of the series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots$$

by the sum of its first n terms. In particular, estimate the accuracy of such an approximation for $n=1,000$.

2505.** Estimate the error due to replacing the sum of the series

$$1+2\left(\frac{1}{4}\right)^2+3\left(\frac{1}{4}\right)^4+\ldots+n\left(\frac{1}{4}\right)^{2n-2}+\ldots$$

by the sum of its first n terms.

2506. How many terms of the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ does one have to take to compute its sum to two decimal places? to three decimals?

2507. How many terms of the series $\sum\limits_{n=1}^{\infty}\frac{n}{(2n+1)5^n}$ does one have to take to compute its sum to two decimal places? to three? to four?

2508*. Find the sum of the series $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}+\ldots$

2509. Find the sum of the series

$$\sqrt[3]{x}+(\sqrt[5]{x}-\sqrt[3]{x})+(\sqrt[7]{x}-\sqrt[5]{x})+\ldots+(\sqrt[2k+1]{x}-\sqrt[2k-1]{x})+\ldots$$

Sec. 2. Functional Series

1°. Region of convergence. The set of values of the argument x for which the functional series

$$f_1(x)+f_2(x)+\ldots+f_n(x)+\ldots \qquad (1)$$

converges is called the *region of convergence* of this series. The function

$$S(x)=\lim_{n\to\infty}S_n(x),$$

where $S_n(x)=f_1(x)+f_2(x)+\ldots+f_n(x)$, and x belongs to the region of convergence, is called the *sum* of the series; $R_n(x)=S(x)-S_n(x)$ is the *remainder* of the series.

In the simplest cases, it is sufficient, when determining the region of convergence of a series (1), to apply to this series certain convergence tests, holding x constant.

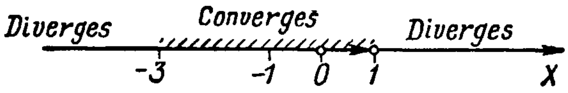

Fig. 104

Example 1. Determine the region of convergence of the series

$$\frac{x+1}{1\cdot 2}+\frac{(x+1)^2}{2\cdot 2^2}+\frac{(x+1)^3}{3\cdot 2^3}+\ldots+\frac{(x+1)^n}{n\cdot 2^n}+\ldots \qquad (2)$$

Solution. Denoting by u_n the general term of the series, we will have

$$\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim_{n\to\infty}\frac{|x+1|^{n+1}2^n n}{2^{n+1}(n+1)|x|^n}=\frac{|x+1|}{2}.$$

Using d'Alembert's test, we can assert that the series converges (and converges absolutely), if $\frac{|x+1|}{2}<1$, that is, if $-3<x<1$; the series diverges, if $\frac{|x+1|}{2}>1$, that is, if $-\infty<x<-3$ or $1<x<\infty$ (Fig. 104). When $x=1$ we get the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\ldots$, which diverges, and when $x=-3$ we have the series $-1+\frac{1}{2}-\frac{1}{3}+\ldots$, which (in accord with the Leibniz test) converges (conditionally).

Thus, the series converges when $-3\leqslant x<1$.

2°. Power series. For any *power series*

$$c_0+c_1(x-a)+c_2(x-a)^2+\ldots+c_n(x-a)^n+\ldots \qquad (3)$$

(c_n and a are real numbers) there exists an interval (the *interval of convergence*) $|x-a|<R$ with centre at the point $x=a$, with in which the series (3) converges absolutely; for $|x-a|>R$ the series diverges. In special cases, the *radius of convergence* R may also be equal to 0 and ∞. At the end-points of the interval of convergence $x=a\pm R$, the power series may either converge or diverge. The interval of convergence is ordinarily determined with the help of the d'Alembert or Cauchy tests, by applying them to a series, the terms of which are the absolute values of the terms of the given series (3).

Applying to the series of absolute values

$$|c_0|+|c_1||x-a|+\ldots+|c_n||x-a|^n+\ldots$$

the convergence tests of d'Alembert and Cauchy, we get, respectively, for the radius of convergence of the power series (3), the formulas

$$R=\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|c_n|}} \quad \text{and} \quad R=\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|.$$

However, one must be very careful in using them because the limits on the right frequently do not exist. For example, if an infinitude of coefficients c_n

vanishes [as a particular instance, this occurs if the series contains terms with only even or only odd powers of $(x-a)$], one cannot use these formulas. It is then advisable, when determining the interval of convergence, to apply the d'Alembert or Cauchy tests directly, as was done when we investigated the series (2), without resorting to general formulas for the radius of convergence.

If $z=x+iy$ is a complex variable, then for the power series

$$c_0+c_1(z-z_0)+c_2(z-z_0)^2+\ldots+c_n(z-z_0)^n+\ldots \tag{4}$$

$(c_n=a_n+ib_n,\ z_0=x_0+iy_0)$ there exists a certain circle *(circle of convergence)* $|z-z_0|<R$ with centre at the point $z=z_0$, inside which the series converges absolutely; for $|z-z_0|>R$ the series diverges. At points lying on the circumference of the circle of convergence, the series (4) may both converge and diverge. It is customary to determine the circle of convergence by means of the d'Alembert or Cauchy tests applied to the series

$$|c_0|+|c_1|\cdot|z-z_0|+|c_2|\cdot|z-z_0|^2+\ldots+|c_n|\cdot|z-z_0|^n+\ldots,$$

whose terms are absolute values of the terms of the given series. Thus, for example, by means of the d'Alembert test it is easy to see that the circle of convergence of the series

$$\frac{z+1}{1\cdot 2}+\frac{(z+1)^2}{2\cdot 2^2}+\frac{(z+1)^3}{3\cdot 2^3}+\ldots+\frac{(z+1)^n}{n\cdot 2^n}+\ldots$$

is determined by the inequality $|z+1|<2$ [it is sufficient to repeat the calculations carried out on page 305 which served to determine the interval of convergence of the series (2), only here x is replaced by z]. The centre of the circle of convergence lies at the point $z=-1$, while the radius R of this circle (the radius of convergence) is equal to 2.

3°. Uniform convergence. The functional series (1) converges uniformly on some interval if, no matter what $\varepsilon>0$, it is possible to find an N such that does not depend on x and that when $n>N$ for all x of the given interval we have the inequality $|R_n(x)|<\varepsilon$, where $R_n(x)$ is the remainder of the given series.

If $|f_n(x)|\leqslant c_n$ $(n=1, 2, \ldots)$ when $a\leqslant x\leqslant b$ and the number series $\sum\limits_{n=1}^{\infty} c_n$ converges, then the functional series (1) converges on the interval $[a, b]$ absolutely and uniformly *(Weierstrass' test)*.

The power series (3) converges absolutely and uniformly on any interval lying within its interval of convergence. The power series (3) may be termwise differentiated and integrated within its interval of convergence (for $|x-a|<R$); that is, if

$$c_0+c_1(x-a)+c_2(x-a)^2+\ldots+c_n(x-a)^n+\ldots=f(x), \tag{5}$$

then for any x of the interval of convergence of the series (3), we have

$$c_1+2c_2(x-a)+\ldots+nc_n(x-a)^{n-1}+\ldots=f'(x), \tag{6}$$

$$\int_{x_0}^{x} c_0dx+\int_{x_0}^{x} c_1(x-a)\,dx+\int_{x_0}^{x} c_2(x-a)^2\,dx+\ldots+\int_{x_0}^{x} c_n(x-a)^n\,dx+\ldots=$$

$$=\sum_{n=0}^{\infty} c_n\,\frac{(x-a)^{n+1}\,(x_0-a)^{n+1}}{n+1}=\int_{x_0}^{x} f(x)\,dx \tag{7}$$

[the number x_0 also belongs to the interval of convergence of the series (3)]. Here, the series (6) and (7) have the same interval of convergence as the series (3).

Find the region of convergence of the series:

2510. $\sum_{n=1}^{\infty} \frac{1}{n^x}$.

2511. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^x}$.

2512. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{\ln x}}$.

2513. $\sum_{n=1}^{\infty} \frac{\sin (2n-1) x}{(2n-1)^2}$.

2514. $\sum_{n=0}^{\infty} 2^n \sin \frac{x}{3^n}$.

2515**. $\sum_{n=0}^{\infty} \frac{\cos nx}{e^{nx}}$.

2516. $\sum_{n=0}^{\infty} (-1)^{n+1} e^{-n \sin x}$.

2517. $\sum_{n=1}^{\infty} \frac{n!}{x^n}$.

2518. $\sum_{n=1}^{\infty} \frac{1}{n!\, x^n}$.

2519. $\sum_{n=1}^{\infty} \frac{1}{(2n-1)\, x^n}$.

2520. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(x-2)^n}$.

2521. $\sum_{n=0}^{\infty} \frac{2n+1}{(n+1)^5 x^{2n}}$.

2522. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 3^n (x-5)^n}$.

2523. $\sum_{n=1}^{\infty} \frac{n^n}{x^{n^n}}$.

2524*. $\sum_{n=1}^{\infty} \left(x^n + \frac{1}{2^n x^n}\right)$.

2525. $\sum_{n=-1}^{\infty} x^n$.

Find the interval of convergence of the power series and test the convergence at the end-points of the interval of convergence:

2526. $\sum_{n=0}^{\infty} x^n$.

2527. $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 2^n}$.

2528. $\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$.

2529. $\sum_{n=1}^{\infty} \frac{2^{n-1} x^{2n-1}}{(4n-3)^2}$.

2530. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$.

2531. $\sum_{n=0}^{\infty} \frac{(n+1)^5 x^{2n}}{2n+1}$.

2532. $\sum_{n=0}^{\infty} (-1)^n (2n+1)^2 x^n$.

2533. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$.

2534. $\sum_{n=1}^{\infty} n!\, x^n$.

2535. $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$.

2536. $\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^{2n-1}x^n$.

2537. $\sum_{n=0}^{\infty}3^{n^2}x^{n^2}$.

2538. $\sum_{n=1}^{\infty}\frac{n}{n+1}\left(\frac{x}{2}\right)^n$.

2539. $\sum_{n=1}^{\infty}\frac{n!\,x^n}{n^n}$.

2540. $\sum_{n=2}^{\infty}\frac{x^{n-1}}{n\cdot 3^n\cdot\ln n}$.

2541. $\sum_{n=1}^{\infty}x^{n!}$.

2542**. $\sum_{n=1}^{\infty}n!\,x^{n!}$.

2543*. $\sum_{n=1}^{\infty}\frac{x^{n^2}}{2^{n-1}n^n}$.

2544*. $\sum_{n=1}^{\infty}\frac{x^{n^n}}{n^n}$.

2545. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-5)^n}{n\cdot 3^n}$.

2546. $\sum_{n=1}^{\infty}\frac{(x-3)^n}{n\cdot 5^n}$.

2547. $\sum_{n=1}^{\infty}\frac{(x-1)^{2n}}{n\cdot 9^n}$.

2548. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-2)^{2n}}{2n}$.

2549. $\sum_{n=1}^{\infty}\frac{(x+3)^n}{n^2}$.

2550. $\sum_{n=1}^{\infty}n^n(x+3)^n$.

2551. $\sum_{n=1}^{\infty}\frac{(x+5)^{2n-1}}{2n\cdot 4^n}$.

2552. $\sum_{n=1}^{\infty}\frac{(x-2)^n}{(2n-1)\,2^n}$.

2553. $\sum_{n=1}^{\infty}(-1)^{n+1}\times$

$\times\frac{(2n-1)^{2n}(x-1)^n}{(3n-2)^{2n}}$.

2554. $\sum_{n=1}^{\infty}\frac{n!\,(x+3)^n}{n^n}$.

2555. $\sum_{n=1}^{\infty}\frac{(x+1)^n}{(n+1)\ln^2(n+1)}$.

2556. $\sum_{n=1}^{\infty}\frac{(x-3)^{2n}}{(n+1)\ln(n+1)}$.

2557. $\sum_{n=1}^{\infty}(-1)^{n+1}\times$

$\times\frac{(x-2)^n}{(n+1)\ln(n+1)}$.

2558. $\sum_{n=1}^{\infty}\frac{(x+2)^{n^2}}{n^n}$.

2559*. $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}(x-1)^n$.

2560. $\sum_{n=1}^{\infty}\frac{(2n-1)^n(x+1)^n}{2^{n-1}\cdot n^n}$.

2561. $\sum_{n=0}^{\infty}(-1)^n\frac{\sqrt[3]{n+2}}{n+1}\times$

$\times(x-2)^n$.

2562. $\sum_{n=0}^{\infty}\frac{(3n-2)(x-3)^n}{(n+1)^2\,2^{n+1}}$.

2563. $\sum_{n=0}^{\infty}(-1)^n\frac{(x-3)^n}{(2n+1)\sqrt{n+1}}$.

Determine the circle of convergence:

2564. $\sum_{n=0}^{\infty} i^n z^n$. **2566.** $\sum_{n=1}^{\infty} \frac{(z-2i)^n}{n\cdot 3^n}$.

2565. $\sum_{n=0}^{\infty} (1+ni) z^n$. **2567.** $\sum_{n=0}^{\infty} \frac{z^{2n}}{2^n}$.

2568. $(1+2i)+(1+2i)(3+2i)z+\ldots+$
$+(1+2i)(3+2i)\ldots(2n+1+2i)z^n+\ldots$

2569. $1+\frac{z}{1-i}+\frac{z^2}{(1-i)(1-2i)}+\ldots$
$\ldots+\frac{z^n}{(1-i)(1-2i)\ldots(1-ni)}+\ldots$

2570. $\sum_{n=0}^{\infty} \left(\frac{1+2ni}{n+2i}\right)^n z^n$.

2571. Proceeding from the definition of uniform convergence, prove that the series

$$1+x+x^2+\ldots+x^n+\ldots$$

does not converge uniformly in the interval $(-1, 1)$, but converges uniformly on any subinterval within this interval.

Solution. Using the formula for the sum of a geometric progression, we get, for $|x|<1$,

$$R_n(x)=x^{n+1}+x^{n+2}+\ldots=\frac{x^{n+1}}{1-x}.$$

Within the interval $(-1, 1)$ let us take a subinterval $[-1+\alpha, 1-\alpha]$, where α is an arbitrarily small positive number. In this subinterval $|x|\leqslant 1-\alpha$, $|1-x|\geqslant\alpha$ and, consequently,

$$|R_n(x)|\leqslant\frac{(1-\alpha)^{n+1}}{\alpha}.$$

To prove the uniform convergence of the given series over the subinterval $[-1+\alpha, 1-\alpha]$, it must be shown that for any $\varepsilon>0$ it is possible to choose an N dependent only on ε such that for any $n>N$ we will have the inequality $|R_n(x)|<\varepsilon$ for *all* x of the subinterval under consideration.

Taking any $\varepsilon>0$, let us require that $\frac{(1-\alpha)^{n+1}}{\alpha}<\varepsilon$; whence $(1-\alpha)^{n+1}<\varepsilon\alpha$, $(n+1)\ln(1-\alpha)<\ln(\varepsilon\alpha)$, that is, $n+1>\frac{\ln(\varepsilon\alpha)}{\ln(1-\alpha)}$ [since $\ln(1-\alpha)<0$] and $n>\frac{\ln(\varepsilon\alpha)}{\ln(1-\alpha)}-1$. Thus, putting $N=\frac{\ln(\varepsilon\alpha)}{\ln(1-\alpha)}-1$, we are convinced that when $n>N$, $|R_n(x)|$ is indeed less than ε for all x of the subinterval $[-1+\alpha, 1-\alpha]$ and the uniform convergence of the given series on any subinterval within the interval $(-1, 1)$ is thus proved.

As for the entire interval $(-1, 1)$, it contains points that are arbitrarily close to $x=1$, and since $\lim_{x\to 1} R_n(x)=\lim_{x\to 1}\frac{x^{n+1}}{1-x}=\infty$, no matter how large n is, points x will be found for which $R_n(x)$ is greater than any arbitrarily large number Hence, it is impossible to choose an N such that for $n > N$ we would have the inequality $|R_n(x)| < \varepsilon$ at *all* points of the interval $(-1, 1)$, and this means that the convergence of the series in the interval $(-1, 1)$ is not uniform.

2572. Using the definition of uniform convergence, prove that:

a) the series

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots$$

converges uniformly in any finite interval;

b) the series

$$\frac{x^2}{1}-\frac{x^4}{2}+\frac{x^6}{3}-\ldots+\frac{(-1)^{n-1}x^{2n}}{n}+\ldots$$

converges uniformly throughout the interval of convergence $(-1, 1)$;

c) the series

$$1+\frac{1}{2^x}+\frac{1}{3^x}+\ldots+\frac{1}{n^x}+\ldots$$

converges uniformly in the interval $(1+\delta, \infty)$ where δ is any positive number;

d) the series

$$(x^2-x^4)+(x^4-x^6)+(x^6-x^8)+\ldots+(x^{2n}-x^{2n+2})+\ldots$$

converges not only within the interval $(-1, 1)$, but at the extremities of this interval, however the convergence of the series in $(-1, 1)$ is nonuniform.

Prove the uniform convergence of the functional series in the indicated intervals:

2573. $\sum\limits_{n=1}^{\infty}\frac{x^n}{n^2}$ on the interval $[-1, 1]$.

2574. $\sum\limits_{n=1}^{\infty}\frac{\sin nx}{2^n}$ over the entire number scale.

2575. $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{\sqrt{n}}$ on the interval $[0, 1]$.

Applying termwise differentiation and integration, find the sums of the series:

2576. $x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots+\frac{x^n}{n}+\ldots$

2577. $x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots$

2578. $x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots+\frac{x^{2n-1}}{2n-1}+\ldots$

2579. $x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots+(-1)^{n-1}\frac{x^{2n-1}}{2n-1}+\ldots$

2580. $1+2x+3x^2+\ldots+(n+1)x^n+\ldots$

2581. $1-3x^2+5x^4-\ldots+(-1)^{n-1}(2n-1)x^{2n-2}+\ldots$

2582. $1\cdot 2+2\cdot 3x+3\cdot 4x^2+\ldots+n(n+1)x^{n-1}+\ldots$

Find the sums of the series:

2583. $\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}+\ldots+\frac{n}{x^n}+\ldots$

2584. $x+\frac{x^5}{5}+\frac{x^9}{9}+\ldots+\frac{x^{4n-3}}{4n-3}+\ldots$

2585*. $1-\frac{1}{3\cdot 3}+\frac{1}{5\cdot 3^2}-\frac{1}{7\cdot 3^3}+\ldots+\frac{(-1)^{n-1}}{(2n-1)\,3^{n-1}}+\ldots$

2586. $\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\ldots+\frac{2n-1}{2^n}+\ldots$

Sec. 3. Taylor's Series

1°. Expanding a function in a power series. If a function $f(x)$ can be expanded, in some neighbourhood $|x-a|<R$ of the point a, in a series of powers of $x-a$, then this series (called *Taylor's series*) is of the form

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\ldots \qquad (1)$$

When $a=0$ the Taylor series is also called a *Maclaurin's series*. Equation (1) holds if when $|x-a|<R$ the *remainder term* (or simply remainder) of the Taylor series

$$R_n(x)=f(x)-\left[f(a)\sum_{k=1}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k\right]\longrightarrow 0$$

as $n\longrightarrow\infty$.

To evaluate the remainder, one can make use of the formula

$$R_n(x)=\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}[a+\theta(x-a)],\ \text{where}\ 0<\theta<1 \qquad (2)$$

(Lagrange's form).

Example 1. Expand the function $f(x)=\cosh x$ in a series of powers of x.

Solution. We find the derivatives of the given function $f(x)=\cosh x$, $f'(x)=\sinh x$, $f''(x)=\cosh x$, $f'''(x)=\sinh x$, ...; generally, $f^{(n)}(x)=\cosh x$, if n is even, and $f^{(n)}(x)=\sinh x$, if n is odd. Putting $a=0$, we get $f(0)=1$, $f'(0)=0$, $f''(0)=1$, $f'''(0)=0$, ...; generally, $f^{(n)}(0)=1$, if n is even, and $f^{(n)}(0)=0$ if n is odd. Whence, from (1), we have:

$$\cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots+\frac{x^{2n}}{(2n)!}+\ldots \qquad (3)$$

To determine the interval of convergence of the series (3) we apply the d'Alembert test. We have

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}:\frac{x^{2n}}{(2n)!}\right|=\lim_{n\to\infty}\frac{x^2}{(2n+1)(2n+2)}=0$$

for any x. Hence, the series converges in the interval $-\infty < x < \infty$. The remainder term, in accord with formula (2), has the form:

$$R_n(x)=\frac{x^{n+1}}{(n+1)!}\cosh\theta x, \text{ if } n \text{ is odd, and}$$

$$R_n(x)=\frac{x^{n+1}}{(n+1)!}\sinh\theta x, \text{ if } n \text{ is even.}$$

Since $0>\theta>1$, it follows that

$$|\cosh\theta x|=\frac{e^{\theta x}+e^{-\theta x}}{2}\leqslant e^{|x|}, \qquad |\sinh\theta x|=\left|\frac{e^{\theta x}-e^{-\theta x}}{2}\right|\leqslant e^{|x|},$$

and therefore $|R_n(x)|\leqslant\frac{|x|^{n+1}}{(n+1)!}e^{|x|}$. A series with the general term $\frac{|x|^n}{n!}$ converges for any x (this is made immediately evident with the help of d'Alembert's test); therefore, in accord with the necessary condition for convergence,

$$\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!}=0,$$

and consequently $\lim\limits_{n\to\infty}R_n(x)=0$ for any x. This signifies that the sum of the series (3) for any x is indeed equal to $\cosh x$.

2°. **Techniques employed for expanding in power series.**

Making use of the principal expansions

$$\text{I. } e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots \quad (-\infty<x<\infty),$$

$$\text{II. } \sin x=\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\ldots \quad (-\infty<x<\infty),$$

$$\text{III. } \cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots+(-1)^n\frac{x^{2n}}{(2n)!}+\ldots \quad (-\infty<x<\infty),$$

$$\text{IV. } (1+x)^m=1+\frac{m}{1!}x+\frac{m(m-1)}{2!}x^2+\ldots$$

$$\ldots+\frac{m(m-1)\ldots(m-n+1)}{n!}x^n+\ldots \quad (-1<x<1)\,^{*}),$$

$$\text{V } \ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots \quad (-1<x\leqslant 1),$$

and also the formula for the sum of a geometric progression, it is possible, in many cases, simply to obtain the expansion of a given function in a power series, without having to investigate the remainder term. It is sometimes advisable to make use of termwise differentiation or integration when expanding a function in a series. When expanding rational functions in power series it is advisable to decompose these functions into partial fractions.

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*) On the boundaries of the interval of convergence (i. e., when $x=-1$ and $x=1$) the expansion IV behaves as follows: for $m\geqslant 0$ it converges absolutely on both boundaries; for $0>m>-1$ it diverges when $x=-1$ and conditionally converges when $x=1$; for $m\leqslant -1$ it diverges on both boundaries.

Example 2. Expand in powers of x *) the function

$$f(x)=\frac{3}{(1-x)(1+2x)}.$$

Solution. Decomposing the function into partial fractions, we will have

$$f(x)=\frac{1}{1-x}+\frac{2}{1+2x}.$$

Since

$$\frac{1}{1-x}=1+x+x^2+\ldots=\sum_{n=0}^{\infty}x^n \tag{4}$$

and

$$\frac{1}{1+2x}=1-2x+(2x)^2-\ldots=\sum_{n=0}^{\infty}(-1)^n 2^n x^n, \tag{5}$$

it follows that we finally get

$$f(x)=\sum_{n=0}^{\infty}x^n+2\sum_{n=0}^{\infty}(-1)^n 2^n x^n=\sum_{n=0}^{\infty}[1+(-1)^n 2^{n+1}]\,x^n. \tag{6}$$

The geometric progressions (4) and (5) converge, respectively, when $|x|<1$ and $|x|<\frac{1}{2}$; hence, formula (6) holds for $|x|<\frac{1}{2}$, i.e., when $-\frac{1}{2}<x<\frac{1}{2}$.

3°. **Taylor's series for a function of two variables.** Expanding a function of two variables $f(x, y)$ into a *Taylor's series* in the neighbourhood of a point (a, b) has the form

$$f(x, y)=f(a, b)+\frac{1}{1!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]f(a, b)+\frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x}+\right.$$
$$\left.+(y-b)\frac{\partial}{\partial y}\right]^2 f(a, b)+\ldots+\frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^n f(a, b)+\ldots \tag{7}$$

If $a=b=0$, the Taylor series is then called a *Maclaurin's series*. Here the notation is as follows:

$$\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]f(a, b)=\left.\frac{\partial f(x, y)}{\partial x}\right|_{\substack{x=a\\ y=b}}(x-a)+\left.\frac{\partial f(x, y)}{\partial y}\right|_{\substack{x=a\\ y=b}}(y-b);$$

$$\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^2 f(a, b)=\left.\frac{\partial^2 f(x, y)}{\partial x^2}\right|_{\substack{x=a\\ y=b}}(x-a)^2+$$
$$+2\left.\frac{\partial^2 f(x, x)}{\partial x\,\partial y}\right|_{\substack{x=a\\ y=b}}(x-a)(y-b)+\left.\frac{\partial^2 f(x, y)}{\partial y^2}\right|_{\substack{x=a\\ y=b}}(y-b)^2 \text{ and so forth.}$$

*) Here and henceforward we mean "in positive integral powers".

The expansion (7) occurs if the remainder term of the series

$$R_n(x, y) = f(x, y) - \left\{ f(a, b) + \sum_{k=1}^{n} \frac{1}{k!} \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} \right]^k f(a, b) \right\} \longrightarrow 0$$

as $n \longrightarrow \infty$. The remainder term may be represented in the form

$$R_n(x, y) + \frac{1}{(n+1)!} \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} \right]^{n+1} f(x, y) \Bigg|_{\substack{x=a+\theta(x-a)\\ y=b+\theta(y-b)}},$$

where $0 < \theta < 1$.

Expand the indicated functions in positive integral powers of x, find the intervals of convergence of the resulting series and investigate the behaviour of their remainders:

2587. $a^x \ (a > 0)$.

2588. $\sin\left(x + \frac{\pi}{4}\right)$.

2589. $\cos(x+a)$.

2590. $\sin^2 x$.

2591*. $\ln(2+x)$.

Making use of the principal expansions I-V and a geometric progression, write the expansion, in powers of x, of the following functions, and indicate the intervals of convergence of the series:

2592. $\frac{2x-3}{(x-1)^2}$.

2593. $\frac{3x-5}{x^2-4x+3}$.

2594. xe^{-2x}.

2595. e^{x^2}.

2596. $\sinh x$.

2597. $\cos 2x$.

2598. $\cos^2 x$.

2599. $\sin 3x + x\cos 3x$.

2600. $\frac{x}{9+x^2}$.

2601. $\frac{1}{\sqrt{4-x^2}}$.

2602. $\ln\frac{1+x}{1-x}$.

2603. $\ln(1+x-2x^2)$.

Applying differentiation, expand the following functions in powers of x, and indicate the intervals in which these expansions occur:

2604. $(1+x)\ln(1+x)$.

2605. $\arctan x$

2606. $\arcsin x$.

2607. $\ln(x+\sqrt{1+x^2})$.

Applying various techniques, expand the given functions in powers of x and indicate the intervals in which these expansions occur:

2608. $\sin^2 x \cos^2 x$.

2609. $(1+x)e^{-x}$.

2610. $(1+e^x)^3$.

2611. $\sqrt[3]{8+x}$.

2612. $\frac{x^2-3x+1}{x^2-5x+6}$.

2613. $\cosh^3 x$.

2614. $\frac{1}{4-x^4}$.

2615. $\ln(x^2+3x+2)$.

2616. $\int\limits_0^x \frac{\sin x}{x}\,dx$.

2617. $\int\limits_0^x e^{-x^2}\,dx$.

2618. $\int\limits_0^x \frac{\ln(1+x)\,dx}{x}$.

2619. $\int\limits_0^x \frac{dx}{\sqrt{1-x^4}}$.

Write the first three nonzero terms of the expansion of the following functions in powers of x:

2620. $\tan x$.

2621. $\tanh x$.

2622. $e^{\cos x}$.

2623. $\sec x$.

2624. $\ln\cos x$.

2625. $e^x \sin x$.

2626*. Show that for computing the length of an ellipse it is possible to make use of the approximate formula

$$s \approx 2\pi a\left(1-\frac{\varepsilon^2}{4}\right),$$

where ε is the eccentricity and $2a$ is the major axis of the ellipse.

2627. A heavy string hangs, under its own weight, in a catenary line $y=a\cosh\frac{x}{a}$, where $a=\frac{H}{q}$ and H is the horizontal tension of the string, while q is the weight of unit length. Show that for small x, to the order of x^4, it may be taken that the string hangs in a parabola $y=a+\frac{x^2}{2a}$.

2628. Expand the function x^3-2x^2-5x-2 in a series of powers of $x+4$.

2629. $f(x)=5x^3-4x^2-3x+2$. Expand $f(x+h)$ in a series of powers of h

2630. Expand $\ln x$ in a series of powers of $x-1$.

2631. Expand $\frac{1}{x}$ in a series of powers of $x-1$.

2632. Expand $\frac{1}{x^2}$ in a series of powers of $x+1$.

2633. Expand $\frac{1}{x^2+3x+2}$ in a series of powers of $x+4$.

2634. Expand $\frac{1}{x^2+4x+7}$ in a series of powers of $x+2$.

2635. Expand e^x in a series of powers of $x+2$.

2636. Expand $\sqrt{x}$ in a series of powers of $x-4$.

2637. Expand $\cos x$ in a series of powers of $x-\frac{\pi}{2}$.

2638. Expand $\cos^2 x$ in a series of powers of $x-\frac{\pi}{4}$.

2639*. Expand $\ln x$ in a series of powers of $\frac{1-x}{1+x}$.

2640. Expand $\frac{x}{\sqrt{1+x}}$ in a series of powers of $\frac{x}{1+x}$.

2641. What is the magnitude of the error if we put approximately

$$e \approx 2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}?$$

2642. To what degree of accuracy will we calculate the number $\frac{\pi}{4}$, if we make use of the series

$$\operatorname{arc\,tan} x = x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots,$$

by taking the sum of its first five terms when $x=1$?

2643*. Calculate the number $\frac{\pi}{6}$ to three decimals by expanding the function $\arcsin x$ in a series of powers of x (see Example 2606).

2644. How many terms do we have to take of the series

$$\cos x = 1-\frac{x^2}{2!}+\ldots,$$

in order to calculate $\cos 18°$ to three decimal places?

2645. How many terms do we have to take of the series

$$\sin x = x-\frac{x^3}{3!}+\ldots,$$

to calculate $\sin 15°$ to four decimal places?

2646. How many terms of the series

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots$$

have to be taken to find the number e to four decimal places?

2647. How many terms of the series

$$\ln(1+x) = x-\frac{x^2}{2}+\ldots,$$

do we have to take to calculate $\ln 2$ to two decimals? to 3 decimals?

2648. Calculate $\sqrt[3]{7}$ to two decimals by expanding the function $\sqrt[3]{8+x}$ in a series of powers of x.

2649. Find out the origin of the approximate formula $\sqrt{a^2+x} \approx a+\frac{x}{2a}$ $(a>0)$, evaluate it by means of $\sqrt{23}$, putting $a=5$, and estimate the error.

2650. Calculate $\sqrt[4]{19}$ to three decimals.

2651. For what values of x does the approximate formula

$$\cos x \approx 1 - \frac{x^2}{2}$$

yield an error not exceeding 0.01? 0.001? 0.0001?

2652. For what values of x does the approximate formula

$$\sin x \approx x$$

yield an error that does not exceed 0.01? 0.001?

2653. Evaluate $\int_0^{1/2} \frac{\sin x}{x}\,dx$ to four decimals.

2654. Evaluate $\int_0^{1} e^{-x^2}\,dx$ to four decimals.

2655. Evaluate $\int_0^{1} \sqrt[3]{x}\cos x\,dx$ to three decimals.

2656. Evaluate $\int_0^{1} \frac{\sin x}{\sqrt{x}}\,dx$ to three decimals.

2657. Evaluate $\int_0^{1/4} \sqrt{1+x^3}\,dx$ to four decimals.

2658. Evaluate $\int_0^{1/9} \sqrt{x}\,e^x\,dx$ to three decimals.

2659. Expand the function $\cos(x-y)$ in a series of powers of x and y, find the region of convergence of the resulting series and investigate the remainder.

Write the expansions, in powers of x and y, of the following functions and indicate the regions of convergence of the series:

2660. $\sin x \cdot \sin y$.

2661. $\sin(x^2+y^2)$.

2662*. $\frac{1-x+y}{1+x-y}$.

2663*. $\ln(1-x-y+xy)$.

2664*. $\arctan\frac{x+y}{1-xy}$.

2665. $f(x, y) = ax^2 + 2bxy + cy^2$. Expand $f(x+h, y+k)$ in powers of h and k.

2666. $f(x, y) = x^3 - 2y^3 + 3xy$. Find the increment of this function when passing from the values $x=1$, $y=2$ to the values $x=1+h$, $y=2+k$.

2667. Expand the function e^{x+y} in powers of $x-2$ and $y+2$.

2668. Expand the function $\sin(x+y)$ in powers of x and $y-\frac{\pi}{2}$.

Write the first three or four terms of a power-series expansion in x and y of the functions:

2669. $e^x \cos y$.

2670. $(1+x)^{1+y}$.

Sec. 4. Fourier Series

1°. Dirichlet's theorem. We say that a function $f(x)$ satisfies the *Dirichlet conditions* in an interval (a, b) if, in this interval, the function

1) is uniformly bounded; that is $|f(x)| \leqslant M$ when $a < x < b$, where M is constant;

2) has no more than a finite number of points of discontinuity and all of them are of the first kind [i.e., at each discontinuity ξ the function $f(x)$ has a finite limit on the left $f(\xi-0) = \lim\limits_{\varepsilon \to 0} f(\xi-\varepsilon)$ and a finite limit on the right $f(\xi+0) = \lim\limits_{\varepsilon \to 0} f(\xi+\varepsilon)$ $(\varepsilon > 0)$];

3) has no more than a finite number of points of strict extremum.

Dirichlet's theorem asserts that a function $f(x)$, which in the interval $(-\pi, \pi)$ satisfies the Dirichlet conditions at any point x of this interval at which $f(x)$ is continuous, may be expanded in a trigonometric *Fourier series*:

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots + a_n \cos nx + b_n \sin nx + \ldots, \quad (1)$$

where the *Fourier coefficients* a_n and b_n are calculated from the formulas

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \; (n=0, 1, 2, \ldots); \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \; (n=1, 2, \ldots).$$

If x is a point of discontinuity, belonging to the interval $(-\pi, \pi)$, of a function $f(x)$, then the sum of the Fourier series $S(x)$ is equal to the arithmetical mean of the left and right limits of the function:

$$S(x) = \frac{1}{2}[f(x-0) + f(x+0)].$$

At the end-points of the interval $x = -\pi$ and $x = \pi$,

$$S(-\pi) = S(\pi) = \frac{1}{2}[f(-\pi+0) + f(\pi-0)].$$

2°. Incomplete Fourier series. If a function $f(x)$ is even [i. e., $f(-x) = f(x)$], then in formula (1)

$$b_n = 0 \quad (n = 1, 2, \ldots)$$

and

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx \quad (n=0, 1, 2, \ldots).$$

If a function $f(x)$ is odd [i.e., $f(-x)=-f(x)$], then $a_n=0$ ($n=0, 1, 2 \ldots$) and

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \quad (n=1, 2, \ldots).$$

A function specified in an interval $(0, \pi)$ may, at our discretion, be continued in the interval $(-\pi, 0)$ either as an even or an odd function; hence, it may be expanded in the interval $(0, \pi)$ in an incomplete Fourier series of sines or of cosines of multiple arcs.

3°. **Fourier series of a period $2l$.** If a function $f(x)$ satisfies the Dirichlet conditions in some interval $(-l, l)$ of length $2l$, then at the discontinuities of the function belonging to this interval the following expansion holds:

$$f(x)=\frac{a_0}{2}+a_1\cos\frac{\pi x}{l}+b_1\sin\frac{\pi x}{l}+a_2\cos\frac{2\pi x}{l}+b_2\sin\frac{2\pi x}{l}+\ldots$$
$$\ldots+a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}+\ldots,$$

where

$$\left.\begin{aligned} a_n&=\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \quad (n=0, 1, 2, \ldots),\\ b_n&=\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx \quad (n=1, 2, \ldots).\end{aligned}\right\} \qquad (2)$$

At the points of discontinuity of the function $f(x)$ and at the end-points $x=\pm l$ of the interval, the sum of the Fourier series is defined in a manner similar to that which we have in the expansion in the interval $(-\pi, \pi)$.

In the case of an expansion of the function $f(x)$ in a Fourier series in an arbitrary interval $(a, a+2l)$ of length $2l$, the limits of integration in formulas (2) should be replaced respectively by a and $a+2l$

Expand the following functions in a Fourier series in the interval $(-\pi, \pi)$, determine the sum of the series at the points of discontinuity and at the end-points of the interval ($x=-\pi$, $x=\pi$), construct the graph of the function itself and of the sum of the corresponding series [outside the interval $(-\pi, \pi)$ as well]:

2671. $f(x)=\begin{cases} c_1 & \text{when } -\pi<x\leqslant 0,\\ c_2 & \text{when } 0<x<\pi.\end{cases}$

Consider the special case when $c_1=-1$, $c_2=1$.

2672. $f(x)=\begin{cases} ax & \text{when } -\pi<x\leqslant 0,\\ bx & \text{when } 0\leqslant x<\pi.\end{cases}$

Consider the special cases: a) $a=b=1$; b) $a=-1$, $b=1$; c) $a=0$, $b=1$; d) $a=1$, $b=0$.

2673. $f(x)=x^2$.

2674. $f(x)=e^{ax}$.

2675. $f(x)=\sin ax$.

2676. $f(x)=\cos ax$.

2677. $f(x)=\sinh ax$.

2678. $f(x)=\cosh ax$.

2679. Expand the function $f(x)=\frac{\pi-x}{2}$ in a Fourier series in the interval $(0, 2\pi)$.

2680. Expand the function $f(x)=\frac{\pi}{4}$ in sines of multiple arcs in the interval $(0, \pi)$. Use the expansion obtained to sum the number series:

a) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$; b) $1+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}-\ldots$;

c) $1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\ldots$

Take the functions indicated below and expand them, in the interval $(0, \pi)$, into incomplete Fourier series: a) of sines of multiple arcs, b) of cosines of multiple arcs. Sketch graphs of the functions and graphs of the sums of the corresponding series in their domains of definition.

2681. $f(x)=x$. Find the sum of the following series by means of the expansion obtained:

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$$

2682. $f(x)=x^2$. Find the sums of the following number series by means of the expansion obtained:

$$1)\ 1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots;\quad 2)\ 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$$

2683. $f(x)=e^{ax}$.

2684. $f(x)=\begin{cases} 1 \text{ when } 0<x<\frac{\pi}{2}, \\ 0 \text{ when } \frac{\pi}{2}\leqslant x<\pi. \end{cases}$

2685. $f(x)=\begin{cases} x \text{ when } 0<x\leqslant\frac{\pi}{2}, \\ \pi-x \text{ when } \frac{\pi}{2}<x<\pi. \end{cases}$

Expand the following functions, in the interval $(0, \pi)$, in sines of multiple arcs:

2686. $f(x)=\begin{cases} x \text{ when } 0<x\leqslant\frac{\pi}{2}, \\ 0 \text{ when } \frac{\pi}{2}<x<\pi. \end{cases}$

2687. $f(x)=x(\pi-x)$.

2688. $f(x)=\sin\frac{x}{2}$.

Expand the following functions, in the interval $(0, \pi)$, in cosines of multiple arcs:

2689. $f(x)=\begin{cases} 1 \text{ when } 0<x\leqslant h, \\ 0 \text{ when } h<x<\pi. \end{cases}$

2690. $f(x)=\begin{cases} 1-\frac{x}{2h} & \text{when } 0<x\leqslant 2h, \\ 0 & \text{when } 2h<x<\pi. \end{cases}$

2691. $f(x)=x\sin x$.

2692. $f(x)=\begin{cases} \cos x & \text{when } 0<x\leqslant \frac{\pi}{2}, \\ -\cos x & \text{when } \frac{\pi}{2}<x<\pi. \end{cases}$

2693. Using the expansions of the functions x and x^2 in the interval $(0, \pi)$ in cosines of multiple arcs (see Problems 2681 and 2682), prove the equality

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}=\frac{3x^2-6\pi x+2\pi^2}{12} \quad (0\leqslant x\leqslant \pi).$$

2694**. Prove that if the function $f(x)$ is even and we have $f\left(\frac{\pi}{2}+x\right)=-f\left(\frac{\pi}{2}-x\right)$, then its Fourier series in the interval $(-\pi, \pi)$ represents an expansion in cosines of odd multiple arcs, and if the function $f(x)$ is odd and $f\left(\frac{\pi}{2}+x\right)=f\left(\frac{\pi}{2}-x\right)$, then in the interval $(-\pi, \pi)$ it is expanded in sines of odd multiple arcs.

Expand the following functions in Fourier series in the indicated intervals:

2695. $f(x)=|x| \quad (-1<x<1)$.

2696. $f(x)=2x \quad (0<x<1)$.

2697. $f(x)=e^x \quad (-l<x<l)$.

2698. $f(x)=10-x \quad (5<x<15)$.

Expand the following functions, in the indicated intervals, in incomplete Fourier series: a) in sines of multiple arcs, and b) in cosines of multiple arcs:

2699. $f(x)=1 \quad (0<x<1)$.

2700. $f(x)=x \quad (0<x<l)$.

2701. $f(x)=x^2 \quad (0<x<2\pi)$.

2702. $f(x)=\begin{cases} x & \text{when } 0<x\leqslant 1, \\ 2-x & \text{when } 1<x<2. \end{cases}$

2703. Expand the following function in cosines of multiple arcs in the interval $\left(\frac{3}{2}, 3\right)$:

$$f(x)=\begin{cases} 1 & \text{when } \frac{3}{2}<x\leqslant 2, \\ 3-x & \text{when } 2<x<3. \end{cases}$$

Chapter IX

DIFFERENTIAL EQUATIONS

Sec. 1. Verifying Solutions. Forming Differential Equations of Families of Curves. Initial Conditions

1°. Basic concepts. An equation of the type

$$F(x, y, y', \ldots, y)^{(n)}=0, \qquad (1)$$

where $y=y(x)$ is the sought-for function, is called a *differential equation of order n*. The function $y=\varphi(x)$, which converts equation (1) into an identity, is called the *solution* of the equation, while the graph of this function is called an *integral curve*. If the solution is represented implicitly, $\Phi(x, y)=0$, then it is usually called an *integral*

Example 1. Check that the function $y=\sin x$ is a solution of the equation

$$y''+y=0.$$

Solution. We have:

$$y'=\cos x, \quad y''=-\sin x$$

and, consequently,

$$y''+y=-\sin x+\sin x\equiv 0.$$

The integral

$$\Phi(x, y, C_1, \ldots, C_n)=0 \qquad (2)$$

of the differential equation (1), which contains n independent arbitrary constants $C_1, \ldots, C_n$ and is equivalent (in the given region) to equation (1), is called the *general integral* of this equation (in the respective region). By assigning definite values to the constants $C_1, \ldots, C_n$ in (2), we get *particular integrals*.

Conversely, if we have a family of curves (2) and eliminate the parameters $C_1, \ldots, C_n$ from the system of equations

$$\Phi=0, \ \frac{d\Phi}{dx}=0, \ \ldots, \ \frac{d^n\Phi}{dx^n}=0,$$

we, generally speaking, get a differential equation of type (1) whose general integral in the corresponding region is the relation (2).

Example 2. Find the differential equation of the family of parabolas

$$y=C_1(x-C_2)^2. \qquad (3)$$

Solution. Differentiating equation (3) twice, we get:

$$y'=2C_1(x-C_2) \quad \text{and} \quad y''=2C_1. \qquad (4)$$

Eliminating the parameters C_1 and C_2 from equations (3) and (4), we obtain the desired differential equation

$$2yy''=y'^2.$$

It is easy to verify that the function (3) converts this equation into an identity.

2°. **Initial conditions.** If for the desired particular solution $y=y(x)$ of a differential equation

$$y^{(n)}=f(x, y, y', \ldots, y^{(n-1)}) \qquad (5)$$

the *initial conditions*

$$y(x_0)=y_0, \quad y'(x_0)=y'_0, \quad \ldots, \quad y^{(n-1)}(x_0)=y_0^{(n-1)}$$

are given and we know the *general solution* of equation (5)

$$y=\varphi(x, C_1, \ldots, C_n),$$

then the arbitrary constants $C_1, \ldots, C_n$ are determined (if this is possible) from the system of equations

$$\left.\begin{aligned} y_0 &= \varphi(x_0, C_1, \ldots, C_n), \\ y'_0 &= \varphi'_x(x_0, C_1, \ldots, C_n), \\ &\ldots\ldots\ldots\ldots \\ y_0^{(n-1)} &= \varphi_{x^{n-1}}^{(n-1)}(x_0, C_1, \ldots, C_n). \end{aligned}\right\}$$

Example 3. Find the curve of the family

$$y=C_1e^x+C_2e^{-2x}, \qquad (6)$$

for which $y(0)=1$, $y'(0)=-2$.

Solution. We have:

$$y'=C_1e^x-2C_2e^{-2x} \qquad (7)$$

Putting $x=0$ in formulas (6) and (7), we obtain

$$1=C_1+C_2, \quad -2=C_1-2C_2,$$

whence

$$C_1=0, \quad C_2=1$$

and, hence,

$$y=e^{-2x}.$$

Determine whether the indicated functions are solutions of the given differential equations:

2704. $xy'=2y$, $y=5x^2$.

2705. $y'^2=x^2+y^2$, $y=\frac{1}{x}$.

2706. $(x+y)\,dx+x\,dy=0$, $y=\frac{C^2-x^2}{2x}$.

2707. $y''+y=0$, $y=3\sin x-4\cos x$.

2708. $\frac{d^2x}{dt^2}+\omega^2x=0$, $x=C_1\cos\omega t+C_2\sin\omega t$.

2709. $y''-2y'+y=0$; a) $y=xe^x$, b) $y=x^2e^x$.

2710. $y''-(\lambda_1+\lambda_2)y'+\lambda_1\lambda_2y=0$,
$y=C_1e^{\lambda_1x}+C_2e^{\lambda_2x}$.

Show that for the given differential equations the indicated relations are integrals:

2711. $(x-2y)y'=2x-y$, $x^2-xy+y^2=C^2$.

2712. $(x-y+1)y'=1, \quad y=x+Ce^y$.
2713. $(xy-x)y''+xy'^2+yy'-2y'=0, \quad y=\ln(xy)$.

Form differential equations of the given families of curves (C, C_1, C_2, C_3 are arbitrary constants):

2714. $y=Cx$.
2715. $y=Cx^2$.
2716. $y^2=2Cx$.
2717. $x^2+y^2=C^2$.
2718. $y=Ce^x$.
2719. $x^3=C(x^2-y^2)$.
2720. $y^2+\frac{1}{x}=2+Ce^{-\frac{y^2}{2}}$.
2721. $\ln\frac{x}{y}=1+ay$ (a is a parameter).
2722. $(y-y_0)^2=2px$ (y_0, p are parameters).
2723. $y=C_1e^{2x}+C_2e^{-x}$.
2724. $y=C_1\cos 2x+C_2\sin 2x$.
2725. $y=(C_1+C_2x)e^x+C_3$.

2726. Form the differential equation of all straight lines in the xy-plane.

2727. Form the differential equation of all parabolas with vertical axis in the xy-plane.

2728. Form the differential equation of all circles in the xy-plane.

For the given families of curves find the lines that satisfy the given initial conditions:

2729. $x^2-y^2=C$, $y(0)=5$.
2730. $y=(C_1+C_2x)e^{2x}, \quad y(0)=0, \quad y'(0)=1$.
2731. $y=C_1\sin(x-C_2), \ y(\pi)=1, \quad y'(\pi)=0$.
2732. $y=C_1e^{-x}+C_2e^x+C_3e^{2x}$;
$y(0)=0, \quad y'(0)=1, \ y''(0)=-2$.

Sec. 2. First-Order Differential Equations

1°. Types of first-order differential equations. A differential equation of the first order in an unknown function y, solved for the derivative y', is of the form

$$y'=f(x, y), \tag{1}$$

where $f(x, y)$ is the given function. In certain cases it is convenient to consider the variable x as the sought-for function, and to write (1) in the form

$$x'=g(x, y), \tag{1'}$$

where $g(x, y)=\frac{1}{f(x, y)}$.

Taking into account that $y'=\frac{dy}{dx}$ and $x'=\frac{dx}{dy}$, the differential equations (1) and (1') may be written in the symmetric form

$$P(x, y)\,dx+Q(x, y)\,dy=0, \tag{2}$$

where $P(x, y)$ and $Q(x, y)$ are known functions.

By solutions to (2) we mean functions of the form $y=\varphi(x)$ or $x=\psi(y)$ that satisfy this equation. The general integral of equations (1) and (1'), or equation (2), is of the form

$$\Phi(x, y, C)=0,$$

where C is an arbitrary constant.

2°. Direction field. The set of directions

$$\tan\alpha=f(x, y)$$

is called a direction field of the differential equation (1) and is ordinarily depicted by means of short lines or arrows inclined at an angle α.

Curves $f(x, y)=k$, at the points of which the inclination of the field has a constant value, equal to k, are called *isoclines*. By constructing the isoclines and direction field, it is possible, in the simplest cases, to give a

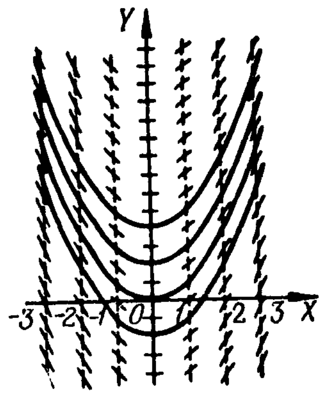

Fig. 105

rough sketch of the field of integral curves, regarding the latter as curves which at each point have the given direction of the field.

Example 1. Using the method of isoclines, construct the field of integral curves of the equation

$$y'=x.$$

Solution. By constructing the isoclines $x=k$ (straight lines) and the direction field, we obtain approximately the field of integral curves (Fig. 105). The family of parabolas

$$y=\frac{x^2}{2}+C$$

is the general solution.

Using the method of isoclines, make approximate constructions of fields of integral curves for the indicated differential equations:

2733. $y'=-x$.

2734. $y'=-\frac{x}{y}$.

2735. $y'=1+y^2$.

2736. $y'=\frac{x+y}{x-y}$.

2737. $y'=x^2+y^2$.

3°. Cauchy's theorem. If a function $f(x, y)$ is continuous in some region $U\{a<x<A,\ b<y<B\}$ and in this region has a bounded derivative $f'_y(x, y)$, then through each point (x_0, y_0) that belongs to U there passes one and only one integral curve $y=\varphi(x)$ of the equation (1) $[\varphi(x_0)=y_0]$.

4°. Euler's broken-line method. For an approximate construction of the integral curve of equation (1) passing through a given point $M_0(x_0, y_0)$, we replace the curve by a broken line with vertices $M_i(x_i, y_i)$, where

$$x_{i+1}=x_i+\Delta x_i,\quad y_{i+1}=y_i+\Delta y_i,$$
$$\Delta x_i=h \text{ (one step of the process)},$$
$$\Delta y_i=hf(x_i, y_i)\quad (i=0, 1, 2, \ldots).$$

Example 2. Using Euler's method for the equation

$$y'=\frac{xy}{2},$$

find $y(1)$, if $y(0)=1$ $(h=0.1)$.

We construct the table:

i	x_i	y_i	$\Delta y_i=\frac{x_i y_i}{20}$
0	0	1	0
1	0.1	1	0 005
2	0.2	1.005	0.010
3	0.3	1 015	0 015
4	0 4	1.030	0 021
5	0.5	1.051	0 026
.6	0.6	1.077	0 032
7	0.7	1.109	0.039
8	0.8	1.148	0 046
9	0.9	1.194	0.054
10	1.0	1.248	

Thus, $y(1)=1.248$. For the sake of comparison, the exact value is $y(1)=e^{\frac{1}{4}}\approx 1.284$

Using Euler's method, find the particular solutions to the given differential equations for the indicated values of x:

2738. $y'=y$, $y(0)=1$; find $y(1)$ $(h=0.1)$.

2739. $y'=x+y$, $y(1)=1$; find $y(2)$, $(h=0.1)$.

2740. $y'=-\frac{y}{1+x}$, $y(0)=2$; find $y(1)$ $(h=0.1)$.

2741. $y'=y-\frac{2x}{y}$, $y(0)=1$; find $y(1)$ $(h=0.2)$.

Sec. 3. First-Order Differential Equations with Variables Separable. Orthogonal Trajectories

1°. First-order equations with variables separable. An equation with *variables separable* is a first-order equation of the type

$$y' = f(x)\,g(y) \tag{1}$$

or

$$X(x)\,Y(y)\,dx + X_1(x)\,Y_1(y)\,dy = 0 \tag{1'}$$

Dividing both sides of equation (1) by $g(y)$ and multiplying by dx, we get $\frac{dy}{g(y)} = f(x)\,dx$ Whence, by integrating, we get the general integral of equation (1) in the form

$$\int \frac{dy}{g(y)} = \int f(x)\,dx + C \tag{2}$$

Similarly, dividing both sides of equation (1') by $X_1(x)\,Y(y)$ and integrating, we get the general integral of (1') in the form

$$\int \frac{X(x)}{X_1(x)}\,dx + \int \frac{Y_1(y)}{Y(y)}\,dy = C \tag{2'}$$

If for some value $y = y_0$ we have $g(y_0) = 0$, then the function $y = y_0$ is also (as is directly evident) a solution of equation (1) Similarly, the straight lines $x = a$ and $y = b$ will be the integral curves of equation (1'), if a and b are, respectively, the roots of the equations $X_1(x) = 0$ and $Y(y) = 0$, by the left sides of which we had to divide the initial equation.

Example 1. Solve the equation

$$y' = -\frac{y}{x}. \tag{3}$$

In particular, find the solution that satisfies the initial conditions

$$y(1) = 2$$

Solution. Equation (3) may be written in the form

$$\frac{dy}{dx} = -\frac{y}{x}.$$

Whence, separating variables, we have

$$\frac{dy}{y} = -\frac{dx}{x}$$

and, consequently,

$$\ln|y| = -\ln|x| + \ln C_1,$$

where the arbitrary constant $\ln C_1$ is taken in logarithmic form. After taking antilogarithms we get the general solution

$$y = \frac{C}{x}, \tag{4}$$

where $C = \pm C_1$.

When dividing by y we could lose the solution $y = 0$, but the latter is contained in the formula (4) for $C = 0$.

Utilizing the given initial conditions, we get $C=2$; and, hence, the desired particular solution is

$$y=\frac{2}{x}.$$

2° Certain differential equations that reduce to equations with variables separable. Differential equations of the form

$$y'=f(ax+by+c) \quad (b \neq 0)$$

reduce to equations of the form (1) by means of the substitution $u=ax+by+c$, where u is the new sought-for function

3° Orthogonal trajectories are curves that intersect the lines of the given family $\Phi(x, y, a)=0$ (a is a parameter) at a right angle. If $F(x, y, y')=0$ is the differential equation of the family, then

$$F\left(x, y, -\frac{1}{y'}\right)=0$$

is the differential equation of the orthogonal trajectories.

Example 2. Find the orthogonal trajectories of the family of ellipses

$$x^2+2y^2=a^2. \tag{5}$$

Solution Differentiating the equation (5), we find the differential equation of the family

$$x+2yy'=0.$$

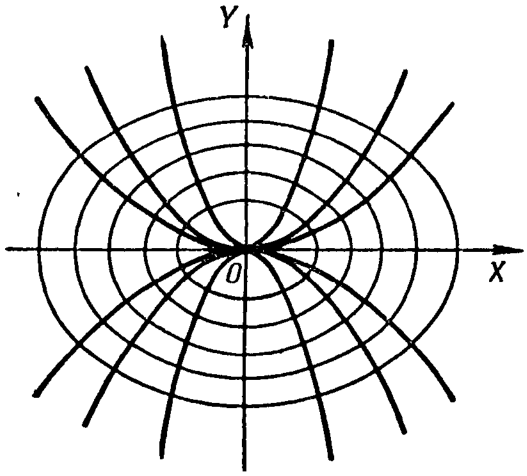

Fig. 106

Whence, replacing y' by $-\frac{1}{y'}$, we get the differential equation of the orthogonal trajectories

$$x-\frac{2y}{y'}=0 \quad \text{or} \quad y'=\frac{2y}{x}.$$

Integrating, we have $y=Cx^2$ (family of parabolas) (Fig. 106).

4°. Forming differential equations. When forming differential equations in geometrical problems, we can frequently make use of the geometrical meaning of the derivative as the tangent of an angle formed by the tangent line to the curve in the positive x-direction. In many cases this makes it possible straightway to establish a relationship between the ordinate y of the desired curve, its abscissa x, and the tangent of the angle of the tangent line y', that is to say, to obtain the differential equation. In other instances (see Problems 2783, 2890, 2895), use is made of the geometrical significance of the definite integral as the area of a curvilinear trapezoid or the length of an arc. In this case, by hypothesis we have a simple integral equation (since the desired function is under the sign of the integral); however, we can readily pass to a differential equation by differentiating both sides.

Example 3. Find a curve passing through the point (3,2) for which the segment of any tangent line contained between the coordinate axes is divided in half at the point of tangency.

Solution. Let $M(x,y)$ be the mid-point of the tangent line AB, which by hypothesis is the point of tangency (the points A and B are points of intersection of the tangent line with the y- and x-axes). It is given that $OA=2y$ and $OB=2x$. The slope of the tangent to the curve at $M(x, y)$ is

$$\frac{dy}{dx}=-\frac{OA}{OB}=-\frac{y}{x}.$$

This is the differential equation of the sought-for curve. Transforming, we get

$$\frac{dx}{x}+\frac{dy}{y}=0$$

and, consequently,

$$\ln x+\ln y=\ln C \text{ or } xy=C.$$

Utilizing the initial condition, we determine $C=3\cdot 2=6$. Hence, the desired curve is the hyperbola $xy=6$.

Solve the differential equations:

2742. $\tan x\sin^2 y\,dx+\cos^2 x\cot y\,dy=0$.

2743. $xy'-y=y^3$.

2744. $xyy'=1-x^2$.

2745. $y-xy'=a(1+x^2y')$.

2746. $3e^x\tan y\,dx+(1-e^x)\sec^2 y\,dy=0$.

2747. $y'\tan x=y$.

Find the particular solutions of equations that satisfy the indicated initial conditions:

2748. $(1+e^x)\,y\,y'=e^x$; $y=1$ when $x=0$.

2749. $(xy^2+x)\,dx+(x^2y-y)\,dy=0$; $y=1$ when $x=0$.

2750. $y'\sin x=y\ln y$; $y=1$ when $x=\frac{\pi}{2}$.

Solve the differential equations by changing the variables:

2751. $y'=(x+y)^2$.

2752. $y=(8x+2y+1)^2$.

2753. $(2x+3y-1)\,dx+(4x+6y-5)\,dy=0$.

2754. $(2x-y)\,dx+(4x-2y+3)\,dy=0$.

In Examples 2755 and 2756, pass to polar coordinates:

2755. $y' = \frac{\sqrt{x^2+y^2}-x}{y}$.

2756. $(x^2+y^2)\,dx - xy\,dy = 0$.

2757*. Find a curve whose segment of the tangent is equal to the distance of the point of tangency from the origin.

2758. Find the curve whose segment of the normal at any point of a curve lying between the coordinate axes is divided in two at this point.

2759. Find a curve whose subtangent is of constant length a.

2760. Find a curve which has a subtangent twice the abscissa of the point of tangency.

2761*. Find a curve whose abscissa of the centre of gravity of an area bounded by the coordinate axes, by this curve and the ordinate of any of its points is equal to 3/4 the abscissa of this point.

2762. Find the equation of a curve that passes through the point (3,1), for which the segment of the tangent between the point of tangency and the x-axis is divided in half at the point of intersection with the y-axis.

2763. Find the equation of a curve which passes through the point (2,0), if the segment of the tangent to the curve between the point of tangency and the y-axis is of constant length 2.

Find the orthogonal trajectories of the given families of curves (a is a parameter), construct the families and their orthogonal trajectories.

2764. $x^2+y^2=a^2$.

2765. $y^2=ax$.

2766. $xy=a$.

2767. $(x-a)^2+y^2=a^2$.

Sec. 4. First-Order Homogeneous Differential Equations

1°. Homogeneous equations. A differential equation

$$P(x, y)\,dx + Q(x, y)\,dy = 0 \quad (1)$$

is called *homogeneous*, if $P(x, y)$ and $Q(x, y)$ are homogeneous functions of the same degree. Equation (1) may be reduced to the form

$$y' = f\left(\frac{y}{x}\right);$$

and by means of the substitution $y = xu$, where u is a new unknown function, it is transformed to an equation with variables separable. We can also apply the substitution $x = yu$.

Example 1. Find the general solution to the equation

$$y' = e^{\frac{y}{x}} + \frac{y}{x}.$$

Solution. Put $y=ux$; then $u+xu'=e^u+u$ or

$$e^{-u}du=\frac{dx}{x}.$$

Integrating, we get $u=-\ln\ln\frac{C}{x}$, whence

$$y=-x\ln\ln\frac{C}{x}.$$

2°. Equations that reduce to homogeneous equations.
If

$$y'=f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right) \qquad (2)$$

and $\delta=\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix}\neq 0$, then, putting into equation (2) $x=u+\alpha$, $y=v+\beta$, where the constants α and β are found from the following system of equations,

$$a_1\alpha+b_1\beta+c_1=0, \quad a_2\alpha+b_2\beta+c_2=0,$$

we get a homogeneous differential equation in the variables u and v. If $\delta=0$, then, putting in (2) $a_1x+b_1y=u$, we get an equation with variables separable.

Integrate the differential equations:

2768. $y'=\frac{y}{x}-1$.

2769. $y'=-\frac{x+y}{x}$.

2770. $(x-y)y\,dx-x^2dy=0$.

2771. For the equation $(x^2+y^2)\,dx-2xy\,dy=0$ find the family of integral curves, and also indicate the curves that pass through the points (4,0) and (1,1), respectively.

2772. $y\,dx+(2\sqrt{xy}-x)\,dy=0$.

2773. $x\,dy-y\,dx=\sqrt{x^2+y^2}dx$.

2774. $(4x^2+3xy+y^2)\,dx+(4y^2+3xy+x^2)\,dy=0$.

2775. Find the particular solution of the equation $(x^2-3y^2)\,dx+2xy\,dy=0$, provided that $y=1$ when $x=2$.

Solve the equations:

2776. $(2x-y+4)\,dy+(x-2y+5)\,dx=0$.

2777. $y'=\frac{1-3x-3y}{1+x+y}$.

2778. $y'=\frac{x+2y+1}{2x+4y+3}$.

2779. Find the equation of a curve that passes through the point (1,0) and has the property that the segment cut off by the tangent line on the y-axis is equal to the radius vector of the point of tangency.

2780**. What shape should the reflector of a search light have so that the rays from a point source of light are reflected as a parallel beam?

2781. Find the equation of a curve whose subtangent is equal to the arithmetic mean of the coordinates of the point of tangency.

2782. Find the equation of a curve for which the segment cut off on the y-axis by the normal at any point of the curve is equal to the distance of this point from the origin.

2783*. Find the equation of a curve for which the area contained between the x-axis, the curve and two ordinates, one of which is a constant and the other a variable, is equal to the ratio of the cube of the variable ordinate to the appropriate abscissa.

2784. Find a curve for which the segment on the y-axis cut off by any tangent line is equal to the abscissa of the point of tangency.

Sec. 5. First-Order Linear Differential Equations. Bernoulli's Equation

1°. Linear equations. A differential equation of the form

$$y' + P(x)\cdot y = Q(x) \qquad (1)$$

of degree one in y and y' is called *linear*.

If a function $Q(x) \equiv 0$, then equation (1) takes the form

$$y' + P(x)\cdot y = 0 \qquad (2)$$

and is called a *homogeneous linear* differential equation. In this case, the variables may be separated, and we get the general solution of (2) in the form

$$y = C\cdot e^{-\int P(x)\,dx}. \qquad (3)$$

To solve the inhomogeneous linear equation (1), we apply a method that is called *variation of parameters*, which consists in first finding the general solution of the respective homogeneous linear equation, that is, relationship (3). Then, assuming here that C is a function of x, we seek the solution of the inhomogeneous equation (1) in the form of (3). To do this, we put into (1) y and y' which are found from (3), and then from the differential equation thus obtained we determine the function $C(x)$. We thus get the general solution of the inhomogeneous equation (1) in the form

$$y = C(x)\cdot e^{-\int P(x)\,dx}.$$

Example 1. Solve the equation

$$y' = \tan x\cdot y + \cos x. \qquad (4)$$

Solution. The corresponding homogeneous equation is

$$y' - \tan x\cdot y = 0.$$

Solving it we get:

$$y = C\cdot\frac{1}{\cos x}.$$

Considering C as a function of x, and differentiating, we find:

$$y=\frac{1}{\cos x}\cdot\frac{dC}{dx}+\frac{\sin x}{\cos^2 x}\cdot C.$$

Putting y and y' into (4), we get:

$$\frac{1}{\cos x}\cdot\frac{dC}{dx}+\frac{\sin x}{\cos^2 x}\cdot C=\tan x\cdot\frac{C}{\cos x}+\cos x, \quad \text{or} \quad \frac{dC}{dx}=\cos^2 x,$$

whence

$$C(x)=\int \cos^2 x\,dx=\frac{1}{2}x+\frac{1}{4}\sin 2x+C_1.$$

Hence, the general solution of equation (4) has the form

$$y=\left(\frac{1}{2}x+\frac{1}{4}\sin 2x+C_1\right)\cdot\frac{1}{\cos x}.$$

In solving the linear equation (1) we can also make use of the substitution

$$y=uv, \tag{5}$$

where u and v are functions of x. Then equation (1) will have the form

$$[u'+P(x)u]\,v+v'u=Q(x). \tag{6}$$

If we require that

$$u'+P(x)u=0, \tag{7}$$

then from (7) we find u, and from (6) we find v; hence, from (5) we find y.

2°. Bernoulli's equation. A first-order equation of the form

$$y'+P(x)y=Q(x)y^{\alpha},$$

where $\alpha \neq 0$ and $\alpha \neq 1$, is called *Bernoulli's equation*. It is reduced to a linear equation by means of the substitution $z=y^{1-\alpha}$. It is also possible to apply directly the substitution $y=uv$, or the method of variation of parameters.

Example 2. Solve the equation

$$y'=\frac{4}{x}y+x\sqrt{y}.$$

Solution. This is Bernoulli's equation. Putting

$$y=u\cdot v,$$

we get

$$u'v+v'u=\frac{4}{x}uv+x\sqrt{uv} \quad \text{or} \quad v\left(u'-\frac{4}{x}u\right)+v'u=x\sqrt{uv}. \tag{8}$$

To determine the function u we require that the relation

$$u'-\frac{4}{x}u=0$$

be fulfilled, whence we have

$$u=x^4.$$

Putting this expression into (8), we get

$$v'x^4=x\sqrt{vx^4},$$

whence we find v:

$$v=\left(\frac{1}{2}\ln x+c\right)^2,$$

and, consequently, the general solution is obtained in the form

$$y=x^4\left(\frac{1}{2}\ln x+C\right)^2.$$

Find the general integrals of the equations:

2785. $\frac{dy}{dx}-\frac{y}{x}=x$.

2786. $\frac{dy}{dx}+\frac{2y}{x}=x^3$.

2787*. $(1+y^2)\,dx=(\sqrt{1+y^2}\sin y-xy)\,dy$.

2788. $y^2dx-(2xy+3)\,dy=0$.

Find the particular solutions that satisfy the indicated conditions:

2789. $xy'+y-e^x=0$; $y=b$ when $x=a$.

2790. $y'-\frac{y}{1-x^2}-1-x=0$; $y=0$ when $x=0$.

2791. $y'-y\tan x=\frac{1}{\cos x}$; $y=0$ when $x=0$.

Find the general solutions of the equations:

2792. $\frac{dy}{dx}+\frac{y}{x}=-xy^2$.

2793. $2xy\frac{dy}{dx}-y^2+x=0$.

2794. $y\,dx+\left(x-\frac{1}{2}x^3y\right)dy=0$.

2795. $3x\,dy=y\,(1+x\sin x-3y^3\sin x)\,dx$.

2796. Given three particular solutions y, y_1, y_2 of a linear equation. Prove that the expression $\frac{y_2-y}{y-y_1}$ remains unchanged for any x. What is the geometrical significance of this result?

2797. Find the curves for which the area of a triangle formed by the x-axis, a tangent line and the radius vector of the point of tangency is constant.

2798. Find the equation of a curve, a segment of which, cut off on the x-axis by a tangent line, is equal to the square of the ordinate of the point of tangency.

2799. Find the equation of a curve, a segment of which, cut off on the y-axis by a tangent line, is equal to the subnormal.

2800. Find the equation of a curve, a segment of which, cut off on the y-axis by a tangent line, is proportional to the square of the ordinate of the point of tangency.

2801. Find the equation of the curve for which the segment of the tangent is equal to the distance of the point of intersection of this tangent with the x-axis from the point $M(0,a)$.

Sec. 6. Exact Differential Equations. Integrating Factor

1°. Exact differential equations. If for the differential equation

$$P(x, y)\,dx + Q(x, y)\,dy = 0 \qquad (1)$$

the equality $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ is fulfilled, then equation (1) may be written in the form $dU(x, y) = 0$ and is then called an *exact differential equation*. The general integral of equation (1) is $U(x, y) = C$. The function $U(x, y)$ is determined by the technique given in Ch. VI, Sec. 8, or from the formula

$$U = \int_{x_0}^{x} P(x, y)\,dx + \int_{y_0}^{y} Q(x_0, y)\,dy$$

(see Ch. VII, Sec. 9).

Example 1. Find the general integral of the differential equation

$$(3x^2 + 6xy^2)\,dx + (6x^2y + 4y^3)\,dy = 0.$$

Solution. This is an exact differential equation, since $\frac{\partial(3x^2 + 6xy^2)}{\partial y} = \frac{\partial(6x^2y + 4y^3)}{\partial x} = 12xy$ and, hence, the equation is of the form $dU = 0$.

Here,

$$\frac{\partial U}{\partial x} = 3x^2 + 6xy^2 \quad \text{and} \quad \frac{\partial U}{\partial y} = 6x^2y + 4y^3;$$

whence

$$U = \int (3x^2 + 6xy^2)\,dx + \varphi(y) = x^3 + 3x^2y^2 + \varphi(y).$$

Differentiating U with respect to y, we find $\frac{\partial U}{\partial y} = 6x^2y + \varphi'(y) = 6x^2y + 4y^3$ (by hypothesis); from this we get $\varphi'(y) = 4y^3$ and $\varphi(y) = y^4 + C_0$. We finally get $U(x, y) = x^3 + 3x^2y^2 + y^4 + C_0$, consequently, $x^3 + 3x^2y^2 + y^4 = C$ is the sought-for general integral of the equation.

2°. Integrating factor. If the left side of equation (1) is not a total (exact) differential and the conditions of the Cauchy theorem are fulfilled, then there exists a function $\mu = \mu(x, y)$ *(integrating factor)* such that

$$\mu(P\,dx + Q\,dy) = dU. \qquad (2)$$

Whence it is found that the function μ satisfies the equation

$$\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q).$$

The integrating factor μ is readily found in two cases:

1) $\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = F(x)$, then $\mu = \mu(x)$;

2) $\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = F_1(y)$, then $\mu = \mu(y)$.

Example 2. Solve the equation $\left(2xy+x^2y+\frac{y^3}{3}\right)dx+(x^2+y^2)\,dy=0$.

Solution. Here $P=2xy+x^2y+\frac{y^3}{3}$, $Q=x^2+y^2$

and $\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=\frac{2x+x^2+y^2-2x}{x^2+y^2}=1$, hence, $\mu=\mu(x)$.

Since $\frac{\partial(\mu P)}{\partial y}=\frac{\partial(\mu Q)}{\partial x}$ or $\mu\frac{\partial P}{\partial y}=\mu\frac{\partial Q}{\partial x}+Q\frac{d\mu}{dx}$,
it follows that

$$\frac{d\mu}{\mu}=\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)dx=dx \text{ and } \ln\mu=x,\ \mu=e^x.$$

Multiplying the equation by $\mu=e^x$, we obtain

$$e^x\left(2xy+x^2y+\frac{y^3}{3}\right)dx+e^x(x^2+y^2)\,dy=0$$

which is an exact differential equation. Integrating it, we get the general integral

$$ye^x\left(x^2+\frac{y^2}{3}\right)=C.$$

Find the general integrals of the equations:

2802 $(x+y)\,dx+(x+2y)\,dy=0$.

2803. $(x^2+y^2+2x)\,dx+2xy\,dy=0$.

2804. $(x^3-3xy^2+2)\,dx-(3x^2y-y^2)\,dy=0$.

2805. $x\,dx-y\,dy=\frac{x\,dy-y\,dx}{x^2+y^2}$.

2806. $\frac{2x\,dx}{y^3}+\frac{y^2-3x^2}{y^4}\,dy=0$.

2807. Find the particular integral of the equation

$$\left(x+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0,$$

which satisfies the initial condition $y(0)=2$.

Solve the equations that admit of an integrating factor of the form $\mu=\mu(x)$ or $\mu=\mu(y)$:

2808. $(x+y^2)\,dx-2xy\,dy=0$.

2809. $y(1+xy)\,dx-x\,dy=0$.

2810. $\frac{y}{x}dx+(y^3-\ln x)dy=0$.

2811. $(x\cos y-y\sin y)\,dy+(x\sin y+y\cos y)\,dx=0$.

Sec. 7. First-Order Differential Equations not Solved for the Derivative

1°. First-order differential equations of higher powers. If an equation

$$F(x, y, y') = 0, \qquad (1)$$

which for example is of degree two in y', then by solving (1) for y' we get two equations:

$$y' = f_1(x, y), \quad y' = f_2(x, y). \qquad (2)$$

Thus, generally speaking, through each point $M_0(x_0, y_0)$ of some region of a plane there pass two integral curves. The general integral of equation (1) then, generally speaking, has the form

$$\Phi(x, y, C) \equiv \Phi_1(x, y, C)\,\Phi_2(x, y, C) = 0, \qquad (3)$$

where Φ_1 and Φ_2 are the general integrals of equations (2).

Besides, there may be a *singular integral* for equation (1). Geometrically, a singular integral is the envelope of a family of curves (3) and may be obtained by eliminating C from the system of equations

$$\Phi(x, y, C) = 0, \quad \Phi'_C(x, y, C) = 0 \qquad (4)$$

or by eliminating $p = y'$ from the system of equations

$$F(x, y, p) = 0, \quad F'_p(x, y, p) = 0. \qquad (5)$$

We note that the curves defined by the equations (4) or (5) are not always solutions of equation (1); therefore, in each case, a check is necessary.

Example 1. Find the general and singular integrals of the equation

$$xy'^2 + 2xy' - y = 0.$$

Solution. Solving for y' we have two homogeneous equations:

$$y' = -1 + \sqrt{1 + \frac{y}{x}}, \quad y' = -1 - \sqrt{1 + \frac{y}{x}},$$

defined in the region

$$x(x + y) > 0,$$

the general integrals of which are

$$\left(\sqrt{1 + \frac{y}{x}} - 1\right)^2 = \frac{C}{x}, \qquad \left(\sqrt{1 + \frac{y}{x}} + 1\right)^2 = \frac{C}{x}$$

or

$$(2x + y - C) - 2\sqrt{x^2 + xy} = 0, \qquad (2x + y - C) + 2\sqrt{x^2 + xy} = 0.$$

Multiplying, we get the general integral of the given equation

$$(2x + y - C)^2 - 4(x^2 + xy) = 0$$

or

$$(y - C)^2 = 4Cx$$

(a family of parabolas).

Differentiating the general integral with respect to C and eliminating C, we find the singular integral

$$y + x = 0.$$

(It may be verified that $y + x = 0$ is the solution of this equation.)

It is also possible to find the singular integral by differentiating $xp^2+2xp-y=0$ with respect to p and eliminating p.

2°. **Solving a differential equation by introducing a parameter.** If a first-order differential equation is of the form

$$x=\varphi(y,\ y'),$$

then the variables y and x may be determined from the system of equations

$$\frac{1}{p}=\frac{\partial\varphi}{\partial y}+\frac{\partial\varphi}{\partial}\frac{dp}{dy},\quad x=\varphi(y,\ p),$$

where $p=y'$ plays the part of a parameter.

Similarly, if $y=\psi(x,\ y')$, then x and y are determined from the system of equations

$$p=\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial p}\frac{dp}{dx},\qquad y=\psi(x,\ p).$$

Example 2. Find the general and singular integrals of the equation

$$y=y'^2-xy'+\frac{x^2}{2}.$$

Solution. Making the substitution $y'=p$, we rewrite the equation in the form

$$y=p^2-xp+\frac{x^2}{2}.$$

Differentiating with respect to x and considering p a function of x, we have

$$p=2p\frac{dp}{dx}-p-x\frac{dp}{dx}+x$$

or $\frac{dp}{dx}(2p-x)=(2p-x)$, or $\frac{dp}{dx}=1$. Integrating we get $p=x+C$. Substituting into the original equation, we have the general solution

$$y=(x+C)^2-x(x+C)+\frac{x^2}{2} \text{ or } y=\frac{x^2}{2}+Cx+C^2.$$

Differentiating the general solution with respect to C and eliminating C, we obtain the singular solution: $y=\frac{x^2}{4}$. (It may be verified that $y=\frac{x^2}{4}$ is the solution of the given equation.)

If we equate to zero the factor $2p-x$, which was cancelled out, we get $p=\frac{x}{2}$ and, putting p into the given equation, we get $y=\frac{x^2}{4}$, which is the same singular solution.

Find the general and singular integrals of the equations: (In Problems 2812 and 2813 construct the field of integral curves.)

2812. $y'^2-\frac{2y}{x}y'+1=0$.

2813. $4y'^2-9x=0$.

2814. $yy'^2-(xy+1)y'+x=0$.
2815. $yy'^2-2xy'+y=0$.
2816. Find the integral curves of the equation $y'^2+y^2=1$ that pass through the point $M\left(0, \frac{1}{2}\right)$.

Introducing the parameter $y'=p$, solve the equations:

2817. $x=\sin y'+\ln y'$.
2818. $y=y'^2e^{y'}$.
2819. $y=y'^2+2\ln y'$.
2820. $4y=x^2+y'^2$.
2821. $e^x=\frac{y^2+y'^2}{2y'}$.

Sec. 8. The Lagrange and Clairaut Equaticns

1°. **Lagrange's equation.** An equation of the form

$$y=x\varphi(p)+\psi(p), \qquad (1)$$

where $p=y'$ is called *Lagrange's equation* Equation (1) is reduced to a linear equation in x by differentiation and taking into consideration that $dy=p\,dx$:

$$p\,dx=\varphi(p)\,dx+[x\varphi'(p)+\psi'(p)]\,dp. \qquad (2)$$

If $p\not\equiv\varphi(p)$, then from (1) and (2) we get the general solution in parametric form:

$$x=Cf(p)+g(p), \quad y=[Cf(p)+g(p)]\,\varphi(p)+\psi(p),$$

where p is a parameter and $f(p)$, $g(p)$ are certain known functions. Besides, there may be a singular solution that is found in the usual way.

2°. **Clairaut's equation.** If in equation (1) $p\equiv\varphi(p)$, then we get *Clairaut's equation*

$$y=xp+\psi(p).$$

Its general solution is of the form $y=Cx+\psi(C)$ (a family of straight lines). There is also a *particular solution* (envelope) that results by eliminating the parameter p from the system of equations

$$\begin{cases} x=-\psi'(p), \\ y=px+\psi(p). \end{cases}$$

Example. Solve the equation

$$y=2y'x+\frac{1}{y'}. \qquad (3)$$

Solution. Putting $y'=p$ we have $y=2px+\frac{1}{p}$; differentiating and replacing dy by $p\,dx$, we get

$$p\,dx=2p\,dx+2x\,dp-\frac{dp}{p^2}$$

or

$$\frac{dx}{dp}=-\frac{2}{p}x+\frac{1}{p^3}.$$

Solving this linear equation, we will have

$$x=\frac{1}{p^2}(\ln p+C).$$

Hence, the general integral will be

$$\begin{cases} x=\frac{1}{p^2}(\ln p+C), \\ y=2px+\frac{1}{p}. \end{cases}$$

To find the singular integral, we form the system

$$y=2px+\frac{1}{p}, \quad 0=2x-\frac{1}{p^2}$$

in the usual way. Whence

$$x=\frac{1}{2p^2}, \quad y=\frac{2}{p}$$

and, consequently,

$$y=\pm 2\sqrt{2x}.$$

Putting y into (3) we are convinced that the function obtained is not a solution and, therefore, equation (3) does not have a singular integral.

Solve the Lagrange equations:

2822. $y=\frac{1}{2}x\left(y'+\frac{y}{y'}\right)$.

2823. $y=y'+\sqrt{1-y'^2}$.

2824. $y=(1+y')x+y'^2$.

2825*. $y=-\frac{1}{2}y'(2x+y')$.

Find the general and singular integrals of the Clairaut equations and construct the field of integral curves:

2826. $y=xy'+y'^2$.

2827. $y=xy'+y'$.

2828. $y=xy'+\sqrt{1+(y')^2}$.

2829. $y=xy'+\frac{1}{y'}$.

2830. Find the curve for which the area of a triangle formed by a tangent at any point and by the coordinate axes is constant.

2831. Find the curve if the distance of a given point to any tangent to this curve is constant.

2832. Find the curve for which the segment of any of its tangents lying between the coordinate axes has constant length l.

Sec. 9. Miscellaneous Exercises on First-Order Differential Equations

2833. Determine the types of differential equations and indicate methods for their solution:

a) $(x+y)y'=x\arctan\frac{y}{x}$;

b) $(x-y)y'=y^2$;

c) $y'=2xy+x^3$;

d) $y'=2xy+y^3$;

e) $xy'+y=\sin y$;

f) $(y-xy')^2=y'^3$;

g) $y=xe^{y'}$;

h) $(y'-2xy)\sqrt{y}=x^3$;

i) $y' = (x+y)^2$;
j) $x \cos y' + y \sin y' = 1$;
k) $(x^2 - xy) y' = y^4$;
l) $(x^2 + 2xy^3)\, dx + (y^2 + 3x^2y^2)\, dy = 0$;
m) $(x^3 - 3xy)\, dx + (x^2 + 3)\, dy = 0$;
n) $(xy^2 + \ln x)\, dx = y^2 dy$.

Solve the equations:

2834. a) $\left(x - y \cos \frac{y}{x}\right) dx + x \cos \frac{y}{x}\, dy = 0$;

b) $x \ln \frac{x}{y}\, dy - y\, dx = 0$.

2835. $x\, dx = \left(\frac{x^2}{y} - y^3\right) dy$.

2836. $(2xy^2 - y)\, dx + x\, dy = 0$.

2837. $xy' + y = xy^2 \ln x$.

2838. $y = xy' + y' \ln y'$.

2839. $y = xy' + \sqrt{-ay'}$.

2840. $x^2 (y+1)\, dx + (x^3 - 1)(y - 1)\, dy = 0$.

2841. $(1 + y^2)(e^{2x}\, dx - e^y\, dy) - (1 + y)\, dy = 0$.

2842. $y' - y \frac{2x-1}{x^2} = 1$.

2843. $ye^y = (y^3 + 2xe^y) y'$.

2844. $y' + y \cos x = \sin x \cos x$.

2845. $(1 - x^2) y' + xy = a$.

2846. $xy' - \frac{y}{x+1} - x = 0$.

2847. $y' (x \cos y + a \sin 2y) = 1$.

2848. $(x^2y - x^2 + y - 1)\, dx + (xy + 2x - 3y - 6)\, dy = 0$.

2849. $y' = \left(1 + \frac{y-1}{2x}\right)^2$.

2850. $xy^3\, dx = (x^2y + 2)\, dy$.

2851. $y' = \frac{3x^2}{x^3 + y + 1}$.

2852. $2dx + \sqrt{\frac{x}{y}}\, dy - \sqrt{\frac{y}{x}}\, dx = 0$.

2853. $y' = \frac{y}{x} + \tan \frac{y}{x}$.

2854. $yy' + y^2 = \cos x$.

2855. $x\, dy + y\, dx = y^2\, dx$.

2856. $y' (x + \sin y) = 1$.

2857. $y \frac{dp}{dy} = -p + p^2$.

2858. $x^3\, dx - (x^4 + y^3)\, dy = 0$.

2859. $x^2y'^2 + 3xyy' + 2y^2 = 0$.

2860. $\frac{x\, dx + y\, dy}{\sqrt{x^2 + y^2}} + \frac{x\, dy - y\, dx}{y^2} = 0$.

2861. $e^y\, dx + (xe^y - 2y)\, dy = 0$.

2862. $y = 2xy' + \sqrt{1 + y'^2}$.

2863. $y' = \frac{y}{x}(1 + \ln y - \ln x)$.

2864. $(2e^x + y^4)\, dy - ye^x\, dx = 0$.

2865. $y' = 2\left(\frac{y+2}{x+y-1}\right)^2$.

2866. $xy (xy^2 + 1)\, dy - dx = 0$.

2867. $a (xy' + 2y) = xyy'$.

2868. $x\, dy - y\, dx = y^2\, dx$.

2869. $(x^2-1)^{3/2}\,dy+(x^3+3xy\sqrt{x^2-1})\,dx=0$.

2870. $\tan x\frac{dy}{dx}-y=a$.

2871. $\sqrt{a^2+x^2}\,dy+(x+y-\sqrt{a^2+x^2})\,dx=0$.

2872. $xyy'^2-(x^2+y^2)\,y'+xy=0$.

2873. $y=xy'+\frac{1}{y'^2}$.

2874. $(3x^2+2xy-y^2)\,dx+(x^2-2xy-3y^2)\,dy=0$.

2875. $2yp\frac{dp}{dy}=3p^2+4y^2$.

Find solutions to the equations for the indicated initial conditions:

2876. $y'=\frac{y+1}{x}$; $y=0$ for $x=1$.

2877. $e^{x-y}y'=1$; $y=1$ for $x=1$.

2878. $\cot x y'+y=2$; $y=2$ for $x=0$.

2879. $e^y(y'+1)=1$; $y=0$ for $x=0$.

2880. $y'+y=\cos x$; $y=\frac{1}{2}$ for $x=0$.

2881. $y'-2y=-x^2$; $y=\frac{1}{4}$ for $x=0$.

2882. $y'+y=2x$; $y=-1$ for $x=0$.

2883. $xy'=y$; a) $y=1$ for $x=1$; b) $y=0$ for $x=0$.

2884. $2xy'=y$; a) $y=1$ for $x=1$; b) $y=0$ for $x=0$.

2885. $2xyy'+x^2-y^2=0$; a) $y=0$ for $x=0$; b) $y=1$ for $x=0$; c) $y=0$ for $x=1$.

2886. Find the curve passing through the point (0, 1), for which the subtangent is equal to the sum of the coordinates of the point of tangency.

2887. Find a curve if we know that the sum of the segments cut off on the coordinate axes by a tangent to it is constant and equal to $2a$.

2888. The sum of the lengths of the normal and subnormal is equal to unity. Find the equation of the curve if it is known that the curve passes through the coordinate origin.

2889*. Find a curve whose angle formed by a tangent and the radius vector of the point of tangency is constant.

2890. Find a curve knowing that the area contained between the coordinate axes, this curve and the ordinate of any point on it is equal to the cube of the ordinate.

2891. Find a curve knowing that the area of a sector bounded by the polar axis, by this curve and by the radius vector of any point of it is proportional to the cube of this radius vector.

2892. Find a curve, the segment of which, cut off by the tangent on the x-axis, is equal to the length of the tangent.

2893. Find the curve, of which the segment of the tangent contained between the coordinate axes is divided into half by the parabola $y^2 = 2x$.

2894. Find the curve whose normal at any point of it is equal to the distance of this point from the origin.

2895*. The area bounded by a curve, the coordinate axes, and the ordinate of some point of the curve is equal to the length of the corresponding arc of the curve. Find the equation of this curve if it is known that the latter passes through the point (0, 1).

2896. Find the curve for which the area of a triangle formed by the x-axis, a tangent, and the radius vector of the point of tangency is constant and equal to a^2.

2897. Find the curve if we know that the mid-point of the segment cut off on the x-axis by a tangent and a normal to the curve is a constant point $(a, 0)$.

When forming first-order differential equations, particularly in physical problems, it is frequently advisable to apply the so-called *method of differentials*, which consists in the fact that approximate relationships between infinitesimal increments of the desired quantities (these relationships are accurate to infinitesimals of higher order) are replaced by the corresponding relationships between their differentials. This does not affect the result.

Problem. A tank contains 100 litres of an aqueous solution containing 10 kg of salt. Water is entering the tank at the rate of 3 litres per minute, and the mixture is flowing out at 2 litres per minute. The concentration is maintained uniform by stirring. How much salt will the tank contain at the end of one hour?

Solution. The concentration c of a substance is the quantity of it in unit volume. If the concentration is uniform, then the quantity of substance in volume V is cV.

Let the quantity of salt in the tank at the end of t minutes be x kg. The quantity of solution in the tank at that instant will be $100+t$ litres, and, consequently, the concentration $c=\frac{x}{100+t}$ kg per litre.

During time dt, $2dt$ litres of the solution flows out of the tank (the solution contains $2c\,dt$ kg of salt). Therefore, a change of dx in the quantity of salt in the tank is given by the relationship

$$-dx = 2c\,dt = \frac{2x}{100+t}\,dt.$$

This is the sought-for differential equation. Separating variables and integrating, we obtain

$$\ln x = -2\ln(100+t) + \ln C$$

or

$$x = \frac{C}{(100+t)^2}.$$

The constant C is found from the fact that when $t=0$, $x=10$, that is, $C=100{,}000$. At the expiration of one hour, the tank will contain $x=\frac{100{,}000}{160^2}\approx 3.9$ kilograms of salt.

2898*. Prove that for a heavy liquid rotating about a vertical axis the free surface has the form of a paraboloid of revolution.

2899*. Find the relationship between the air pressure and the altitude if it is known that the pressure is 1 kgf on 1 cm^2 at sea level and 0.92 kgf on 1 cm^2 at an altitude of 500 metres.

2900*. According to Hooke's law an elastic band of length l increases in length klF (k = const) due to a tensile force F. By how much will the band increase in length due to its weight W if the band is suspended at one end? (The initial length of the band is l.)

2901. Solve the same problem for a weight P suspended from the end of the band.

When solving Problems 2902 and 2903, make use of Newton's law, by which the rate of cooling of a body is proportional to the difference of temperatures of the body and the ambient medium.

2902. Find the relationship between the temperature T and the time t if a body, heated to T_0 degrees, is brought into a room at constant temperature (a degrees).

2903. During what time will a body heated to 100° cool off to 30° if the temperature of the room is 20° and during the first 20 minutes the body cooled to 60°?

2904. The retarding action of friction on a disk rotating in a liquid is proportional to the angular velocity of rotation. Find the relationship between the angular velocity and time if it is known that the disk began rotating at 100 rpm and after one minute was rotating at 60 rpm.

2905*. The rate of disintegration of radium is proportional to the quantity of radium present. Radium disintegrates by one half in 1600 years. Find the percentage of radium that has disintegrated after 100 years.

2906*. The rate of outflow of water from an aperture at a vertical distance h from the free surface is defined by the formula

$$v = c\sqrt{2gh},$$

where $c \approx 0.6$ and g is the acceleration of gravity.

During what period of time will the water filling a hemispherical boiler of diameter 2 metres flow out of it through a circular opening of radius 0.1 m in the bottom.

2907*. The quantity of light absorbed in passing through a thin layer of water is proportional to the quantity of incident light and to the thickness of the layer. If one half of the original quantity of light is absorbed in passing through a three-metre-thick layer of water, what part of this quantity will reach a depth of 30 metres?

2908*. The air resistance to a body falling with a parachute is proportional to the square of the rate of fall. Find the limiting velocity of descent.

2909*. The bottom of a tank with a capacity of 300 litres is covered with a mixture of salt and some insoluble substance. Assuming that the rate at which the salt dissolves is proportional to the difference between the concentration at the given time and the concentration of a saturated solution (1 kg of salt per 3 litres of water) and that the given quantity of pure water dissolves 1/3 kg of salt in 1 minute, find the quantity of salt in solution at the expiration of one hour.

2910*. The electromotive force e in a circuit with current i, resistance R and self-induction L is made up of the voltage drop Ri and the electromotive force of self-induction $L\frac{di}{dt}$. Determine the current i at time t if $e=E\sin\omega t$ (E and ω are constants) and $i=0$ when $t=0$.

Sec. 10. Higher-Order Differential Equations

1°. The case of direct integration. If

$$y^{(n)}=f(x),$$

then

$$y=\underbrace{\int dx\int \dots \int}_{n \text{ times}} f(x)\,dx+C_1x^{n-1}+C_2x^{n-2}+\dots+C_n.$$

2°. Cases of reduction of order. 1) If a differential equation does not contain y explicitly, for instance,

$$F(x,\ y',\ y'')=0,$$

then, assuming $y'=p$, we get an equation of an order one unit lower:

$$F(x,\ p,\ p')=0.$$

Example 1. Find the particular solution of the equation

$$xy''+y'+x=0,$$

that satisfies the conditions

$$y=0,\ y'=0 \text{ when } x=0.$$

Solution. Putting $y'=p$, we have $y''=p'$, whence

$$xp'+p+x=0.$$

Solving the latter equation as a linear equation in the function p, we get

$$px=C_1-\frac{x^2}{2}.$$

From the fact that $y'=p=0$ when $x=0$, we have $0=C_1-0$, i.e., $C_1=0$. Hence,

$$p=-\frac{x}{2}$$

or

$$\frac{dy}{dx}=-\frac{x}{2},$$

whence, integrating once again, we obtain

$$y=-\frac{x^2}{4}+C_2$$

Putting $y=0$ when $x=0$, we find $C_2=0$. Hence, the desired particular solution is

$$y=-\frac{1}{4}x^2.$$

2) If a differential equation does not contain x explicitly, for instance,

$$F(y,\ y',\ y'')=0$$

then, putting $y'=p$, $y''=p\frac{dp}{dy}$, we get an equation of an order one unit lower:

$$F\left(y,\ p,\ p\frac{dp}{dy}\right)=0.$$

Example 2. Find the particular solution of the equation

$$yy''-y'^2=y^4$$

provided that $y=1$, $y'=0$ when $x=0$.

Solution. Put $y'=p$, then $y''=p\frac{dp}{dy}$ and our equation becomes

$$yp\frac{dp}{dy}-p^2=y^4.$$

We have obtained an equation of the Bernoulli type in p (y is considered the argument). Solving it, we find

$$p=\pm y\sqrt{C_1+y^2}.$$

From the fact that $y'=p=0$ when $y=1$, we have $C_1=-1$. Hence,

$$p=\pm y\sqrt{y^2-1}$$

or

$$\frac{dy}{dx}=\pm y\sqrt{y^2-1}.$$

Integrating, we have

$$\operatorname{arc\,cos}\frac{1}{y}\pm x=C_2.$$

Putting $y=1$ and $x=0$, we obtain $C_2=0$, whence $\frac{1}{y}=\cos x$ or $y=\sec x$.

Solve the following equations:

2911. $y'' = \frac{1}{x}$.

2912. $y'' = -\frac{2}{2y^3}$.

2913. $y'' = 1 - y'^2$.

2914. $xy'' + y' = 0$.

2915. $yy'' = y'^2$.

2916. $yy'' + y'^2 = 0$.

2917. $(1 + x^2) y'' + y'^2 + 1 = 0$.

2918. $y' (1 + y'^2) = ay''$.

2919. $x^2 y'' + xy' = 1$.

2920. $yy'' = y^2 y' + y'^2$.

2921. $yy'' - y' (1 + y') = 0$.

2922. $y'' = -\frac{x}{y'}$.

2923. $(x + 1) y'' - (x + 2) y' + x + 2 = 0$.

2924. $xy'' = y' \ln \frac{y'}{x}$.

2925. $y' + \frac{1}{4} y''^2 = xy''$.

2926. $xy''' + y'' = 1 + x$.

2927. $y'''^2 + y''^2 = 1$.

Find the particular solutions for the indicated initial conditions:

2928. $(1 + x^2) y'' - 2xy' = 0$; $y = 0$, $y' = 3$ for $x = 0$.

2929. $1 + y'^2 = 2yy''$; $y = 1$, $y' = 1$ for $x = 1$.

2930. $yy'' + y'^2 = y'^3$; $y = 1$, $y' = 1$ for $x = 0$.

2931. $xy'' = y'$; $y = 0$, $y' = 0$ for $x = 0$.

Find the general integrals of the following equations:

2932. $yy' = \sqrt{y^2 + y'^2}\, y'' - y' y''$.

2933. $yy'' = y'^2 + y' \sqrt{y^2 + y'^2}$.

2934. $y'^2 - yy'' = y^2 y'$.

2935. $yy'' + y'^2 - y'^3 \ln y = 0$.

Find solutions that satisfy the indicated conditions:

2936. $y'' y^3 = 1$; $y = 1$, $y' = 1$ for $x = \frac{1}{2}$.

2937. $yy'' + y'^2 = 1$; $y = 1$, $y' = 1$ for $x = 0$.

2938. $xy'' = \sqrt{1 + y'^2}$; $y = 0$ for $x = 1$; $y = 1$ for $x = e^2$.

2939. $y'' (1 + \ln x) + \frac{1}{x} \cdot y' = 2 + \ln x$; $y = \frac{1}{2}$, $y' = 1$ for $x = 1$.

2940. $y'' = \frac{y'}{x} \left(1 + \ln \frac{y'}{x} \right)$; $y = \frac{1}{2}$, $y' = 1$ for $x = 1$.

2941. $y'' - y'^2 + y' (y - 1) = 0$; $y = 2$, $y' = 2$ for $x = 0$.

2942. $3y' y'' = y + y'^3 + 1$; $y = -2$, $y' = 0$ for $x = 0$.

2943. $y^2 + y'^2 - 2yy'' = 0$; $y = 1$, $y' = 1$ for $x = 0$.

2944. $yy' + y'^2 + yy'' = 0$; $y = 1$ for $x = 0$ and $y = 0$ for $x = -1$.

2945. $2y' + (y'^2 - 6x) \cdot y'' = 0$; $y = 0$, $y' = 2$ for $x = 2$.
2946. $y'y^2 + yy'' - y'^2 = 0$; $y = 1$, $y' = 2$ for $x = 0$.
2947. $2yy'' - 3y'^2 = 4y^2$; $y = 1$, $y' = 0$ for $x = 0$.
2948. $2yy'' + y^2 - y'^2 = 0$; $y = 1$, $y' = 1$ for $x = 0$.
2949. $y'' = y'^2 - y$; $y = -\frac{1}{4}$, $y' = \frac{1}{2}$ for $x = 1$.
2950. $y'' + \frac{1}{y^2} e^{y^2} y' - 2yy'^2 = 0$; $y = 1$, $y' = e$ for $x = -\frac{1}{2e}$.
2951. $1 + yy'' + y'^2 = 0$; $y = 0$, $y' = 1$ for $x = 1$.
2952. $(1 + yy')\, y'' = (1 + y'^2)\, y'$; $y = 1$, $y' = 1$ for $x = 0$.
2953. $(x + 1)\, y'' + xy'^2 = y'$; $y = -2$, $y' = 4$ for $x = 1$.

Solve the equations:
2954. $y' = xy''^2 + y''^2$.
2955. $y' = xy'' + y'' - y''^2$.
2956. $y'''^2 = 4y''$.

2957. $yy'y'' = y'^3 + y''^2$. Choose the integral curve passing through the point (0, 0) and tangent, at it, to the straight line $y + x = 0$.

2958. Find the curves of constant radius of curvature.

2959. Find a curve whose radius of curvature is proportional to the cube of the normal.

2960. Find a curve whose radius of curvature is equal to the normal.

2961. Find a curve whose radius of curvature is double the normal.

2962. Find the curves whose projection of the radius of curvature on the y-axis is a constant.

2963. Find the equation of the cable of a suspension bridge on the assumption that the load is distributed uniformly along the projection of the cable on a horizontal straight line. The weight of the cable is neglected.

2964*. Find the position of equilibrium of a flexible nontensile thread, the ends of which are attached at two points and which has a constant load q (including the weight of the thread) per unit length.

2965*. A heavy body with no initial velocity is sliding along an inclined plane. Find the law of motion if the angle of inclination is α, and the coefficient of friction is μ.

(Hint. The frictional force is μN, where N is the force of reaction of the plane.)

2966*. We may consider that the air resistance in free fall is proportional to the square of the velocity. Find the law of motion if the initial velocity is zero.

2967*. A motor-boat weighing 300 kgf is in rectilinear motion with initial velocity 66 m/sec. The resistance of the water is proportional to the velocity and is 10 kgf at 1 metre/sec. How long will it be before the velocity becomes 8 m/sec?

Sec. 11. Linear Differential Equations

1°. **Homogeneous equations**. The functions $y_1=\varphi_1(x)$, $y_2=\varphi_2(x)$, ..., $y_n=\varphi_n(x)$ are called *linearly dependent* if there are constants $C_1, C_2, \ldots, C_n$ not all equal to zero, such that

$$C_1y_1+C_2y_2+\ldots+C_ny_n\equiv 0;$$

otherwise, these functions are called *linearly independent*.

The general solution of a *homogeneous linear* differential equation

$$y^{(n)}+P_1(x)\,y^{(n-1)}+\ldots+P_n(x)\,y=0 \qquad (1)$$

with continuous coefficients $P_i(x)$ $(i=1, 2, \ldots, n)$ is of the form

$$y=C_1y_1+C_2y_2+\ldots+C_ny_n,$$

where $y_1, y_2, \ldots, y_n$ are linearly independent solutions of equation (1) (*fundamental system of solutions*).

2°. **Inhomogeneous equations**. The general solution of an *inhomogeneous linear* differential equation

$$y^{(n)}+P_1(x)\,y^{(n-1)}+\ldots+P_n(x)\,y=f(x) \qquad (2)$$

with continuous coefficients $P_i(x)$ and the right side $f(x)$ has the form

$$y=y_0+Y,$$

where y_0 is the general solution of the corresponding homogeneous equation (1) and Y is a particular solution of the given inhomogeneous equation (2).

If the fundamental system of solutions $y_1, y_2, \ldots, y_n$ of the homogeneous equation (1) is known, then the general solution of the corresponding inhomogeneous equation (2) may be found from the formula

$$y=C_1(x)\,y_1+C_2(x)\,y_2+\ldots+C_n(x)\,y_n,$$

where the functions $C_i(x)$ $(i=1, 2, \ldots, n)$ are determined from the following system of equations:

$$\left.\begin{array}{l} C_1'(x)\,y_1+C_2'(x)\,y_2+\ldots+C_n'(x)\,y_n=0,\\ C_1'(x)\,y_1'+C_2'(x)\,y_2'+\ldots+C_n'(x)\,y_n'=0,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ C_1'(x)\,y_1^{(n-2)}+C_2'(x)\,y_2^{(n-2)}+\ldots+C_n'(x)\,y_n^{(n-2)}=0,\\ C_1'(x)\,y_1^{(n-1)}+C_2'(x)\,y_2^{(n-1)}+\ldots+C_n'(x)\,y_n^{(n-1)}=f(x) \end{array}\right\} \qquad (3)$$

(the *method of variation of parameters*).

Example. Solve the equation

$$xy''+y'=x^2. \qquad (4)$$

Solution. Solving the homogeneous equation

$$xy''+y'=0,$$

we get

$$y=C_1\ln x+C_2. \qquad (5)$$

Hence, it may be taken that

$$y_1=\ln x \text{ and } y_2=1$$

and the solution of equation (4) may be sought in the form

$$y=C_1(x)\ln x+C_2(x).$$

Forming the system (3) and taking into account that the reduced form of the equation (4) is $y''+\frac{y'}{x}=x$, we obtain

$$\begin{cases} C_1'(x)\ln x+C_2'(x)\,1=0, \\ C_1'(x)\,\frac{1}{x}+C_2'(x)\,0=x. \end{cases}$$

Whence

$$C_1(x)=\frac{x^3}{3}+A \quad \text{and} \quad C_2(x)=-\frac{x^3}{3}\ln x+\frac{x^3}{9}+B$$

and, consequently,

$$y=\frac{x^3}{9}+A\ln x+B,$$

where A and B are arbitrary constants.

2968. Test the following systems of functions for linear relationships:

a) $x,\ x+1$;
b) $x^2,\ -2x^2$;
c) $0,\ 1,\ x$;
d) $x,\ x+1,\ x+2$;
e) $x,\ x^2,\ x^3$;
f) $e^x,\ e^{2x},\ e^{3x}$;
g) $\sin x,\ \cos x,\ 1$;
h) $\sin^2 x,\ \cos^2 x,\ 1$.

2969. Form a linear homogeneous differential equation, knowing its fundamental system of equations:

a) $y_1=\sin x,\ y_2=\cos x$;
b) $y_1=e^x,\ y_2=xe^x$;
c) $y_1=x,\ y_2=x^2$,
d) $y_1=e^x,\ y_2=e^x\sin x,\ y_3=e^x\cos x$.

2970. Knowing the fundamental system of solutions of a linear homogeneous differential equation

$$y_1=x,\ y_2=x^2,\ y_3=x^3,$$

find its particular solution y that satisfies the initial conditions

$$y|_{x=1}=0,\quad y'|_{x=1}=-1,\quad y''|_{x=1}=2.$$

2971*. Solve the equation

$$y'' + \frac{2}{x} y' + y = 0,$$

knowing its particular solution $y_1 = \frac{\sin x}{x}$.

2972. Solve the equation

$$x^2 (\ln x - 1) y'' - xy' + y = 0,$$

knowing its particular solution $y_1 = x$.

By the method of variation of parameters, solve the following inhomogeneous linear equations.

2973. $x^2 y'' - xy' = 3x^3$.

2974*. $x^2 y'' + xy' - y = x^2$.

2975. $y''' + y' = \sec x$.

Sec. 12. Linear Differential Equations of Second Order with Constant Coefficients

1°. Homogeneous equations. A second-order linear equation with constant coefficients p and q without the right side is of the form

$$y'' + py' + qy = 0 \qquad (1)$$

If k_1 and k_2 are roots of the characteristic equation

$$\varphi(k) \equiv k^2 + pk + q = 0, \qquad (2)$$

then the general solution of equation (1) is written in one of the following three ways:

1) $y = C_1 e^{k_1 x} + C_2 e^{k_2 x}$ if k_1 and k_2 are real and $k_1 \neq k_2$;
2) $y = e^{k_1 x}(C_1 + C_2 x)$ if $k_1 = k_2$;
3) $y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$ if $k_1 = \alpha + \beta i$ and $k_2 = \alpha - \beta i$ $(\beta \neq 0)$.

2°. Inhomogeneous equations. The general solution of a linear inhomogeneous differential equation

$$y'' + py' + qy = f(x) \qquad (3)$$

may be written in the form of a sum:

$$y = y_0 + Y,$$

where y_0 is the general solution of the corresponding equation (1) without right side and determined from formulas (1) to (3), and Y is a particular solution of the given equation (3).

The function Y may be found by the *method of undetermined coefficients* in the following simple cases:

1. $f(x) = e^{ax} P_n(x)$, where $P_n(x)$ is a polynomial of degree n.

If a is not a root of the characteristic equation (2), that is, $\varphi(a) \neq 0$, then we put $Y = e^{ax} Q_n(x)$ where $Q_n(x)$ is a polynomial of degree n with undetermined coefficients.

If a is a root of the characteristic equation (2), that is, $\varphi(a) = 0$, then $Y = x^r e^{ax} Q_n(x)$, where r is the multiplicity of the root a ($r = 1$ or $r = 2$).

2. $f(x) = e^{ax}[P_n(x) \cos bx + Q_m(x) \sin bx]$.

If $\varphi(a \pm bi) \neq 0$, then we put

$$Y = e^{ax}[S_N(x)\cos bx + T_N(x)\sin bx],$$

where $S_N(x)$ and $T_N(x)$ are polynomials of degree N-max $\{n, m\}$.

But if $\varphi(a \pm bi) = 0$, then

$$Y = x^r e^{ax}[S_N(x)\cos bx + T_N(x)\sin bx],$$

where r is the multiplicity of the roots $a \pm bi$ (for second-order equations, $r = 1$).

In the general case, the *method of variation of parameters* (see Sec. 11) is used to solve equation (3).

Example 1. Find the general solution of the equation $2y'' - y' - y = 4xe^{2x}$.

Solution. The characteristic equation $2k^2 - k - 1 = 0$ has roots $k_1 = 1$ and $k_2 = -\frac{1}{2}$. The general solution of the corresponding homogeneous equation (first type) is $y_0 = C_1e^x + C_2e^{-\frac{x}{2}}$. The right side of the given equation is $f(x) = 4xe^{2x} = e^{ax}P_n(x)$. Hence, $Y = e^{2x}(Ax + B)$, since $n = 1$ and $r = 0$. Differentiating Y twice and putting the derivatives into the given equation, we obtain:

$$2e^{2x}(4Ax + 4B + 4A) - e^{2x}(2Ax + 2B + A) - e^{2x}(Ax + B) = 4xe^{2x}.$$

Cancelling out e^{2x} and equating the coefficients of identical powers of x and the absolute terms on the left and right of the equality, we have $5A = 4$ and $7A + 5B = 0$, whence $A = \frac{4}{5}$ and $B = -\frac{28}{25}$.

Thus, $Ye^{2x}\left(\frac{4}{5}x - \frac{28}{25}\right)$, and the general solution of the given equation is

$$y = C_1e^x + C_2e^{-\frac{1}{2}} + e^{2x}\left(\frac{4}{5}x - \frac{28}{25}\right).$$

Example 2. Find the general solution of the equation $y'' - 2y' + y = xe^x$.

Solution. The characteristic equation $k^2 - 2k + 1 = 0$ has a double root $k = 1$ The right side of the equation is of the form $f(x) = xe^x$; here, $a = 1$ and $n = 1$. The particular solution is $Y = x^2e^x(Ax + B)$, since a coincides with the double root $k = 1$ and, consequently, $r = 2$.

Differentiating Y twice, substituting into the equation, and equating the coefficients, we obtain $A = \frac{1}{6}$, $B = 0$. Hence, the general solution of the given equation will be written in the form

$$y = (C_1 + C_2x)e^x + \frac{1}{6}x^3e^x.$$

Example 3. Find the general solution of the equation $y'' + y = x\sin x$.

Solution. The characteristic equation $k^2 + 1 = 0$ has roots $k_1 = i$ and $k_2 = -i$. The general solution of the corresponding homogeneous equation will [see 3, where $\alpha = 0$ and $\beta = 1$] be

$$y_0 = C_1\cos x + C_2\sin x.$$

The right side is of the form

$$f(x) = e^{ax}[P_n(x)\cos bx + Q_m(x)\sin bx],$$

where $a=0$, $b=1$, $P_n(x)=0$, $Q_m(x)=x$. To this side there corresponds the particular solution Y,

$$Y=x[(Ax+B)\cos x+(Cx+D)\sin x]$$

(here, $N=1$, $a=0$, $b=1$, $r=1$).

Differentiating twice and substituting into the equation, we equate the coefficients of both sides in $\cos x$, $x\cos x$, $\sin x$, and $x\sin x$. We then get four equations $2A+2D=0$, $4C=0$, $-2B+2C=0$, $-4A=1$, from which we determine $A=-\frac{1}{4}$, $B=0$, $C=0$, $D=\frac{1}{4}$. Therefore, $Y=-\frac{x^2}{4}\cos x+\frac{x}{4}\sin x$.

The general solution is

$$y=C_1\cos x+C_2\sin x-\frac{x^2}{4}\cos x+\frac{x}{4}\sin x.$$

3°. **The principle of superposition of solutions.** If the right side of equation (3) is the sum of several functions

$$f(x)=f_1(x)+f_2(x)+\ldots+f_n(x)$$

and Y_i $(i=1, 2, 3, \ldots, n)$ are the corresponding solutions of the equations

$$y''+py'+qy=f_i(x) \qquad (i=1, 2, \ldots, n),$$

then the sum

$$y=Y_1+Y_2+\ldots+Y_n$$

is the solution of equation (3).

Find the general solutions of the equations:

2976. $y''-5y'+6y=0$.

2977. $y''-9y=0$.

2978. $y''-y'=0$.

2979. $y''+y=0$.

2980. $y''-2y'+2y=0$.

2981. $y''+4y'+13y=0$.

2982. $y''+2y'+y=0$.

2983. $y''-4y'+2y=0$.

2984. $y''+ky=0$.

2985. $y=y''+y'$.

2986. $\frac{y'-y}{y''}=3$.

Find the particular solutions that satisfy the indicated conditions:

2987. $y''-5y'+4y=0$; $y=5$, $y'=8$ for $x=0$

2988. $y''+3y'+2y=0$; $y=1$, $y'=-1$ for $x=0$.

2989. $y''+4y=0$; $y=0$, $y'=2$ for $x=0$.

2990. $y''+2y'=0$; $y=1$, $y'=0$ for $x=0$

2991. $y''=\frac{y}{a^2}$; $y=a$, $y'=0$ for $x=0$.

2992. $y''+3y'=0$; $y=0$ for $x=0$ and $y=0$ for $x=3$.

2993. $y''+\pi^2y=0$; $y=0$ for $x=0$ and $y=0$ for $x=1$.

2994. Indicate the type of particular solutions for the given inhomogeneous equations:

a) $y''-4y=x^2e^{2x}$;

b) $y''+9y=\cos 2x$;

c) $y''-4y'+4y=\sin 2x+e^{2x}$;
d) $y''+2y'+2y=e^{x}\sin x$;
e) $y''-5y'+6y=(x^2+1)e^{x}+xe^{2x}$;
f) $y''-2y'+5y=xe^{x}\cos 2x-x^2e^{x}\sin 2x$.

Find the general solutions of the equations:

2995. $y''-4y'+4y=x^2$.
2996. $y''-y'+y=x^3+6$.
2997. $y''+2y'+y=e^{2x}$.
2998. $y''-8y'+7y=14$.
2999. $y''-y=e^{x}$.
3000. $y''+y=\cos x$.
3001. $y''+y'-2y=8\sin 2x$.
3002. $y''+y'-6y=xe^{2x}$.
3003. $y''-2y'+y=\sin x+\sinh x$.
3004. $y''+y'=\sin^2 x$.
3005. $y''-2y'+5y=e^{x}\cos 2x$.
3006. Find the solution of the equation $y''+4y=\sin x$ that satisfies the conditions $y=1$, $y'=1$ for $x=0$.

Solve the equations:

3007. $\frac{d^2x}{dt^2}+\omega^2x=A\sin pt$. Consider the cases: 1) $p\neq\omega$; 2) $p=\omega$.
3008. $y''-7y'+12y=-e^{4x}$.
3009. $y''-2y'=x^2-1$.
3010. $y''-2y'+y=2e^{x}$.
3011. $y''-2y'=e^{2x}+5$.
3012. $y''-2y'-8y=e^{x}-8\cos 2x$.
3013. $y''+y'=5x+2e^{x}$.
3014. $y''-y'=2x-1-3e^{x}$.
3015. $y''+2y'+y=e^{x}+e^{-x}$.
3016. $y''-2y'+10y=\sin 3x+e^{x}$.
3017. $y''-4y'+4y=2e^{2x}+\frac{x}{2}$.
3018. $y''-3y'=x+\cos x$.
3019. Find the solution to the equation $y''-2y'=e^{2x}+x^2-1$ that satisfies the conditions $y=\frac{1}{8}$, $y'=1$ for $x=0$.

Solve the equations:

3020. $y''-y=2x\sin x$.
3021. $y''-4y=e^{2x}\sin 2x$.
3022. $y''+4y=2\sin 2x-3\cos 2x+1$.
3023. $y''-2y'+2y=4e^{x}\sin x$.
3024. $y''=xe^{x}+y$.
3025. $y''+9y=2x\sin x+xe^{3x}$.

3026. $y''-2y'-3y=x(1+e^{2x})$.
3027. $y''-2y'=3x+2xe^x$.
3028. $y''-4y'+4y=xe^{2x}$.
3029. $y''+2y'-3y=2xe^{-3x}+(x+1)e^x$.
3030*. $y''+y=2x\cos x\cos 2x$.
3031. $y''-2y=2xe^x(\cos x-\sin x)$.

Applying the method of variation of parameters, solve the following equations:

3032. $y''+y=\tan x$.
3033. $y''+y=\cot x$.
3034. $y''-2y'+y=\frac{e^x}{x}$.
3035. $y''+2y'+y=\frac{e^{-x}}{x}$.
3036. $y''+y=\frac{1}{\cos x}$.
3037. $y''+y=\frac{1}{\sin x}$.
3038. a) $y''-y=\tanh x$.
b) $y''-2y=4x^2e^{x^2}$.

3039. Two identical loads are suspended from the end of a spring. Find the equation of motion that will be performed by one of these loads if the other falls.

Solution. Let the increase in the length of the spring under the action of one load in a state of rest be a and the mass of the load, m. Denote by x the coordinate of the load reckoned vertically from the position of equilibrium in the case of a single load. Then

$$m\frac{d^2x}{dt^2}=mg-k(x+a),$$

where, obviously, $k=\frac{mg}{a}$ and, consequently, $\frac{d^2x}{dt^2}=-\frac{g}{a}x$. The general solution is $x=C_1\cos\sqrt{\frac{g}{a}}\,t+C_2\sin\sqrt{\frac{g}{a}}\,t$. The initial conditions yield $x=a$ and $\frac{dx}{dt}=0$ when $t=0$; whence $C_1=a$ and $C_2=0$; and so

$$x=a\cos\sqrt{\frac{g}{a}}\,t.$$

3040*. The force stretching a spring is proportional to the increase in its length and is equal to 1 kgf when the length increases by 1 cm. A load weighing 2 kgf is suspended from the spring. Find the period of oscillatory motion of the load if it is pulled downwards slightly and then released.

3041*. A load weighing $P=4$ kgf is suspended from a spring and increases the length of the spring by 1 cm. Find the law of motion of the load if the upper end of the spring performs a vertical harmonic oscillation $y=2\sin 30t$ cm and if at the initial instant the load was at rest (resistance of the medium is neglected).

3042. A material point of mass m is attracted by each of two centres with a force proportional to the distance (the constant of proportionality is k). Find the law of motion of the point knowing that the distance between the centres is $2b$, at the initial instant the point was located on the line connecting the centres (at a distance c from its midpoint) and had a velocity of zero.

3043. A chain of length 6 metres is sliding from a support without friction. If the motion begins when 1 m of the chain is hanging from the support, how long will it take for the entire chain to slide down?

3044*. A long narrow tube is revolving with constant angular velocity ω about a vertical axis perpendicular to it. A ball inside the tube is sliding along it without friction. Find the law of motion of the ball relative to the tube, considering that

a) at the initial instant the ball was at a distance a from the axis of rotation; the initial velocity of the ball was zero;

b) at the initial instant the ball was located on the axis of rotation and had an initial velocity v_0.

Sec. 13. Linear Differential Equations of Order Higher than Two with Constant Coefficients

1°. Homogeneous equations. The fundamental system of solutions y_1, y_2, ..., y_n of a homogeneous linear equation with constant coefficients

$$y^{(n)}+a_1y^{(n-1)}+\ldots+a_{n-1}y'+a_ny=0 \qquad (1)$$

is constructed on the basis of the character of the roots of the *characteristic equation*

$$k^n+a_1k^{n-1}+\ldots+a_{n-1}k+a_n=0. \qquad (2)$$

Namely, 1) if k is a real root of the equation (2) of multiplicity m, then to this root there correspond m linearly independent solutions of equation (1):

$$y_1=e^{kx},\ y_2=xe^{kx},\ \ldots,\ y_m=x^{m-1}e^{kx};$$

2) if $\alpha\pm\beta i$ is a pair of complex roots of equation (2) of multiplicity m, then to the latter there correspond $2m$ linearly independent solutions of equation (1):

$$y_1=e^{\alpha x}\cos\beta x,\ y_2=e^{\alpha x}\sin\beta x,\ y_3=xe^{\alpha x}\cos\beta x,\ y_4=xe^{\alpha x}\sin\beta x,\ \ldots$$
$$\ldots,\ y_{2m-1}=x^{m-1}e^{\alpha x}\cos\beta x,\ y_{2m}=x^{m-1}e^{\alpha x}\sin\beta x.$$

2°. Inhomogeneous equations. A particular solution of the inhomogeneous equation

$$y^{(n)}+a_1y^{(n-1)}+\ldots+a_{n-1}y'+a_ny=f(x) \qquad (3)$$

is sought on the basis of rules 2° and 3° of Sec. 12.

Find the general solutions of the equations:

3045. $y''' - 13y'' + 12y' = 0$.

3046. $y''' - y' = 0$.

3047. $y''' + y = 0$.

3048. $y^{IV} - 2y'' = 0$.

3049. $y''' - 3y'' + 3y' - y = 0$.

3050. $y^{IV} + 4y = 0$.

3051. $y^{IV} + 8y'' + 16y = 0$.

3052. $y^{IV} + y' = 0$.

3053. $y^{IV} - 2y'' + y = 0$.

3054. $y^{IV} - a^4 y = 0$.

3055. $y^{IV} - 6y'' + 9y = 0$.

3056. $y^{IV} + a^2 y'' = 0$.

3057. $y^{IV} + 2y''' + y'' = 0$.

3058. $y^{IV} + 2y'' + y = 0$.

3059. $y^{(n)} + \frac{n}{1} y^{(n-1)} + \frac{n(n-1)}{1\cdot 2} y^{(n-2)} + \ldots + \frac{n}{1} y' + y = 0$.

3060. $y^{IV} - 2y''' + y'' = e^x$.

3061. $y^{IV} - 2y''' + y'' = x^3$.

3062. $y''' - y = x^3 - 1$.

3063. $y^{IV} + y''' = \cos 4x$.

3064. $y''' + y'' = x^2 + 1 + 3xe^x$.

3065. $y''' + y'' + y' + y = xe^x$.

3066. $y''' + y' = \tan x \sec x$.

3067. Find the particular solution of the equation

$$y''' + 2y'' + 2y' + y = x$$

that satisfies the initial conditions $y(0) = y'(0) = y''(0) = 0$.

Sec. 14. Euler's Equations

A linear equation of the form

$$(ax+b)^n y^{(n)} + A_1 (ax+b)^{n-1} y^{(n-1)} + \ldots + A_{n-1}(ax+b) y + A_n y = f(x), \quad (1)$$

where $a, b, A_1, \ldots, A_{n-1}, A_n$ are constants, is called *Euler's equation*.

Let us introduce a new independent variable t, putting

$$ax + b = e^t.$$

Then

$$y' = ae^{-t}\frac{dy}{dt}, \quad y'' = a^2 e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right),$$

$$y''' = a^3 e^{-3t}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right) \text{ and so forth}$$

and Euler's equation is transformed into a linear equation with constant coefficients.

Example 1. Solve the equation $x^2y'' + xy' + y = 1$.

Solution. Putting $x = e^t$, we get

$$\frac{dy}{dx} = e^{-t}\frac{dy}{dt}, \quad \frac{d^2y}{dx^2} = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

Consequently, the given equation takes on the form

$$\frac{d^2y}{dt^2} + y = 1,$$

whence

$$y = C_1 \cos t + C_2 \sin t + 1$$

or

$$y = C_1 \cos(\ln x) + C_2 \sin(\ln x) + 1.$$

For the homogeneous Euler equation

$$x^n y^{(n)} + A_1 x^{n-1} y^{(n-1)} + \ldots + A_{n-1} x y' + A_n y = 0 \qquad (2)$$

the solution may be sought in the form

$$y = x^k. \qquad (3)$$

Putting into (2) y, y', ..., $y^{(n)}$ found from (3), we get a characteristic equation from which we can find the exponent k.

If k is a real root of the characteristic equation of multiplicity m, then to it correspond m linearly independent solutions

$$y_1 = x^k,\ y_2 = x^k \cdot \ln x,\ y_3 = x^k (\ln x)^2,\ \ldots,\ y_m = x^k (\ln x)^{m-1}.$$

If $\alpha \pm \beta i$ is a pair of complex roots of multiplicity m, then to it there correspond $2m$ linearly independent solutions

$$y_1 = x^\alpha \cos(\beta \ln x),\ y_2 = x^\alpha \sin(\beta \ln x),\ y_3 = x^\alpha \ln x \cos(\beta \ln x),$$
$$y_4 = x^\alpha \ln x \cdot \sin(\beta \ln x),\ \ldots,\ y_{2m-1} = x^\alpha (\ln x)^{m-1} \cos(\beta \ln x),$$
$$y_{2m} = x^\alpha (\ln x)^{m-1} \sin(\beta \ln x).$$

Example 2. Solve the equation

$$x^2 y'' - 3xy' + 4y = 0.$$

Solution. We put

$$y = x^k,\quad y' = kx^{k-1},\quad y'' = k(k-1)x^{k-2}.$$

Substituting into the given equation, after cancelling out x^k, we get the characteristic equation

$$k^2 - 4k + 4 = 0.$$

Solving it we find

$$k_1 = k_2 = 2.$$

Hence, the general solution will be

$$y = C_1 x^2 + C_2 x^2 \ln x.$$

Solve the equations:

3068. $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = 0.$

3069. $x^2 y'' - xy' - 3y = 0.$

3070. $x^2 y'' + xy' + 4y = 0.$

3071. $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0.$

3072. $(3x+2)y'' + 7y' = 0.$

3073. $y'' = \frac{2y}{x^2}.$

3074. $y'' + \frac{y'}{x} + \frac{y}{x^2} = 0.$

3075. $x^2 y'' - 4xy' + 6y = x.$

3076. $(1+x)^2 y'' - 3(1+x)y' + 4y = (1+x)^3.$

3077. Find the particular solution of the equation

$$x^2y'' - xy' + y = 2x$$

that satisfies the initial conditions $y=0$, $y'=1$ when $x=1$.

Sec. 15. Systems of Differential Equations

Method of elimination. To find the solution, for instance, of a normal system of two first-order differential equations, that is, of a system of the form

$$\frac{dy}{dx}=f(x, y, z), \qquad \frac{dz}{dx}=g(x, y, z), \tag{1}$$

solved for the derivatives of the desired functions, we differentiate one of them with respect to x. We have, for example,

$$\frac{d^2y}{dx^2}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}f+\frac{\partial f}{\partial z}g. \tag{2}$$

Determining z from the first equation of the system (1) and substituting the value found,

$$z=\varphi\left(x, y, \frac{dy}{dx}\right) \tag{3}$$

into equation (2), we get a second-order equation with one unknown function y. Solving it, we find

$$y=\psi(x, C_1, C_2), \tag{4}$$

where C_1 and C_2 are arbitrary constants. Substituting function (4) into formula (3), we determine the function z without new integrations. The set of formulas (3) and (4), where y is replaced by ψ, yields the *general solution of the system* (1).

Example. Solve the system

$$\begin{cases} \dfrac{dy}{dx}+2y+4z=1+4x, \\ \dfrac{dz}{dx}+y-z=\dfrac{3}{2}x^2. \end{cases}$$

Solution. We differentiate the first equation with respect to x:

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+4\frac{dz}{dx}=4.$$

From the first equation we determine $z=\frac{1}{4}\left(1+4x-\frac{dy}{dx}-2y\right)$ and then from the second we will have $\frac{dz}{dx}=\frac{3}{2}x^2+x+\frac{1}{4}-\frac{3}{2}y-\frac{1}{4}\frac{dy}{dx}$. Putting z and $\frac{dz}{dx}$ into the equation obtained after differentiation, we arrive at a second-order equation in one unknown y:

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=-6x^2-4x+3.$$

Solving it we find:

$$y=C_1e^{2x}+C_2e^{-3x}+x^2+x,$$

and then

$$z=\frac{1}{4}\left(1+4x-\frac{dy}{dx}-2y\right)=-C_1e^{2x}+\frac{C_2}{4}e^{-3x}-\frac{1}{2}x^2.$$

We can do likewise in the case of a system with a larger number of equations.

Solve the systems:

3078. $\begin{cases} \frac{dy}{dx}=z, \\ \frac{dz}{dx}=-y. \end{cases}$

3079. $\begin{cases} \frac{dy}{dx}=y+5z, \\ \frac{dz}{dx}+y+3z=0. \end{cases}$

3080. $\begin{cases} \frac{dy}{dx}=-3y-z, \\ \frac{dz}{dx}=y-z. \end{cases}$

3081. $\begin{cases} \frac{dx}{dt}=y, \\ \frac{dy}{dt}=z, \\ \frac{dz}{dt}=x. \end{cases}$

3082. $\begin{cases} \frac{dx}{dt}=y+z, \\ \frac{dy}{dt}=x+z, \\ \frac{dz}{dt}=x+y. \end{cases}$

3083. $\begin{cases} \frac{dy}{dx}=y+z, \\ \frac{dz}{dx}=x+y+z. \end{cases}$

3084. $\begin{cases} \frac{dy}{dx}+2y+z=\sin x, \\ \frac{dz}{dx}-4y-2z=\cos x. \end{cases}$

3085. $\begin{cases} \frac{dy}{dx}+3y+4z=2x, \\ \frac{dz}{dx}-y-z=x, \end{cases}$

$y=0$, $z=0$ when $x=0$.

3086. $\begin{cases} \frac{dx}{dt}-4x-y+36t=0, \\ \frac{dy}{dt}+2x-y+2e^t=0, \end{cases}$

$x=0$, $y=1$ when $t=0$.

3087. $\begin{cases} \frac{dy}{dx}=\frac{y^2}{z}, \\ \frac{dz}{dx}=\frac{1}{2}y. \end{cases}$

3088*. a) $\frac{dx}{x^3+3xy^2}=\frac{dy}{2y^3}=\frac{dz}{2y^2z}$;

b) $\frac{dx}{x-y}=\frac{dy}{x+y}=\frac{dz}{z}$;

c) $\frac{dx}{y-z}=\frac{dy}{z-x}=\frac{dz}{x-y}$,

isolate the integral curve passing through the point (1, 1, —2).

3089. $\begin{cases} \frac{dy}{dx}+z=1, \\ \frac{dz}{dx}+\frac{2}{x^2}y=\ln x. \end{cases}$

3090. $\begin{cases} \frac{d^2y}{dx^2}+2y+4z=e^x, \\ \frac{d^2z}{dx^2}-y-3z=-x. \end{cases}$

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3091.** A shell leaves a gun with initial velocity v_0 at an angle α to the horizon. Find the equation of motion if we take the air resistance as proportional to the velocity.

3092*. A material point is attracted by a centre O with a force proportional to the distance. The motion begins from point A at a distance a from the centre with initial velocity v_0 perpendicular to OA. Find the trajectory.

Sec. 16. Integration of Differential Equations by Means of Power Series

If it is not possible to integrate a differential equation with the help of elementary functions, then in some cases its solution may be sought in the form of a power series:

$$y=\sum_{n=0}^{\infty} c_n (x-x_0)^n. \tag{1}$$

The undetermined coefficients c_n ($n=1, 2, \ldots$) are found by putting the series (1) into the equation and equating the coefficients of identical powers of the binomial $x-x_0$ on the left-hand and right-hand sides of the resulting equation.

We can also seek the solution of the equation

$$y'=f(x, y);\quad y(x_0)=y_0 \tag{2}$$

in the form of the Taylor's series

$$y(x)=\sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!}(x-x_0)^n, \tag{3}$$

where $y(x_0)=y_0$, $y'(x_0)=f(x_0, y_0)$ and the subsequent derivatives $y^{(n)}(x_0)$ ($n=2, 3, \ldots$) are successively found by differentiating equation (2) and by putting x_0 in place of x

Example 1. Find the solution of the equation

$$y''-xy=0,$$

if $y=y_0$, $y'=y_0'$ for $x=0$.

Solution. We put

$$y=c_0+c_1x+\ldots+c_nx^n+\ldots,$$

whence, differentiating, we get

$$y''=2\cdot 1c_2+3\cdot 2c_3x+\ldots+n(n-1)c_nx^{n-2}+(n+1)nc_{n+1}x^{n-1}+$$
$$+(n+2)(n+1)c_{n+2}x^n+\ldots$$

Substituting y and y'' into the given equation, we arrive at the identity

$$[2\cdot 1c_2+3\cdot 2c_3x+\ldots+n(n-1)c_nx^{n-2}+(n+1)nc_{n+1}x^{n-1}+$$
$$+(n+2)(n+1)c_{n+2}x^n+\ldots]-x[c_0+c_1x+\ldots+c_nx^n+\ldots]\equiv 0.$$

Collecting together, on the left of this equation, the terms with identical powers of x and equating to zero the coefficients of these powers, we will have

$$c_2=0;\quad 3\cdot 2c_3-c_0=0,\quad c_3=\frac{c_0}{3\cdot 2};\quad 4\cdot 3c_4-c_1=0,\quad c_4=\frac{c_1}{4\cdot 3};\quad 5\cdot 4c_5-c_2=0,$$

$$c_5=\frac{c_2}{5\cdot 4}\text{ and so forth.}$$

Generally,

$$c_{3k}=\frac{c_0}{2\cdot 3\cdot 5\cdot 6\cdot\ldots\cdot(3k-1)\,3k},\qquad c_{3k+1}=\frac{c_1}{3\cdot 4\cdot 6\cdot 7\cdot\ldots\cdot 3k\,(3k+1)},$$

$$c_{3k+2}=0\quad (k=1,\ 2,\ 3,\ \ldots).$$

Consequently,

$$y=c_0\left(1+\frac{x^3}{2\cdot 3}+\frac{x^6}{2\cdot 3\cdot 5\cdot 6}+\ldots+\frac{x^{3k}}{2\cdot 3\cdot 5\cdot 6\cdot\ldots\cdot(3k-1)\,3k}+\ldots\right)+$$
$$+c_1\left(x+\frac{x^4}{3\cdot 4}+\frac{x^7}{3\cdot 4\cdot 6\cdot 7}+\ldots+\frac{x^{3k+1}}{3\cdot 4\cdot 6\cdot 7\cdot\ldots\cdot 3k\,(3k+1)}+\ldots\right), \quad (4)$$

where $c_0=y_0$ and $c_1=y_0'$.

Applying d'Alembert's test, it is readily seen that series (4) converges for $-\infty<x<+\infty$.

Example 2. Find the solution of the equation

$$y'=x+y;\quad y_0=y(0)=1.$$

Solution. We put

$$y=y_0+y_0'x+\frac{y_0''}{2!}x^2+\frac{y_0'''}{3!}x^3+\ldots$$

We have $y_0=1$, $y_0'=0+1=1$. Differentiating equation $y'=x+y$, we successively find $y''=1+y'$, $y_0''=1+1=2$, $y'''=y''$, $y_0'''=2$, etc. Consequently,

$$y=1+x+\frac{2}{2!}x^2+\frac{2}{3!}x^3+\ldots$$

For the example at hand, this solution may be written in final form as

$$y=1+x+2\,(e^x-1-x)\ \text{ or }\ y=2e^x-1-x.$$

The procedure is similar for differential equations of higher orders. Testing the resulting series for convergence is, generally speaking, complicated and is not obligatory when solving the problems of this section.

With the help of power series, find the solutions of the equations for the indicated initial conditions.

In Examples 3097, 3098, 3099, 3101, test the solutions obtained for convergence.

3093. $y'=y+x^2$; $y=-2$ for $x=0$.

3094. $y'=2y+x-1$; $y=y_0$ for $x=1$.

3095. $y'=y^2+x^3$; $y=\frac{1}{2}$ for $x=0$.

3096. $y'=x^2-y^2$; $y=0$ for $x=0$.

3097. $(1-x)\,y'=1+x-y$; $y=0$ for $x=0$.

3098*. $xy''+y=0$; $y=0$, $y'=1$ for $x=0$.
3099. $y''+xy=0$; $y=1$, $y'=0$ for $x=0$.
3100*. $y''+\frac{2}{x}y'+y=0$; $y=1$, $y'=0$ for $x=0$.
3101*. $y''+\frac{1}{x}y'+y=0$; $y=1$, $y'=0$ for $x=0$.
3102. $\frac{d^2x}{dt^2}+x\cos t=0$; $x=a$; $\frac{dx}{dt}=0$ for $t=0$.

Sec. 17. Problems on Fourier's Method

To find the solutions of a linear homogeneous partial differential equation by Fourier's method, first seek the particular solutions of this special-type equation, each of which represents the product of functions that are dependent on one argument only. In the simplest case, there is an infinite set of such solutions u_n $(n=1, 2, \ldots)$, which are linearly independent among themselves in any finite number and which satisfy the given *boundary conditions*. The desired solution u is represented in the form of a series arranged according to these particular solutions:

$$u=\sum_{n=1}^{\infty} C_n u_n. \qquad (1)$$

The coefficients C_n which remain undetermined are found from the *initial conditions*.

Problem. A transversal displacement $u=u(x, t)$ of the points of a string with abscissa x satisfies, at time t, the equation

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}, \qquad (2)$$

where $a^2=\frac{T_0}{\varrho}$ (T_0 is the tensile force and ϱ is the linear density of the string). Find the form of the string at time t if its ends $x=0$ and $x=l$ are

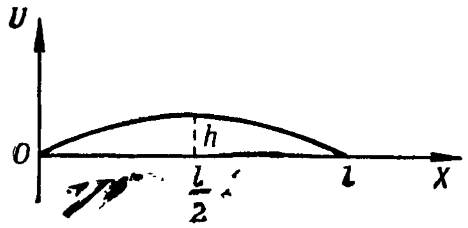

Fig. 107

fixed and at the initial instant, $t=0$, the string had the form of a parabola $u=\frac{4h}{l^2}x(l-x)$ (Fig. 107) and its points had zero velocity.

Solution. It is required to find the solution $u=u(x, t)$ of equation (2) that satisfies the boundary conditions

$$u(0, t)=0, \quad u(l, t)=0 \qquad (3)$$

and the initial conditions

$$u(x, 0)=\frac{4h}{l^2}x(l-x), \quad u_t'(x, 0)=0. \tag{4}$$

We seek the nonzero solutions of equation (2) of the special form

$$u=X(x)\,T(t).$$

Putting this expression into equation (2) and separating the variables, we get

$$\frac{T''(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}. \tag{5}$$

Since the variables x and t are independent, equation (5) is possible only when the general quantity of relation (5) is constant Denoting this constant by $-\lambda^2$, we find two ordinary differential equations:

$$T''(t)+(a\lambda)^2\cdot T(t)=0 \text{ and } X''(x)+\lambda^2X(x)=0.$$

Solving these equations, we get

$$T(t)=A\cos a\lambda t+B\sin a\lambda t,$$
$$X(x)=C\cos\lambda x+D\sin\lambda x,$$

where A, B, C, D are arbitrary constants. Let us determine the constants. From condition (3) we have $X(0)=0$ and $X(l)=0$; hence, $C=0$ and $\sin\lambda l=0$ (since D cannot be equal to zero at the same time as C is zero). For this reason, $\lambda_k=\frac{k\pi}{l}$, where k is an integer. It will readily be seen that we do not lose generality by taking for k only positive values ($k=1, 2, 3, \ldots$).

To every value λ_k there corresponds a particular solution

$$u_k=\left(A_k\cos\frac{ka\pi}{l}t+B_k\sin\frac{ka\pi}{l}t\right)\sin\frac{k\pi x}{l}$$

that satisfies the boundary conditions (3).

We construct the series

$$u=\sum_{k=1}^{\infty}\left(A_k\cos\frac{ka\pi t}{l}+B_k\sin\frac{ka\pi t}{l}\right)\sin\frac{k\pi x}{l},$$

whose sum obviously satisfies equation (2) and the boundary conditions (3).

We choose the constants A_k and B_k so that the sum of the series should satisfy the initial conditions (4). Since

$$\frac{\partial u}{\partial t}=\sum_{k=1}^{\infty}\frac{ka\pi}{l}\left(-A_k\sin\frac{ka\pi t}{l}+B_k\cos\frac{ka\pi t}{l}\right)\sin\frac{k\pi x}{l},$$

it follows that, by putting $t=0$, we obtain

$$u(x, 0)=\sum_{k=1}^{\infty}A_k\sin\frac{k\pi x}{l}\equiv\frac{4h}{l^2}x(l-x)$$

and

$$\frac{\partial u(x, 0)}{\partial t}=\sum_{k=1}^{\infty}\frac{ka\pi}{l}B_k\sin\frac{k\pi x}{l}\equiv 0.$$

Hence, to determine the coefficients A_k and B_k it is necessary to expand in a Fourier series, in sines only, the function $u(x, 0)=\frac{4h}{l^2}x(l-x)$ and the function $\frac{\partial u(x, 0)}{\partial t}\equiv 0$.

From familiar formulas (Ch. VIII, Sec. 4,3°) we have

$$A_k=\frac{2}{l}\int_0^l \frac{4h}{l^2}x(l-x)\sin\frac{k\pi x}{l}\,dx=\frac{32h}{\pi^3k^3},$$

if k is odd, and $A_k=0$ if k is even;

$$\frac{ka\pi}{l}B_k=\frac{2}{l}\int_0^l 0\sin\frac{k\pi x}{l}\,dx=0,\quad B_k=0.$$

The sought-for solution will be

$$u=\frac{32h}{\pi^3}\sum_{n=0}^{\infty}\frac{\cos\frac{(2n+1)a\pi t}{l}}{(2n+1)^3}\sin\frac{(2n+1)\pi x}{l}.$$

3103*. At the initial instant $t=0$, a string, attached at its ends, $x=0$ and $x=l$, had the form of the sine curve $u=A\sin\frac{\pi x}{l}$, and the points of it had zero velocity. Find the form of the string at time t.

3104*. At the initial time $t=0$, the points of a straight string $0<x<l$ receive a velocity $\frac{\partial u}{\partial t}=1$. Find the form of the string at time t if the ends of the string $x=0$ and $x=l$ are fixed (see Problem 3103).

3105*. A string of length $l=100$ cm and attached at its ends, $x=0$ and $x=l$, is pulled out to a distance $h=2$ cm at point $x=50$ cm at the initial time, and is then released without any impulse. Determine the shape of the string at any time t.

3106*. In longitudinal vibrations of a thin homogeneous and rectilinear rod, whose axis coincides with the x-axis, the displacement $u=u(x, t)$ of a cross-section of the rod with abscissa x satisfies, at time t, the equation

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2},$$

where $a^2=\frac{E}{\varrho}$ (E is Young's modulus and ϱ is the density of the rod). Determine the longitudinal vibrations of an elastic horizontal rod of length $l=100$ cm fixed at the end $x=0$ and pulled back at the end $x=100$ by $\Delta l=1$ cm, and then released without impulse.

3107*. For a rectilinear homogeneous rod whose axis coincides with the x-axis, the temperature $u=u(x, t)$ in a cross-section with abscissa x at time t, in the absence of sources of heat, satisfies the equation of heat conduction

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2},$$

where a is a constant. Determine the temperature distribution for any time t in a rod of length 100 cm if we know the initial temperature distribution

$$u(x, 0)=0.01x(100-x).$$

Chapter X

APPROXIMATE CALCULATIONS

Sec. 1. Operations on Approximate Numbers

1°. **Absolute error.** The *absolute error* of an approximate number a which replaces the exact number A is the absolute value of the difference between them. The number Δ, which satisfies the inequality

$$|A-a| \leqslant \Delta, \tag{1}$$

is called the *limiting absolute error*. The exact number A is located within the limits $a-\Delta \leqslant A \leqslant a+\Delta$ or, more briefly, $A=a \pm \Delta$

2°. **Relative error.** By the *relative error* of an approximate number a replacing an exact number A $(A>0)$ we understand the ratio of the absolute error of the number a to the exact number A. The number δ, which satisfies the inequality

$$\frac{|A-a|}{A} \leqslant \delta, \tag{2}$$

is called the *limiting relative error* of the approximate number a. Since in actual practice $A \approx a$, we often take the number $\delta = \frac{\Delta}{a}$ for the limiting relative error.

3°. **Number of correct decimals.** We say that a positive approximate number a written in the form of a decimal expansion has *n correct decimal places in a narrow sense* if the absolute error of this number does not exceed one half unit of the nth decimal place. In this case, when $n>1$ we can take, for the limiting relative error, the number

$$\delta = \frac{1}{2k}\left(\frac{1}{10}\right)^{n-1},$$

where k is the first significant digit of the number a. And conversely, if it is known that $\delta \leqslant \frac{1}{2(k+1)}\left(\frac{1}{10}\right)^{n-1}$, then the number a has n correct decimal places in the narrow meaning of the word. In particular, the number a definitely has n correct decimals in the narrow meaning if $\delta \leqslant \frac{1}{2}\left(\frac{1}{10}\right)^{n}$.

If the absolute error of an approximate number a does not exceed a unit of the last decimal place (such, for example, are numbers resulting from measurements made to a definite accuracy), then it is said that all decimal places of this approximate number are *correct in a broad sense*. If there is a larger number of significant digits in the approximate number, the latter (if it is the final result of calculations) is ordinarily rounded off so that all the remaining digits are correct in the narrow or broad sense.

Henceforth, we shall assume that all digits in the initial data are correct (if not otherwise stated) in the narrow sense. The results of intermediate calculations may contain one or two reserve digits.

We note that the examples of this section are, as a rule, the results of final calculations, and for this reason the answers to them are given as approximate numbers with only correct decimals.

4°. Addition and subtraction of approximate numbers. The limiting absolute error of an algebraic sum of several numbers is equal to the sum of the limiting absolute errors of these numbers. Therefore, in order to have, in the sum of a small number of approximate numbers (all decimal places of which are correct), only correct digits (at least in the broad sense), all summands should be put into the form of that summand which has the smallest number of decimal places, and in each summand a reserve digit should be retained. Then add the resulting numbers as exact numbers, and round off the sum by one decimal place

If we have to add approximate numbers that have not been rounded off, they should be rounded off and one or two reserve digits should be retained. Then be guided by the foregoing rule of addition while retaining the appropriate extra digits in the sum up to the end of the calculations.

Example 1. $215.21+14.182+21.4=215.2(1)+14.1(8)+21.4=250.8$.

The relative error of a sum of positive terms lies between the least and greatest relative errors of these terms.

The relative error of a difference is not amenable to simple counting. Particularly unfavourable in this sense is the difference of two close numbers.

Example 2. In subtracting the approximate numbers 6.135 and 6.131 to four correct decimal places, we get the difference 0.004. The limiting relative error is $\delta=\frac{\frac{1}{2}0.001+\frac{1}{2}0.001}{0.004}=\frac{1}{4}=0.25$. Hence, not one of the decimals of the difference is correct. Therefore, it is always advisable to avoid subtracting close approximate numbers and to transform the given expression, if need be, so that this undesirable operation is omitted.

5°. Multiplication and division of approximate numbers. The limiting relative error of a product and a quotient of approximate numbers is equal to the sum of the limiting relative errors of these numbers. Proceeding from this and applying the rule for the number of correct decimals (3°), we retain in the answer only a definite number of decimals

Example 3. The product of the approximate numbers $25.3\cdot 4.12=104.236$.

Assuming that all decimals of the factors are correct, we find that the limiting relative error of the product is

$$\delta=\frac{1}{2\cdot 2}0.01+\frac{1}{4\cdot 2}0.01\approx 0.003.$$

Whence the number of correct decimals of the product is three and the result, if it is final, should be written as follows: $25.3\cdot 4.12=104$, or more correctly, $25.3\cdot 4.12=104.2\pm 0.3$.

6°. Powers and roots of approximate numbers. The limiting relative error of the mth power of an approximate number a is equal to the m-fold limiting relative error of this number

The limiting relative error of the mth root of an approximate number a is the $\frac{1}{m}$th part of the limiting relative error of the number a.

7°. Calculating the error of the result of various operations on approximate numbers. If $\Delta a_1, \ldots, \Delta a_n$ are the limiting absolute errors of the approximate numbers $a_1, \ldots, a_n$, then the limiting absolute error ΔS of the result

$$S = f(a_1, \ldots, a_n)$$

may be evaluated approximately from the formula

$$\Delta S = \left|\frac{\partial f}{\partial a_1}\right| \Delta a_1 + \ldots + \left|\frac{\partial f}{\partial a_n}\right| \Delta a_n.$$

The limiting relative error S is then equal to

$$\delta S = \frac{\Delta S}{|S|} = \left|\frac{\partial f}{\partial a_1}\right| \cdot \frac{\Delta a_1}{|f|} + \ldots + \left|\frac{\partial f}{\partial a_n}\right| \frac{\Delta a_n}{|f|} =$$
$$= \left|\frac{\partial \ln f}{\partial a_1}\right| \Delta a_1 + \ldots + \left|\frac{\partial \ln f}{\partial a_n}\right| \Delta a_n.$$

Example 4. Evaluate $S = \ln(10.3 + \sqrt{4.4})$; the approximate numbers 10.3 and 4.4 are correct to one decimal place.

Solution. Let us first compute the limiting absolute error ΔS in the general form: $S = \ln(a + \sqrt{b})$, $\Delta S = \frac{1}{a + \sqrt{b}}\left(\Delta a + \frac{1}{2}\frac{\Delta b}{\sqrt{b}}\right)$. We have $\Delta a = \Delta b \approx \frac{1}{20}$; $\sqrt{4.4} = 2.0976\ldots$; we leave 2.1, since the relative error of the approximate number $\sqrt{4.4}$ is equal to $\approx \frac{1}{2} \cdot \frac{1}{40} = \frac{1}{80}$; the absolute error is then equal to $\approx 2\frac{1}{80} = \frac{1}{40}$; we can be sure of the first decimal place. Hence,

$$\Delta S = \frac{1}{10.3 + 2.1}\left(\frac{1}{20} + \frac{1}{2} \cdot \frac{1}{20 \cdot 2.1}\right) = \frac{1}{12.4 \cdot 20}\left(1 + \frac{1}{4.2}\right) = \frac{13}{2604} \approx 0.005.$$

Thus, two decimal places will be correct.

Now let us do the calculations with one reserve decimal:

$\log(10.3 + \sqrt{4.4}) \approx \log 12.4 = 1.093$, $\ln(10.3 + \sqrt{4.4}) \approx 1.093 \cdot 2.303 = 2.517$. And we get the answer: 2.52

8°. Establishing admissible errors of approximate numbers for a given error in the result of operations on them. Applying the formulas of 7° for the quantities ΔS or δS given us and considering all particular differentials $\left|\frac{\partial f}{\partial a_k}\right| \Delta a_k$ or the quantities $\left|\frac{\partial f}{\partial a_k}\right| \frac{\Delta a_k}{|f|}$ equal, we calculate the admissible absolute errors $\Delta a_1, \ldots, \Delta a_n, \ldots$ of the approximate numbers $a_1, \ldots, a_n, \ldots$ that enter into the operations (the *principle of equal effects*).

It should be pointed out that sometimes when calculating the admissible errors of the arguments of a function it is not advantageous to use the principle of equal effects, since the latter may make demands that are practically unfulfilable In these cases it is advisable to make a reasonable redistribution of errors (if this is possible) so that the overall total error does not exceed a specified quantity. Thus, strictly speaking, the problem thus posed is indeterminate.

Example 5. The volume of a "cylindrical segment", that is, a solid cut off a circular cylinder by a plane passing through the diameter of the base (equal to $2R$) at an angle α to the base, is computed from the formula $V = \frac{2}{3} R^2 \tan \alpha$. To what degree of accuracy should we measure the radius

$R \approx 60$ cm and the angle of inclination α so that the volume of the cylindrical segment is found to an accuracy up to 1%?

Solution. If ΔV, ΔR and $\Delta\alpha$ are the limiting absolute errors of the quantities V, R and α, then the limiting relative error of the volume V that we are calculating is

$$\delta = \frac{3\Delta R}{R} + \frac{2\Delta\alpha}{\sin 2\alpha} \leqslant \frac{1}{100}.$$

We assume $\frac{3\Delta R}{R} \leqslant \frac{1}{200}$ and $\frac{2\Delta\alpha}{\sin 2\alpha} \leqslant \frac{1}{200}$. Whence

$$\Delta R \leqslant \frac{R}{600} \approx \frac{60 \text{ cm}}{600} = 1 \text{ mm};$$

$$\Delta\alpha \leqslant \frac{\sin 2\alpha}{400} \leqslant \frac{1}{400} \text{ radian} \approx 9'.$$

Thus, we ensure the desired accuracy in the answer to 1% if we measure the radius to 1 mm and the angle of inclination α to $9'$.

3108. Measurements yielded the following approximate numbers that are correct in the broad meaning to the number of decimal places indicated:

a) $12°07'14''$; b) 38.5 cm; c) 62.215 kg.

Compute their absolute and relative errors.

3109. Compute the absolute and relative errors of the following approximate numbers which are correct in the narrow sense to the decimal places indicated:

a) 241.7; b) 0.035; c) 3.14.

3110. Determine the number of correct (in the narrow sense) decimals and write the approximate numbers:

a) 48.361 for an accuracy of 1%;

b) 14.9360 for an accuracy of 1%;

c) 592.8 for an accuracy of 2%.

3111. Add the approximate numbers, which are correct to the indicated decimals:

a) $25.386 + 0.49 + 3.10 + 0.5$;

b) $1.2 \cdot 10^2 + 41.72 + 0.09$;

c) $38.1 + 2.0 + 3.124$.

3112. Subtract the approximate numbers, which are correct to the indicated decimals:

a) $148.1 - 63.871$; b) $29.72 - 11.25$; c) $34.22 - 34.21$.

3113*. Find the difference of the areas of two squares whose measured sides are 15.28 cm and 15.22 cm (accurate to 0.05 mm).

3114. Find the product of the approximate numbers, which are correct to the indicated decimals:

a) $3.49 \cdot 8.6$; b) $25.1 \cdot 1.743$; c) $0.02 \cdot 16.5$. Indicate the possible limits of the results.

3115. The sides of a rectangle are 4.02 and 4.96 m (accurate to 1 cm). Compute the area of the rectangle.

3116. Find the quotient of the approximate numbers, which are correct to the indicated decimals:

a) 5.684 : 5.032; b) 0.144 : 1.2; c) 216:4.

3117. The legs of a right triangle are 12.10 cm and 25.21 cm (accurate to 0.01 cm). Compute the tangent of the angle opposite the first leg.

3118. Compute the indicated powers of the approximate numbers (the bases are correct to the indicated decimals):

a) 0.4158^2; b) 65.2^3; c) 1.5^2.

3119. The side of a square is 45.3 cm (accurate to 1 mm). Find the area.

3120. Compute the values of the roots (the radicands are correct to the indicated decimals):

a) $\sqrt{2.715}$; b) $\sqrt[3]{65.2}$; c) $\sqrt{81.1}$.

3121. The radii of the bases and the generatrix of a truncated cone are $R = 23.64$ cm ± 0.01 cm; $r = 17.31$ cm ± 0.01 cm; $l = 10.21$ cm ± 0.01 cm; $\pi = 3.14$. Use these data to compute the total surface of the truncated cone. Evaluate the absolute and relative errors of the result.

3122. The hypotenuse of a right triangle is 15.4 cm $\pm$ 0.1 cm; one of the legs is 6.8 cm $\pm$ 0.1 cm. To what degree of accuracy can we determine the second leg and the adjacent acute angle? Find their values.

3123. Calculate the specific weight of aluminium if an aluminium cylinder of diameter 2 cm and altitude 11 cm weighs 93.4 gm. The relative error in measuring the lengths is 0.01, while the relative error in weighing is 0.001.

3124. Compute the current if the electromotive force is equal to 221 volts $\pm$ 1 volt and the resistance is 809 ohms $\pm$ 1 ohm.

3125. The period of oscillation of a pendulum of length l is equal to

$$T = 2\pi \sqrt{\frac{l}{g}},$$

where g is the acceleration of gravity. To what degree of accuracy do we have to measure the length of the pendulum, whose period is close to 2 sec, in order to obtain its oscillation period with a relative error of 0.5%? How accurate must the numbers π and g be taken?

3126. It is required to measure, to within 1%, the lateral surface of a truncated cone whose base radii are 2 m and 1 m, and the generatrix is 5 m (approximately). To what degree of accuracy do we have to measure the radii and the generatrix and to how many decimal places do we have to take the number π?

3127. To determine Young's modulus for the bending of a rod of rectangular cross-section we use the formula

$$E=\frac{1}{4}\cdot\frac{l^3P}{d^3bs},$$

where l is the rod length, b and d are the basis and altitude of the cross-section of the rod, s is the sag, and P the load. To what degree of accuracy do we have to measure the length l and the sag s so that the error E should not exceed 5.5%, provided that the load P is known to 0.1%, and the quantities d and b are known to an accuracy of 1%, $l\approx 50$ cm, $s\approx 2.5$ cm?

Sec. 2. Interpolation of Functions

1°. Newton's interpolation formula. Let $x_0, x_1, \ldots, x_n$ be the tabular values of an argument, the difference of which $h=\Delta x_i$ ($\Delta x_i=x_{i+1}-x_i$; $i=0, 1, \ldots, n-1$) is constant (*table interval*) and $y_0, y_1, \ldots, y_n$ are the corresponding values of the function y. Then the value of the function y for an intermediate value of the argument x is approximately given by *Newton's interpolation formula*

$$y=y_0+q\cdot\Delta y_0+\frac{q(q-1)}{2!}\Delta^2y_0+\ldots+\frac{q(q-1)\ldots(q-n+1)}{n!}\Delta^ny_0, \qquad (1)$$

where $q=\frac{x-x_0}{h}$ and $\Delta y_0=y_1-y_0$, $\Delta^2y_0=\Delta y_1-\Delta y_0, \ldots$ are successive finite differences of the function y. When $x=x_i$ ($i=0, 1, \ldots, n$), the polynomial (1) takes on, accordingly, the tabular values y_i ($i=0, 1, \ldots, n$). As particular cases of Newton's formula we obtain: for $n=1$, *linear interpolation*; for $n=2$, *quadratic interpolation*. To simplify the use of Newton's formula, it is advisable first to set up a table of finite differences.

If $y=f(x)$ is a polynomial of degree n, then

$$\Delta^ny_i=\text{const and } \Delta^{n+1}y_i=0$$

and, hence, formula (1) is exact.

In the general case, if $f(x)$ has a continuous derivative $f^{(n+1)}(x)$ on the interval $[a, b]$, which includes the points $x_0, x_1, \ldots, x_n$ and x, then the error of formula (1) is

$$R_n(x)=y-\sum_{i=0}^{n}\frac{q(q-1)\ldots(q-i+1)}{i!}\Delta^iy_0=$$
$$=h^{n+1}\frac{q(q-1)\ldots(q-n)}{(n+1)!}f^{(n+1)}(\xi), \qquad (2)$$

where ξ is some intermediate value between x_i ($i=0, 1, \ldots, n$) and x. For practical use, the following approximate formula is more convenient:

$$R_n(x)\approx\frac{\Delta^{n+1}y_0}{(n+1)!}q(q-1)\ldots(q-n).$$

If the number n may be any number, then it is best to choose it so that the difference $\Delta^{n+1}y_0 \approx 0$ within the limits of the given accuracy; in other words, the differences $\Delta^n y_0$ should be constant to within the given places of decimals

Example 1. Find sin 26°15′ using the tabular data $\sin 26° = 0.43837$, $\sin 27° = 0.45399$, $\sin 28° = 0.46947$.

Solution. We set up the table

i	x_i	y_i	Δy_i	$\Delta^2 y_i$
0	26°	0 43837	1562	—14
1	27°	0 45399	1548	
2	28°	0 46947		

Here, $h = 60'$, $q = \frac{26°15' - 26°}{60'} = \frac{1}{4}$.

Applying formula (1) and using the first horizontal line of the table, we have

$$\sin 26°15' = 0.43837 + \frac{1}{4}\,0.01562 + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}\cdot(-0.00014) = 0.44229.$$

Let us evaluate the error R_2 Using formula (2) and taking into account that if $y = \sin x$, then $|y^{(n)}| \leqslant 1$, we will have:

$$|R_2| \leqslant \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)}{3!}\left(\frac{\pi}{180}\right)^3 = \frac{7}{128}\cdot\frac{1}{57.33^3} \approx \frac{1}{4}\cdot 10^{-6}.$$

Thus, all the decimals of sin 26°15′ are correct.

Using Newton's formula, it is also possible, from a given intermediate value of the function y, to find the corresponding value of the argument x (*inverse interpolation*). To do this, first determine the corresponding value q by the method of successive approximation, putting

$$q^{(0)} = \frac{y - y_0}{\Delta y_0}$$

and

$$q^{(i+1)} = q^{(0)} - \frac{q^{(i)}(q^{(i)}-1)}{2!}\cdot\frac{\Delta^2 y_0}{\Delta y_0} - \ldots - \frac{q^{(i)}(q^{(i)}-1)\ldots(q^{(i)}-n+1)}{n!}\,\frac{\Delta^n y_0}{\Delta y_0}$$
$$(i = 0, 1, 2, \ldots).$$

Here, for q we take the common value (to the given accuracy!) of two successive approximations $q^{(m)} = q^{(m+1)}$. Whence $x = x_0 + q\cdot h$.

Example 2. Using the table

x	$y = \sinh x$	Δy	$\Delta^2 y$
2 2	4.457	1.009	0.220
2 4	5.466	1 229	
2 6	6.695		

approximate the root of the equation $\sinh x = 5$.

Solution. Taking $y_0=4.457$, we have

$$q^{(0)}=\frac{5-4.457}{1.009}=\frac{0.543}{1.009}=0.538;$$

$$q^{(1)}=q^{(0)}+\frac{q^{(0)}(1-q^{(0)})}{2}\cdot\frac{\Delta^2 y_0}{\Delta y_0}=0.538+\frac{0.538\cdot 0.462}{2}\cdot\frac{0.220}{1.009}=$$
$$=0.538+0.027=0.565;$$

$$q^{(2)}=0.538+\frac{0.565\cdot 0.435}{2}\cdot\frac{0.220}{1.009}=0.538+0.027=0.565.$$

We can thus take

$$x=2.2+0.565\cdot 0.2=2.2+0.113=2.313.$$

2°. Lagrange's interpolation formula. In the general case, a polynomial of degree n, which for $x=x_i$ takes on given values y_i ($i=0, 1, \ldots, n$), is given by the *Lagrange interpolation formula*

$$y=\frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}y_1+\ldots$$
$$\ldots+\frac{(x-x_0)(x-x_1)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}y_k+\ldots$$
$$\ldots+\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}y_n.$$

3128. Given a table of the values of x and y:

x	1	2	3	4	5	6
y	3	10	15	12	9	5

Set up a table of the finite differences of the function y.

3129. Set up a table of differences of the function $y=x^3-5x^2+x-1$ for the values $x=1, 3, 5, 7, 9, 11$. Make sure that all the finite differences of order 3 are equal.

3130*. Utilizing the constancy of fourth-order differences, set up a table of differences of the function $y=x^4-10x^3+2x^2+3x$ for integral values of x lying in the range $1\leqslant x\leqslant 10$.

3131. Given the table

$$\log 1=0.000,$$
$$\log 2=0.301,$$
$$\log 3=0.477,$$
$$\log 4=0.602,$$
$$\log 5=0.699.$$

Use linear interpolation to compute the numbers: $\log 1.7$, $\log 2.5$, $\log 3.1$, and $\log 4.6$.

3132. Given the table

$$\sin 10^\circ = 0.1736, \quad \sin 13^\circ = 0.2250,$$
$$\sin 11^\circ = 0.1908, \quad \sin 14^\circ = 0.2419,$$
$$\sin 12^\circ = 0.2079, \quad \sin 15^\circ = 0.2588.$$

Fill in the table by computing (with Newton's formula, for $n=2$) the values of the sine every half degree.

3133. Form Newton's interpolation polynomial for a function represented by the table

x	0	1	2	3	4
y	1	4	15	40	85

3134*. Form Newton's interpolation polynomial for a function represented by the table

x	2	4	6	8	10
y	3	11	27	50	83

Find y for $x=5.5$. For what x will $y=20$?

3135. A function is given by the table

x	−2	1	2	4
y	25	−8	−15	−23

Form Lagrange's interpolation polynomial and find the value of y for $x=0$.

3136. Experiment has yielded the contraction of a spring (x mm) as a function of the load (P kg) carried by the spring:

x	5	10	15	20	25	30	35	40
P	49	105	172	253	352	473	619	793

Find the load that yields a contraction of the spring by 14 mm.

3137. Given a table of the quantities x and y

x	0	1	3	4	5
y	1	−3	25	129	381

Compute the values of y for $x=0.5$ and for $x=2$: a) by means of linear interpolation; b) by Lagrange's formula.

Sec. 3. Computing the Real Roots of Equations

1°. Establishing initial approximations of roots. The approximation of the roots of a given equation

$$f(x)=0 \qquad (1)$$

consists of two stages: 1) *separating the roots*, that is, establishing the intervals (as small as possible) within which lies one and only one root of equation (1); 2) *computing the roots* to a given degree of accuracy

If a function $f(x)$ is defined and continuous on an interval $[a, b]$ and $f(a)\cdot f(b)<0$, then on $[a, b]$ there is at least one root ξ of equation (1). This root will definitely be the only one if $f'(x)>0$ or $f'(x)<0$ when $a<x<b$.

In approximating the root ξ it is advisable to use millimetre paper and construct a graph of the function $y=f(x)$. The abscissas of the points of intersection of the graph with the x-axis are the roots of the equation $f(x)=0$. It is sometimes convenient to replace the given equation with an equivalent equation $\varphi(x)=\psi(x)$. Then the roots of the equation are found as the abscissas of points of intersection of the graphs $y=\varphi(x)$ and $y=\psi(x)$.

2°. The rule of proportionate parts (chord method). If on an interval $[a, b]$ there is a unique root ξ of the equation $f(x)=0$, where the function $f(x)$ is continuous on $[a, b]$, then by replacing the curve $y=f(x)$ by a chord passing through the points $[a, f(a)]$ and $[b, f(b)]$, we obtain the first approximation of the root

$$c_1=a-\frac{f(a)}{f(b)-f(a)}(b-a). \qquad (2)$$

To obtain a second approximation c_2, we apply formula (2) to that one of the intervals $[a, c_1]$ or $[c_1, b]$ at the ends of which the function $f(x)$ has values of opposite sign. The succeeding approximations are constructed in the same manner. The sequence of numbers c_n $(n=1, 2, \ldots)$ converges to the root ξ, that is,

$$\lim_{n\to\infty} c_n=\xi.$$

Generally speaking, we should continue to calculate the approximations c_1, c_2, ..., until the decimals retained in the answer cease to change (in accord with the specified degree of accuracy!); for intermediate calculations, take one or two reserve decimals This is a general remark.

If the function $f(x)$ has a nonzero continuous derivative $f'(x)$ on the interval $[a, b]$, then to evaluate the absolute error of the approximate root

c_n, we can make use of the formula

$$|\xi - c_n| \leqslant \frac{|f(c_n)|}{\mu},$$

where $\mu = \min\limits_{a \leqslant x \leqslant b} |f'(x)|$.

3°. Newton's method (method of tangents). If $f'(x) \neq 0$ and $f''(x) \neq 0$ for $a \leqslant x \leqslant b$, where $f(a) f(b) < 0$, $f(a) f''(a) > 0$, then the successive approximations x_n ($n = 0, 1, 2, \ldots$) to the root ξ of an equation $f(x) = 0$ are computed from the formulas

$$x_0 = a, \quad x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \quad (n = 1, 2, \ldots). \tag{3}$$

Under the given assumptions, the sequence x_n ($n = 1, 2, \ldots$) is monotonic and

$$\lim_{n \to \infty} x_n = \xi.$$

To evaluate the errors we can use the formula

$$|x_n - \xi| \leqslant \frac{|f(x_n)|}{\mu},$$

where $\mu = \min\limits_{a \leqslant x \leqslant b} |f'(x)|$.

For practical purposes it is more convenient to use the simpler formulas

$$x_0 = a, \quad x_n = x_{n-1} - \alpha f(x_{n-1}) \quad (n = 1, 2, \ldots), \tag{3'}$$

where $\alpha = \frac{1}{f'(a)}$, which yield the same accuracy as formulas (3).

If $f(b) f''(b) > 0$, then in formulas (3) and (3') we should put $x_0 = b$.

4°. Iterative method. Let the given equation be reduced to the form

$$x = \varphi(x), \tag{4}$$

where $|\varphi'(x)| \leqslant r < 1$ (r is constant) for $a \leqslant x \leqslant b$. Proceeding from the initial value x_0, which belongs to the interval $[a, b]$, we build a sequence of numbers $x_1, x_2, \ldots$ according to the following law:

$$x_1 = \varphi(x_0), \quad x_2 = \varphi(x_1), \quad \ldots, \quad x_n = \varphi(x_{n-1}), \quad \ldots \tag{5}$$

If $a \leqslant x_n \leqslant b$ ($n = 1, 2, \ldots$), then the limit

$$\xi = \lim_{n \to \infty} x_n$$

is the *only root* of equation (4) on the interval $[a, b]$; that is, x_n are *successive approximations* to the root ξ.

The evaluation of the absolute error of the nth approximation to x_n is given by the formula

$$|\xi - x_n| \leqslant \frac{|x_{n+1} - x_n|}{1 - r}.$$

Therefore, if x_n and x_{n+1} coincide to within ε, then the limiting absolute error for x_n will be $\frac{\varepsilon}{1-r}$.

In order to transform equation $f(x) = 0$ to (4), we replace the latter with an equivalent equation

$$x = x - \lambda f(x),$$

where the number $\lambda \neq 0$ is chosen so that the function $\frac{d}{dx}[x - \lambda f(x)] = 1 - \lambda f'(x)$

should be small in absolute value in the neighbourhood of the point x_0 [for example, we can put $1-\lambda f'(x_0)=0$].

Example 1. Reduce the equation $2x-\ln x-4=0$ to the form (4) for the initial approximation to the root $x_0=2.5$.

Solution. Here, $f(x)=2x-\ln x-4$; $f'(x)=2-\frac{1}{x}$. We write the equivalent equation $x=x-\lambda(2x-\ln x-4)$ and take 0.5 as one of the suitable values of λ; this number is close to the root of the equation $\left|1-\lambda\left(2-\frac{1}{x}\right)\right|_{x=2.5}=0$, that is, close to $\frac{1}{1.6}\approx 0.6$.

The initial equation is reduced to the form

$$x=x-0.5\,(2x-\ln x-4)$$

or

$$x=2+\frac{1}{2}\ln x.$$

Example 2. Compute, to two decimal places, the root ξ of the preceeding equation that lies between 2 and 3.

Computing the root by the iterative method. We make use of the result of Example 1, putting $x_0=2.5$. We carry out the calculations using formulas (5) with one reserve decimal.

$$x_1=2+\frac{1}{2}\ln 2.5\approx 2.458,$$

$$x_2=2+\frac{1}{2}\ln 2.458\approx 2.450,$$

$$x_3=2+\frac{1}{2}\ln 2.450\approx 2.448,$$

$$x_4=2+\frac{1}{2}\ln 2.448\approx 2.448.$$

And so $\xi\approx 2\,45$ (we can stop here since the third decimal place has become fixed)

Let us now evaluate the error. Here,

$$\varphi(x)=2+\frac{1}{2}\ln x \quad\text{and}\quad \varphi'(x)=\frac{1}{2x}.$$

Considering that all approximations to x_n lie in the interval [2.4, 2.5], we get

$$r=\max|\varphi'(x)|=\frac{1}{2\cdot 2.4}=0.21.$$

Hence, the limiting absolute error in the approximation to x_3 is, by virtue of the remark made above,

$$\Delta=\frac{0.001}{1-0.21}=0.0012\approx 0.001.$$

Thus, the exact root ξ of the equation lies within the limits

$$2\,447<\xi<2.449;$$

we can take $\xi\approx 2.45$, and all the decimals of this approximate number will be correct in the narrow sense.

Calculating the root by Newton's method. Here,

$$f(x)=2x-\ln x-4, \qquad f'(x)=2-\frac{1}{x}, \qquad f''(x)=\frac{1}{x^2}.$$

On the interval $2 \leqslant x \leqslant 3$ we have: $f'(x)>0$ and $f''(x)>0$; $f(2)f(3)<0$; $f(3)f''(3)>0$. Hence, the conditions of 3° for $x_0=3$ are fulfilled.
We take

$$\alpha=\left(2-\frac{1}{3}\right)^{-1}=0.6.$$

We carry out the calculations using formulas (3′) with two reserve decimals:

$$x_1=3-0.6\,(2\cdot 3-\ln 3-4)=2.4592;$$
$$x_2=2.4592-0.6\,(2\cdot 2.4592-\ln 2.4592-4)=2.4481;$$
$$x_3=2.4481-0.6\,(2\cdot 2.4481-\ln 2.4481-4)=2.4477;$$
$$x_4=2.4477-0.6\,(2\cdot 2.4477-\ln 2.4477-4)=2.4475.$$

At this stage we stop the calculations, since the third decimal place does not change any more. The answer is: the root $\xi=2.45$. We omit the evaluation of the error.

5°. The case of a system of two equations. Let it be required to calculate the real roots of a system of two equations in two unknowns (to a given degree of accuracy):

$$\begin{cases} f(x, y)=0, \\ \varphi(x, y)=0, \end{cases} \tag{6}$$

and let there be an initial approximation to one of the solutions (ξ, η) of this system $x=x_0$, $y=y_0$.

This initial approximation may be obtained, for example, graphically, by plotting (in the same Cartesian coordinate system) the curves $f(x, y)=0$ and $\varphi(x, y)=0$ and by determining the coordinates of the points of intersection of these curves.

a) Newton's method. Let us suppose that the functional determinant

$$I=\frac{\partial(f, \varphi)}{\partial(x, y)}$$

does not vanish near the initial approximation $x=x_0$, $y=y_0$. Then by Newton's method the first approximate solution to the system (6) has the form $x_1=x_0+\alpha_0$, $y_1=y_0+\beta_0$, where α_0, β_0 are the solution of the system of two linear equations

$$\begin{cases} f(x_0, y_0)+\alpha_0 f'_x(x_0, y_0)+\beta_0 f'_y(x_0, y_0)=0, \\ \varphi(x_0, y_0)+\alpha_0 \varphi'_x(x_0, y_0)+\beta_0 \varphi'_y(x_0, y_0)=0. \end{cases}$$

The second approximation is obtained in the very same way:

$$x_2=x_1+\alpha_1, \quad y_2=y_1+\beta_1,$$

where α_1, β_1 are the solution of the system of linear equations

$$\begin{cases} f(x_1, y_1)+\alpha_1 f'_x(x_1, y_1)+\beta_1 f'_y(x_1, y_1)=0, \\ \varphi(x_1, y_1)+\alpha_1 \varphi'_x(x_1, y_1)+\beta_1 \varphi'_y(x_1, y_1)=0. \end{cases}$$

Similarly we obtain the third and succeeding approximations.

b) Iterative method. We can also apply the iterative method to solving the system of equations (6), by transforming this system to an equivalent one

$$\begin{cases} x=F(x, y), \\ y=\Phi(x, y) \end{cases} \tag{7}$$

and assuming that

$$|F'_x(x,y)|+|\Phi'_x(x, y)| \leqslant r < 1; \quad |F'_y(x, y)|+|\Phi'_y(x, y)| \leqslant r < 1 \tag{8}$$

in some two-dimensional neighbourhood U of the initial approximation (x_0, y_0), which neighbourhood also contains the exact solution (ξ, η) of the system.

The sequence of approximations (x_n, y_n) $(n=1, 2, \ldots)$, which converges to the solution of the system (7) or, what is the same thing, to the solution of (6), is constructed according to the following law:

$$\begin{aligned} &x_1=F(x_0, y_0), \quad y_1=\Phi(x_0, y_0), \\ &x_2=F(x_1, y_1), \quad y_2=\Phi(x_1, y_1), \\ &x_3=F(x_2, y_2), \quad y_3=\Phi(x_2, y_2), \\ &\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

If all (x_n, y_n) belong to U, then $\lim\limits_{n\to\infty} x_n=\xi$, $\lim\limits_{n\to\infty} y_n=\eta$.

The following technique is advised for transforming the system of equations (6) to (7) with condition (8) observed. We consider the system of equations

$$\begin{cases} \alpha f(x, y)+\beta\varphi(x, y)=0, \\ \gamma f(x, y)+\delta\varphi(x, y)=0, \end{cases}$$

which is equivalent to (6) provided that $\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix} \neq 0$. Rewrite it in the form

$$\begin{aligned} x&=x+\alpha f(x, y)+\beta\varphi(x, y) \equiv F(x, y), \\ y&=y+\gamma f(x, y)+\delta\varphi(x, y) \equiv \Phi(x, y). \end{aligned}$$

Choose the parameters α, β, γ, δ such that the partial derivatives of the functions $F(x, y)$ and $\Phi(x, y)$ will be equal or close to zero in the initial approximation; in other words, we find α, β, γ, δ as approximate solutions of the system of equations

$$\begin{cases} 1+\alpha f'_x(x_0, y_0)+\beta\varphi'_x(x_0, y_0)=0, \\ \alpha f'_y(x_0, y_0)+\beta\varphi'_y(x_0, y_0)=0, \\ \gamma f'_x(x_0, y_0)+\delta\varphi'_x(x_0, y_0)=0, \\ 1+\gamma f'_y(x_0, y_0)+\delta\varphi'_y(x_0, y_0)=0. \end{cases}$$

Condition (8) will be observed in such a choice of parameters α, β, γ, δ on the assumption that the partial derivatives of the functions $f(x, y)$ and $\varphi(x, y)$ do not vary very rapidly in the neighbourhood of the initial approximation (x_0, y_0).

Example 3. Reduce to the form (7) the system of equations

$$\begin{cases} x^2+y^2-1=0, \\ x^3-y=0 \end{cases}$$

given the initial approximation to the root $x_0=0.8$, $y_0=0.55$.

Solution. Here, $f(x, y)=x^2+y^2-1$, $\varphi(x, y)=x^3-y$; $f'_x(x_0, y_0)=1.6$, $f'_y(x_0, y_0)=1.1$; $\varphi'_x(x_0, y_0)=1.92$, $\varphi'_y(x_0, y_0)=-1$.

Write down the system (that is equivalent to the initial one)

$$\begin{cases} \alpha(x^2+y^2-1)+\beta(x^3-y)=0, \\ \gamma(x^2+y^2-1)+\delta(x^3-y)=0 \end{cases} \quad \left(\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix} \neq 0\right)$$

in the form

$$x=x+\alpha(x^2+y^2-1)+\beta(x^3-y),$$
$$y=y+\gamma(x^2+y^2-1)+\delta(x^3-y).$$

For suitable numerical values of α, β, γ and δ choose the solution of the system of equations

$$\begin{cases} 1+1.6\alpha+1.92\beta=0, \\ 1.1\alpha-\beta=0, \\ 1.6\gamma+1.92\delta=0, \\ 1+1.1\gamma-\delta=0; \end{cases}$$

i. e., we put $\alpha\approx-0.3$, $\beta\approx-0.3$, $\gamma\approx-0.5$, $\delta\approx0.4$.

Then the system of equations

$$\begin{cases} x=x-0.3(x^2+y^2-1)-0.3(x^3-y), \\ y=y-0.5(x^2+y^2-1)+0.4(x^3-y), \end{cases}$$

which is equivalent to the initial system, has the form (7); and in a sufficiently small neighbourhood of the point (x_0, y_0) condition (8) will be fulfilled.

Isolate the real roots of the equations by trial and error, and by means of the rule of proportional parts compute them to two decimal places.

3138. $x^3-x+1=0$.

3139. $x^4+0.5x-1.55=0$.

3140. $x^3-4x-1=0$.

Proceeding from the graphically found initial approximations, use Newton's method to compute the real roots of the equations to two decimal places:

3141. $x^3-2x-5=0$.

3142. $2x-\ln x-4=0$.

3143. $2^x=4x$.

3144. $\log x=\frac{1}{x}$.

Utilizing the graphically found initial approximations, use the iterative method to compute the real roots of the equations to two decimal places:

3145. $x^3-5x+0.1=0$.

3146. $4x=\cos x$.

3147. $x^5-x-2=0$.

Find graphically the initial approximations and compute the real roots of the equations and systems to two decimals:

3148. $x^3-3x+1=0$.

3149. $x^3-2x^2+3x-5=0$.

3150. $x^4+x^2-2x-2=0$.

3151. $x\cdot\ln x-14=0$.

3152. $x^3+3x-0.5=0$.

3153. $4x-7\sin x=0$.

3154. $x^x + 2x - 6 = 0$.

3155. $e^x + e^{-3x} - 4 = 0$.

3156. $\begin{cases} x^2 + y^2 - 1 = 0, \\ x^3 - y = 0. \end{cases}$

3157. $\begin{cases} x^2 + y - 4 = 0, \\ y - \log x - 1 = 0. \end{cases}$

3158. Compute to three decimals the smallest positive root of the equation $\tan x = x$.

3159. Compute the roots of the equation $x \cdot \tanh x = 1$ to four decimal places.

Sec. 4. Numerical Integration of Functions

1°. Trapezoidal formula. For the approximate evaluation of the integral

$$\int_a^b f(x)\,dx$$

[$f(x)$ is a function continuous on $[a, b]$] we divide the interval of integration $[a, b]$ into n equal parts and choose the *interval of calculations* $h = \frac{b-a}{n}$. Let $x_i = x_0 + ih$ ($x_0 = a$, $x_n = b$, $i = 0, 1, 2, \ldots, n$) be the abscissas of the partition points, and let $y_i = f(x_i)$ be the corresponding values of the integrand $y = f(x)$. Then the *trapezoidal formula* yields

$$\int_a^b f(x)\,dx \approx h\left(\frac{y_0 + y_n}{2} + y_1 + y_2 + \ldots + y_{n-1}\right) \qquad (1)$$

with an absolute error of

$$R_n \leqslant \frac{h^2}{12}(b-a) \cdot M_2,$$

where $M_2 = \max|f''(x)|$ when $a \leqslant x \leqslant b$.

To attain the specified accuracy ε when evaluating the integral, the interval h is found from the inequality

$$h^2 \leqslant \frac{12\varepsilon}{(b-a)\,M_2}. \qquad (2)$$

That is, h must be of the order of $\sqrt{\varepsilon}$. The value of h obtained is rounded off to the smaller value so that

$$\frac{b-a}{h} = n$$

should be an integer; this is what gives us the number of partitions n. Having established h and n from (1), we compute the integral by taking the values of the integrand with one or two reserve decimal places.

2°. Simpson's formula (parabolic formula). If n is an even number, then in the notation of 1° *Simpson's formula*

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})\right] \qquad (3)$$

holds with an absolute error of

$$R_n \leqslant \frac{h^4}{180}(b-a)\,M_4, \tag{4}$$

where $M_4 = \max|f^{IV}(x)|$ when $a \leqslant x \leqslant b$.

To ensure the specified accuracy ε when evaluating the integral, the interval of calculations h is determined from the inequality

$$\frac{h^4}{180}(b-a)\,M_4 \leqslant \varepsilon. \tag{5}$$

That is, the interval h is of the order $\sqrt[4]{\varepsilon}$. The number h is rounded off to the smaller value so that $n = \frac{b-a}{n}$ is an even integer.

Remark. Since, generally speaking, it is difficult to determine the interval h and the number n associated with it from the inequalities (2) and (5), in practical work h is determined in the form of a rough estimate. Then, after the result is obtained, the number n is doubled; that is, h is halved. If the new result coincides with the earlier one to the number of decimal places that we retain, then the calculations are stopped, otherwise the procedure is repeated, etc.

For an approximate calculation of the absolute error R of Simpson's quadrature formula (3), use can also be made of the *Runge principle*, according to which

$$R = \frac{|\Sigma - \overline{\Sigma}|}{15},$$

where Σ and $\overline{\Sigma}$ are the results of calculations from formula (3) with interval h and $H = 2h$, respectively.

3160. Under the action of a variable force $\overline{F}$ directed along the x-axis, a material point is made to move along the x-axis from $x=0$ to $x=4$. Approximate the work A of a force $\overline{F}$ if a table is given of the values of its modulus F:

x	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
F	1.50	0.75	0.50	0.75	1.50	2.75	4.50	6.75	10.00

Carry out the calculations by the trapezoidal formula and by the Simpson formula.

3161. Approximate $\int_0^1 (3x^2 - 4x)\,dx$ by the trapezoidal formula putting $n = 10$. Evaluate this integral exactly and find the absolute and relative errors of the result. Establish the upper limit Δ of absolute error in calculating for $n = 10$, utilizing the error formula given in the text.

3162. Using the Simpson formula, calculate $\int\limits_0^1 \frac{x\,dx}{x+1}$ to four decimal places, taking $n=10$. Establish the upper limit Δ of absolute error, using the error formula given in the text.

Calculate the following definite integrals to two decimals:

3163. $\int\limits_0^1 \frac{dx}{1+x}$.

3164. $\int\limits_0^1 \frac{dx}{1+x^2}$.

3165. $\int\limits_0^1 \frac{dx}{1+x^3}$.

3166. $\int\limits_1^2 x \log x\,dx$.

3167. $\int\limits_1^2 \frac{\log x}{x}\,dx$.

3168. $\int\limits_0^2 \frac{\sin x}{x}\,dx$.

3169. $\int\limits_0^{\pi} \frac{\sin x}{x}\,dx$.

3170. $\int\limits_1^2 \frac{\cos x}{x}\,dx$.

3171. $\int\limits_0^{\frac{\pi}{2}} \frac{\cos x}{1+x}\,dx$.

3172. $\int\limits_0^1 e^{-x^2}\,dx$.

3173. Evaluate to two decimal places the improper integral $\int\limits_1^{\infty} \frac{dx}{1+x^2}$ by applying the substitution $x=\frac{1}{t}$. Verify the calculations by applying Simpson's formula to the integral $\int\limits_1^b \frac{dx}{1+x^2}$, where b is chosen so that $\int\limits_b^{+\infty} \frac{dx}{1+x^2} < \frac{1}{2}\cdot 10^{-2}$.

3174. A plane figure bounded by a half-wave of the sine curve $y=\sin x$ and the x-axis is in rotation about the x-axis. Using the Simpson formula, calculate the volume of the solid of rotation to two decimal places.

3175*. Using Simpson's formula, calculate to two decimal places the length of an arc of the ellipse $\frac{x^2}{1}+\frac{y^2}{(0.6222)^2}=1$ situated in the first quadrant.

Sec. 5. Numerical Integration of Ordinary Differential Equations

1°. A method of successive approximation (Picard's method). Let there be given a first-order differential equation

$$y'=f(x, y) \tag{1}$$

subject to the initial condition $y=y_0$ when $x=x_0$.

The solution $y(x)$ of (1), which satisfies the given initial condition, can, generally speaking, be represented in the form

$$y(x) = \lim_{i \to \infty} y_i(x) \tag{2}$$

where the *successive approximations* $y_i(x)$ are determined from the formulas

$$y_0(x) = y_0,$$
$$y_i(x) = y_0 + \int_{x_0}^{x} f(x, y_{i-1}(x))\, dx$$
$$(i = 0, 1, 2, \ldots).$$

If the right side $f(x, y)$ is defined and continuous in the neighbourhood

$$R\{|x - x_0| \leqslant a, \quad |y - y_0| \leqslant b\}$$

and satisfies, in this neighbourhood, the *Lipschitz condition*

$$|f(x, y_1) - f(x, y_2)| \leqslant L|y_1 - y_2|$$

(L is constant), then the process of successive approximation (2) definitely converges in the interval

$$|x - x_0| \leqslant h,$$

where $h = \min\limits_R \left(a, \dfrac{b}{M}\right)$ and $M = \max\limits_R |f(x, y)|$. And the error here is

$$R_n = |y(x) - y_n(x)| \leqslant ML^n \frac{|x - x_0|^{n+1}}{(n+1)!},$$

if

$$|x - x_0| \leqslant h.$$

The method of successive approximation *(Picard's method)* is also applicable, with slight modifications, to normal systems of differential equations. Differential equations of higher orders may be written in the form of systems of differential equations.

2°. The Runge-Kutta method. Let it be required, on a given interval $x_0 \leqslant x \leqslant X$, to find the solution $y(x)$ of (1) to a specified degree of accuracy ε.

To do this, we choose the interval of calculations $h = \dfrac{X - x_0}{n}$ by dividing the interval $[x_0, X]$ into n equal parts so that $h^4 < \varepsilon$. The partition points x_i are determined from the formula

$$x_i = x_0 + ih \quad (i = 0, 1, 2, \ldots, n).$$

By the *Runge-Kutta method*, the corresponding values $y_i = y(x_i)$ of the desired function are successively computed from the formulas

$$y_{i+1} = y_i + \Delta y_i,$$
$$\Delta y_i = \frac{1}{6}\left(k_1^{(i)} + 2k_2^{(i)} + 2k_3^{(i)} + k_4^{(i)}\right),$$

where

$$\begin{aligned} &i=0,\ 1,\ 2,\ \ldots,\ n \text{ and} \\ &k_1^{(i)}=f(x_i,\ y_i)\,h, \\ &k_2^{(i)}=f\left(x_i+\frac{h}{2},\ y_i+\frac{k_1^{(i)}}{2}\right)h, \\ &k_3^{(i)}=f\left(x_i+\frac{h}{2},\ y_i+\frac{k_2^{(i)}}{2}\right)h, \\ &k_4^{(i)}=f\left(x_i+h,\ y_i+k_3^{(i)}\right)h. \end{aligned} \tag{3}$$

To check the correct choice of the interval h it is advisable to verify the quantity

$$\theta=\left|\frac{k_2^{(i)}-k_3^{(i)}}{k_1^{(i)}-k_2^{(i)}}\right|.$$

The fraction θ should amount to a few hundredths, otherwise h has to be reduced.

The Runge-Kutta method is accurate to the order of h^4. A rough estimate of the error of the Runge-Kutta method on the given interval $[x_0, X]$ may be obtained by proceeding from the Runge principle:

$$R=\frac{|y_{2m}-\tilde{y}_m|}{15},$$

where $n=2m$, y_{2m} and $\tilde{y}_m$ are the results of calculations using the scheme (3) with interval h and interval $2h$.

The Runge-Kutta method is also applicable for solving systems of differential equations

$$y'=f(x,\ y,\ z),\quad z'=\varphi(x,\ y,\ z) \tag{4}$$

with given initial conditions $y=y_0$, $z=z_0$ when $x=x_0$.

3°. Milne's method. To solve (1) by the *Milne method*, subject to the initial conditions $y=y_0$ when $x=x_0$, we in some way find the successive values

$$y_1=y(x_1),\quad y_2=y(x_2),\quad y_3=y(x_3)$$

of the desired function $y(x)$ [for instance, one can expand the solution $y(x)$ in a series (Ch. IX, Sec. 17) or find these values by the method of successive approximation, or by using the Runge-Kutta method, and so forth]. The approximations $\bar{y}_i$ and $\bar{\bar{y}}_i$ for the following values of y_i ($i=4,\ 5,\ \ldots,\ n$) are successively found from the formulas

$$\left.\begin{aligned} \bar{y}_i&=y_{i-4}+\frac{4h}{3}(2f_{i-3}-f_{i-2}+2f_{i-1}), \\ \bar{\bar{y}}_i&=y_{i-2}+\frac{h}{3}(\bar{f}_i+4f_{i-1}+f_{i-2}), \end{aligned}\right\} \tag{5}$$

where $f_i=f(x_i,\ y_i)$ and $\bar{f}_i=f(x_i,\ \bar{y}_i)$. To check we calculate the quantity

$$\varepsilon_i=\frac{1}{29}|\bar{y}_i-\bar{\bar{y}}_i|. \tag{6}$$

If ε_i does not exceed the unit of the last decimal 10^{-m} retained in the answer for $y(x)$, then for y_i we take $\bar{\bar{y}}_i$ and calculate the next value y_{i+1}, repeating the process. But if $\varepsilon_i > 10^{-m}$, then one has to start from the beginning and reduce the interval of calculations. The magnitude of the initial interval is determined approximately from the inequality $h^4 < 10^{-m}$.

For the case of a solution of the system (4), the Milne formulas are written separately for the functions $y(x)$ and $z(x)$. The order of calculations remains the same.

Example 1. Given a differential equation $y' = y - x$ with the initial condition $y(0) = 1.5$. Calculate to two decimal places the value of the solution of this equation when the argument is $x = 1.5$. Carry out the calculations by a combined Runge-Kutta and Milne method.

Solution. We choose the initial interval h from the condition $h^4 < 0.01$. To avoid involved writing, let us take $h = 0.25$. Then the entire interval of integration from $x = 0$ to $x = 1.5$ is divided into six equal parts of length 0.25 by means of points x_i $(i = 0, 1, 2, 3, 4, 5, 6)$; we denote by y_i and y_i' the corresponding values of the solution y and the derivative y'.

We calculate the first three values of y (not counting the initial one) by the Runge-Kutta method [from formulas (3)]; the remaining three values — y_4, y_5, y_6 — we calculate by the Milne method [from formulas (5)]

The value of y_6 will obviously be the answer to the problem.

We carry out the calculations with two reserve decimals according to a definite scheme consisting of two sequential Tables 1 and 2. At the end of Table 2 we obtain the answer.

Calculating the value y_1. Here, $f(x, y) = -x + y$, $x_0 = 0$, $y_0 = 1.5$

$$h = 0.25. \quad \Delta y_0 = \frac{1}{6}\left(k_1^{(0)} + 2k_2^{(0)} + 2k_3^{(0)} + k_4^{(0)}\right) =$$

$$= \frac{1}{6}(0.3750 + 2\cdot 0.3906 + 2\cdot 0.3926 + 0.4106) = 0.3920;$$

$$k_1^{(0)} = f(x_0, y_0)\, h = (-0 + 1.5000)0.25 = 0.3750;$$

$$k_2^{(0)} = f\left(x_0 + \frac{h}{2}, \quad y_0 + \frac{k_1^{(0)}}{2}\right) h = (-0.125 + 1.5000 + 0.1875)\, 0.25 = 0.3906;$$

$$k_3^{(0)} = f\left(x_0 + \frac{h}{2}, \quad y_0 + \frac{k_2^{(0)}}{2}\right) h = (-0.125 + 1.5000 + 0.1953)\, 0.25 = 0.3926;$$

$$k_4^{(0)} = f(x_0 + h, \quad y_0 + k_3^{(0)})\, h = (-0.25 + 1.5000 + 0.3926)\, 0.25 = 0.4106;$$

$y_1 = y_0 + \Delta y_0 = 1.5000 + 0.3920 = 1.8920$ (the first three decimals in this approximate number are guaranteed).

Let us check:

$$\theta = \left|\frac{k_2^{(0)} - k_3^{(0)}}{k_1^{(0)} - k_2^{(0)}}\right| = \frac{|0.3906 - 0.3926|}{|0.3750 - 0.3906|} \approx \frac{20}{156} = 0.13.$$

By this criterion, the interval h that we chose was rather rough.

Similarly we calculate the values y_2 and y_3. The results are tabulated in Table 1.

Table 1. **Calculating y_1, y_2, y_3 by the Runge-Kutta Method.**
$f(x, y) = -x + y$; $h = 0.25$

Value of i	x_i	y_i	$y'_i = f(x_i, y_i)$	$k_1^{(i)}$	$f\left(x_i+\frac{h}{2}, y_i+\frac{k_1^{(i)}}{2}\right)$	$k_2^{(i)}$
0	0	1.5000	1.5000	0.3750	1.5625	0.3906
1	0.25	1.8920	1.6420	0.4105	1.7223	0.4306
2	0.50	2.3243	1.8243	0.4561	1.9273	0.4818
3	0 75	2.8084	2.0584	0.5146	2.1907	0.5477

Value of i	$f\left(x_i+\frac{h}{2}, y_i+\frac{k_2^{(i)}}{2}\right)$	$k_3^{(i)}$	$f(x_i+h, y_i+k_3^{(i)})$	$k_4^{(i)}$	Δy_i	y_{i+1}
0	1.5703	0.3926	1.6426	0.4106	0.3920	1.8920
1	1.7323	0.4331	1.8251	0.4562	0.4323	2.3243
2	1.9402	0.4850	2.0593	0.5148	0.4841	2.8084
3	2.2073	0.5518	2.3602	0.5900	0.5506	3.3590

Calculating the value of y_4. We have: $f(x, y) = -x + y$, $h = 0.25$, $x_4 = 1$;
$y_0 = 1.5000$, $y_1 = 1.8920$, $y_2 = 2.3243$, $y_3 = 2.8084$;
$y'_0 = 1.5000$, $y'_1 = 1.6420$, $y'_2 = 1.8243$, $y'_3 = 2.0584$.

Applying formulas (5), we find

$$\bar{y}_4 = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3) =$$

$$= 1.5000 + \frac{4 \cdot 0.25}{3}(2 \cdot 1.6420 - 1.8243 + 2 \cdot 2.0584) = 3.3588;$$

$$\bar{y}'_4 = f(x_4, \bar{y}_4) = -1 + 3.3588 = 2.3588;$$

$$\bar{\bar{y}}_4 = y_2 + \frac{h}{3}(\bar{y}'_4 + 4y'_3 + y'_2) = 2.3243 + \frac{0.25}{3}(2.3588 + 4 \cdot 2.0584 + 1.8243) = 3.3590;$$

$$\varepsilon_4 = \frac{|\bar{y}_4 - \bar{\bar{y}}_4|}{29} = \frac{|3.3588 - 3.3590|}{29} = \frac{0.0002}{29} \approx 7 \cdot 10^{-6} < \frac{1}{2} \cdot 0.001;$$

hence, there is no need to reconsider the interval of calculations.

We obtain $y_4 = \bar{\bar{y}}_4 = 3.3590$ (in this approximate number the first three decimals are guaranteed).

Similarly we calculate the values of y_5 and y_6. The results are given in Table 2.

Thus, we finally have

$$y(1.5) = 4.74.$$

4°. Adams' method. To solve (1) by the Adams method on the basis of the initial data $y(x_0) = y_0$ we in some way find the following three values of the desired function $y(x)$:

$$y_1 = y(x_1) = y(x_0 + h),\quad y_2 = y(x_2) = y(x_0 + 2h),\quad y_3 = y(x_3) = y(x_0 + 3h)$$

[these three values may be obtained, for instance, by expanding $y(x)$ in a power series (Ch IX, Sec. 16), or they may be found by the method of successive approximation (1°), or by applying the Runge-Kutta method (2°) and so forth].

With the help of the numbers x_0, x_1, x_2, x_3 and y_0, y_1, y_2, y_3 we calculate q_0, q_1, q_2, q_3, where

$$q_0 = hy_0' = hf(x_0, y_0),\quad q_1 = hy_1' = hf(x_1, y_1),$$
$$q_2 = hy_2' = hf(x_2, y_2),\quad q_3 = hy_3' = hf(x_3, y_3).$$

We then form a *diagonal table* of the finite differences of q:

x	y	$\Delta y = y_{n+1} - y_n$	$y' = f(x, y)$	$q = y'h$	$\Delta q = q_{n+1} - q_n$	$\Delta^2 q = \Delta q_{n+1} - \Delta q_n$	$\Delta^3 q = \Delta^2 q_{n+1} - \Delta^2 q_n$
x_0	y_0	Δy_0	$f(x_0, y_0)$	q_0	Δq_0	$\Delta^2 q_0$	$\Delta^3 q_0$
x_1	y_1	Δy_1	$f(x_1, y_1)$	q_1	Δq_1	$\Delta^2 q_1$	$\Delta^3 q_1$
x_2	y_2	Δy_2	$f(x_2, y_2)$	q_2	Δq_2	$\Delta^2 q_2$	$\Delta^3 q_2$
x_3	y_3	Δy_3	$f(x_3, y_3)$	q_3	Δq_3	$\Delta^2 q_3$	
x_4	y_4	Δy_4	$f(x_4, y_4)$	q_4	Δq_4		
x_5	y_5	Δy_5	$f(x_5, y_5)$	q_5			
x_6	y_6						

The *Adams method* consists in continuing the diagonal table of differences with the aid of the *Adams formula*

$$\Delta y_n = q_n + \frac{1}{2}\Delta q_{n-1} + \frac{5}{12}\Delta^2 q_{n-2} + \frac{3}{8}\Delta^3 q_{n-3}. \qquad (7)$$

Thus, utilizing the numbers q_3, Δq_2, $\Delta^2 q_1$, $\Delta^3 q_0$ situated diagonally in the difference table, we calculate, by means of formula (7) and putting $n=3$ in it, $\Delta y_3 = q_3 + \frac{1}{2}\Delta q_2 + \frac{5}{12}\Delta^2 q_1 + \frac{3}{8}\Delta^3 q_0$. After finding Δy_3, we calculate $y_4 = y_3 + \Delta y_3$. And when we know x_4 and y_4, we calculate $q_4 = hf(x_4, y_4)$, introduce y_4, Δy_3 and q_4 into the difference table and then fill into it the finite differences Δq_3, $\Delta^2 q_2$, $\Delta^3 q_1$, which are situated (together with q_4) along a new diagonal parallel to the first one.

Then, utilizing the numbers of the new diagonal, we use formula (8) (putting $n=4$ in it) to calculate Δy_4, y_5 and q_5 and obtain the next diagonal: q_5, Δq_4, $\Delta^2 q_3$, $\Delta^3 q_2$. Using this diagonal we calculate the value of y_6 of the desired solution $y(x)$, and so forth.

The Adams formula (7) for calculating Δy proceeds from the assumption that the third finite differences $\Delta^3 q$ are constant. Accordingly, the quantity h of the initial interval of calculations is determined from the inequality $h^4 < 10^{-m}$ [if we wish to obtain the value of $y(x)$ to an accuracy of 10^{-m}].

In this sense the Adams formula (7) is equivalent to the formulas of Milne (5) and Runge-Kutta (3).

Evaluation of the error for the Adams method is complicated and for practical purposes is useless, since in the general case it yields results with considerable excess. In actual practice, we follow the course of the third finite differences, choosing the interval h so small that the adjacent differences $\Delta^3 q_i$ and $\Delta^3 q_{i+1}$ differ by not more than one or two units of the given decimal place (not counting reserve desimals).

To increase the accuracy of the result, Adams' formula may be extended by terms containing fourth and higher differences of q, in which case there is an increase in the number of first values of the function y that are needed when we first fill in the table. We shall not here give the Adams formula for higher accuracy.

Example 2. Using the combined Runge-Kutta and Adams method, calculate to two decimal places (when $x=1.5$) the value of the solution of the differential equation $y' = y - x$ with the initial condition $y(0)=1.5$ (see Example 1).

Solution. We use the values y_1, y_2, y_3 that we obtained in the solution of Example 1. Their calculation is given in Table 1.

We calculate the subsequent values y_4, y_5, y_6 by the Adams method (see Tables 3 and 4).

The answer to the problem is $y_6 = 4.74$.

For solving system (4), the Adams formula (7) and the calculation scheme shown in Table 3 are applied separately for both functions $y(x)$ and $z(x)$.

Find three successive approximations to the solutions of the differential equations and systems indicated below.

3176. $y' = x^2 + y^2$; $y(0) = 0$.

3177. $y' = x + y + z$, $z' = y - z$; $y(0) = 1$, $z(0) = -2$.

3178. $y'' = -y$; $y(0) = 0$, $y'(0) = 1$.

Table 2. **Calculating y_4, y_5, y_6 by the Milne Method.**
$f(x, y) = -x + h$; $h = 0.25$. (Italicised figures are input data)

Value of i	x_i	y_i	$y'_i = f(x_i, y_i)$	$\overline{y_i}$	$\overline{y'_i} = f(x_i, \overline{y_i})$	$\overline{\overline{y_i}}$	ε_i	y_i	$y'_i = f(x_i, y_i)$	Reconsider interval of calculations, following indications of formula (6).
0	*0*	*1.5000*	*1.5000*							
1	*0.25*	*1.8920*	*1.6420*							
2	*0.50*	*2.3243*	*1.8243*							
3	*0.75*	*2.8084*	*2.0584*							
4	1.00			3.3588	2.3588	3.3590	$\approx 7 \cdot 10^{-5}$	3.3590	2.3590	Do not reconsider
5	1.25			3.9947	2.7447	3.9950	$\approx 10^{-5}$	3.9950	2.7450	Do not reconsider
6	1.50			4.7402	3.2402	4.7406	$\approx 1.4 \cdot 10^{-5}$	4.7406		Do not reconsider
							Answer:	$y(1.5) = 4.74$		

Table 3. **Basic Table for Calculating y_4, y_5, y_6 by the Adams Method.**

$f(x, y) = -x + y;\ h = 0.25$

(Italicised figures are input data)

Value of i	x_i	y_i	Δy_i	$y'_i = f(x_i, y_i)$	$q_i = y'_i h$	Δq_i	$\Delta^2 q_i$	$\Delta^3 q_i$
0	*0*	*1.5000*		*1.5000*	0.3750	0 0355	0.0101	0 0028
1	*0.25*	*1.8920*		*1.6420*	0.4105	0.0456	0.0129	0 0037
2	*0 50*	*2.3243*		*1.8243*	0.4561	0.0585	0.0166	0.0047
3	*0.75*	*2.8084*	0.5504	2.0584	0.5146	0.0751	0.0213	
4	1 00	3.3588	0.6356	2.3588	0.5897	0.0964		
5	1 25	3.9944	0.7450	2.7444	0.6861			
6	1.50	4.7394						

Answer: 4.74

Table 4 **Auxiliary Table for Calculating by the Adams Method**

$$\Delta y_i = q_i + \frac{1}{2}\Delta q_{i-1} + \frac{5}{12}\Delta^2 q_{i-2} + \frac{3}{8}\Delta^3 q_{i-3}$$

Value of i	q_i	$\frac{1}{2}\Delta q_{i-1}$	$\frac{5}{12}\Delta^2 q_{i-2}$	$\frac{3}{8}\Delta^3 q_{i-3}$	Δy_i
3	0.5146	0.0293	0.0054	0.0011	0.5504
4	0 5897	0.0376	0.0069	0.0014	0.6356
5	0.6861	0.0482	0.0089	0.0018	0.7450

Putting the interval $h=0.2$, use the Runge-Kutta method to calculate approximately the solutions of the given differential equations and systems for the indicated intervals:

3179. $y'=y-x$; $y(0)=1.5$ $(0 \leqslant x \leqslant 1)$.

3180. $y'=\frac{y}{x}-y^2$; $y(1)=1$ $(1 \leqslant x \leqslant 2)$.

3181. $y'=z+1$, $z'=y-x$, $y(0)=1$, $z(0)=1$ $(0 \leqslant x \leqslant 1)$.

Applying a combined Runge-Kutta and Milne method or Runge-Kutta and Adams method, calculate to two decimal places the solutions to the differential equations and systems indicated below for the indicated values of the argument;

3182. $y'=x+y$; $y=1$ when $x=0$. Compute y when $x=0.5$.

3183. $y'=x^2+y$; $y=1$ when $x=0$. Compute y when $x=1$.

3184. $y'=2y-3$; $y=1$ when $x=0$. Compute y when $x=0.5$.

3185. $\begin{cases} y'=-x+2y+z, \\ z'=x+2y+3z; \end{cases}$ $y=2$, $z=-2$ when $x=0$.

Compute y and z when $x=0.5$.

3186. $\begin{cases} y'=-3y-z, \\ z'=y-z; \end{cases}$ $y=2$, $z=-1$ when $x=0$.

Compute y and z when $x=0.5$.

3187. $y''=2-y$; $y=2$, $y'=-1$ when $x=0$.

Compute y when $x=1$.

3188. $y^3y''+1=0$; $y=1$, $y'=0$ when $x=1$.

Compute y when $x=1.5$.

3189. $\frac{d^2x}{dt^2}+\frac{x}{2}\cos 2t=0$; $x=0$, $x'=1$ when $t=0$.

Find $x(\pi)$ and $x'(\pi)$.

Sec. 6. Approximating Fourier Coefficients

Twelve-ordinate scheme. Let $y_n=f(x_n)$ $(n=0, 1, \ldots, 12)$ be the values of the function $y=f(x)$ at equidistant points $x_n=\frac{\pi n}{6}$ of the interval $[0,2\pi]$, and $y_0=y_{12}$. We set up the tables:

	y_0	y_1	y_2	y_3	y_4	y_5	y_6
		y_{11}	y_{10}	y_9	y_8	y_7	
Sums ($\sum$)	u_0	u_1	u_2	u_3	u_4	u_5	u_6
Differences (Δ)		v_1	v_2	v_3	v_4	v_5	

	u_0	u_1	u_2	u_3
	u_6	u_5	u_4	
Sums	s_0	s_1	s_2	s_3
Differences	t_0	t_1	t_2	

	v_1	v_2	v_3
	v_5	v_4	
Sums	σ_1	σ_2	σ_3
Differences	τ_1	τ_2	

The Fourier coefficients a_n, b_n ($n=0$, 1, 2, 3) of the function $y=f(x)$ may be determined approximately from the formulas:

$$\begin{aligned} 6a_0 &= s_0+s_1+s_2+s_3, & 6b_1 &= 0.5\sigma_1+0.866\sigma_2+\sigma_3, \\ 6a_1 &= t_0+0.866t_1+0.5t_2, & 6b_2 &= 0.866(\tau_1+\tau_2), \\ 6a_2 &= s_0-s_3+0.5(s_1-s_2), & 6b_3 &= \sigma_1-\sigma_3, \\ 6a_3 &= t_0-t_2, \end{aligned} \qquad (1)$$

where $0.866=\frac{\sqrt{3}}{2}\approx 1-\frac{1}{10}-\frac{1}{30}$.

We have

$$f(x)\approx\frac{a_0}{2}+\sum_{n=1}^{3}(a_n\cos nx+b_n\sin nx).$$

Other schemes are also used. Calculations are simplified by the use of *patterns*.

Example. Find the Fourier polynomial for the function $y=f(x)$ $(0\leqslant x\leqslant 2\pi)$ represented by the table

y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}
38	38	12	4	14	4	−18	−23	−27	−24	8	32

Solution. We set up the tables:

y	38	38	12	4	14	4	−18
		32	8	−24	−27	−23	
u	38	70	20	−20	−13	−19	−18
v		6	4	28	41	27	

u	38	70	20	−20
	−18	−19	−13	
s	20	51	7	−20
t	56	89	33	

v	6	4	28
	27	41	
σ	33	45	28
τ	−21	−37	

From formulas (1) we have

$$a_0=9.7;\ a_1=24.9;\ a_2=10.3;\ a_3=3.8;$$
$$b_1=13.9;\ b_2=-8.4;\ b_3=0.8.$$

Consequently,

$$f(x)\approx 4.8+(24.9\cos x+13.9\sin x)+(10.3\cos 2x-8.4\sin 2x)+ \\ +(3.8\cos 3x+0.8\sin 3x).$$

Using the 12-ordinate scheme, find the Fourier polynomials for the following functions defined in the interval $(0,2\pi)$ by the tables of their values that correspond to the equidistant values of the argument.

3190.	$y_0 = -7200$	$y_3 = 4300$	$y_6 = 7400$	$y_9 = 7600$
	$y_1 = 300$	$y_4 = 0$	$y_7 = -2250$	$y_{10} = 4500$
	$y_2 = 700$	$y_5 = -5200$	$y_8 = 3850$	$y_{11} = 250$
3191.	$y_0 = 0$	$y_3 = 9.72$	$y_6 = 7.42$	$y_9 = 5.60$
	$y_1 = 6.68$	$y_4 = 8.97$	$y_7 = 6.81$	$y_{10} = 4.88$
	$y_2 = 9.68$	$y_5 = 8.18$	$y_8 = 6.22$	$y_{11} = 3.67$
3192.	$y_0 = 2.714$	$y_3 = 1.273$	$y_6 = 0.370$	$y_9 = -0.357$
	$y_1 = 3.042$	$y_4 = 0.788$	$y_7 = 0.540$	$y_{10} = -0.437$
	$y_2 = 2.134$	$y_5 = 0.495$	$y_8 = 0.191$	$y_{11} = 0.767$

3193. Using the 12-ordinate scheme, evaluate the first several Fourier coefficients for the following functions:

a) $f(x) = \frac{1}{2\pi^2}(x^3 - 3\pi x^2 + 2\pi^2 x) \quad (0 \leqslant x \leqslant 2\pi)$,

b) $f(x) = \frac{1}{\pi^2}(x - \pi)^2 \quad (0 \leqslant x \leqslant 2\pi)$.

ANSWERS

Chapter I

1. Solution. Since $a=(a-b)+b$, then $|a|\leqslant|a-b|+|b|$. Whence $|a-b|\geqslant|a|-|b|$ and $|a-b|=|b-a|\geqslant|b|-|a|$. Hence, $|a-b|\geqslant|a|-|b|$. Besides, $|a-b|=|a+(-b)|\leqslant|a|+|-b|=|a|+|b|$. **3.** a) $-2<x<4$; b) $x<-3$, $x>1$; c) $-1<x<0$; d) $x>0$. **4.** -24; -6; 0; 0; 6. **5.** 1; $1\frac{1}{4}$; $\sqrt{1+x^2}$; $|x|^{-1}\sqrt{1+x^2}$; $1/\sqrt{1+x^2}$. **6.** π; $\frac{\pi}{2}$; 0. **7.** $f(x)=-\frac{5}{3}x+\frac{1}{3}$. **8.** $f(x)=\frac{7}{6}x^2-\frac{13}{6}x+1$. **9.** 0.4. **10.** $\frac{1}{2}(x+|x|)$. **11.** a) $-1\leqslant x<+\infty$; b) $-\infty<x<+\infty$. **12.** $(-\infty,-2)$, $(-2, 2)$, $(2, +\infty)$. **13.** a) $-\infty<x\leqslant-\sqrt{2}$, $\sqrt{2}\leqslant x<+\infty$; b) $x=0$, $|x|\geqslant\sqrt{2}$. **14.** $-1\leqslant x\leqslant 2$. **Solution.** It should be $2+x-x^2\geqslant 0$, or $x^2-x-2\leqslant 0$; that is, $(x+1)(x-2)\leqslant 0$. Whence either $x+1\geqslant 0$, $x-2\leqslant 0$, i. e., $-1\leqslant x\leqslant 2$ or $x+1\leqslant 0$, $x-2\geqslant 0$, i. e., $x\leqslant-1$, $x\geqslant 2$, but this is impossible. Thus, $-1\leqslant x\leqslant 2$. **15.** $-2<x\leqslant 0$. **16.** $-\infty<x\leqslant-1$, $0\leqslant x\leqslant 1$. **17.** $-2<x<2$. **18.** $-1<x<1$, $2<x<+\infty$. **19.** $-\frac{1}{3}\leqslant x\leqslant 1$. **20.** $1\leqslant x\leqslant 100$. **21.** $k\pi\leqslant x\leqslant k\pi+\frac{\pi}{2}$ $(k=0, \pm 1, \pm 2, \ldots)$. **22.** $\varphi(x)=2x^4-5x^2-10$, $\psi(x)=-3x^3+6x$. **23.** a) Even, b) odd, c) even, d) odd, e) odd. **24. Hint.** Utilize the identity $f(x)=\frac{1}{2}[f(x)+f(-x)]+\frac{1}{2}[f(x)-f(-x)]$. **26.** a) Periodic, $T=\frac{2}{3}\pi$, b) periodic, $T=\frac{2\pi}{\lambda}$, c) periodic, $T=\pi$, d) periodic $T=\pi$, e) nonperiodic. **27.** $y=\frac{b}{c}x$, if $0\leqslant x\leqslant c$; $y=b$ if $c<x\leqslant a$; $S=\frac{b}{2c}x^2$ if $0\leqslant x\leqslant c$; $S=bx-\frac{bc}{2}$ if $c<x\leqslant a$. **28.** $m=q_1x$ when $0\leqslant x\leqslant l_1$; $m=q_1l_1+q_2(x-l_1)$ when $l_1<x\leqslant l_1+l_2$; $m=q_1l_1+q_2l_2+q_3(x-l_1-l_2)$ when $l_1+l_2<x\leqslant l_1+l_2+l_3=l$. **29.** $\varphi[\psi(x)]=2^{2x}$; $\psi[\varphi(x)]=2^{x^2}$ **30.** x. **31.** $(x+2)^2$. **37.** $-\frac{\pi}{2}$; 0; $\frac{\pi}{4}$. **38.** a) $y=0$ when $x=-1$, $y>0$ when $x>-1$, $y<0$ when $x<-1$; b) $y=0$ when $x=-1$ and $x=2$, $y>0$ when $-1<x<2$, $y<0$ when $-\infty<x<-1$ and $2<x<+\infty$; c) $y>0$ when $-\infty<x<+\infty$; d) $y=0$ when $x=0$, $x=-\sqrt{3}$ and $x=\sqrt{3}$, $y>0$ when $-\sqrt{3}<x<0$ and $\sqrt{3}<x<+\infty$, $y<0$ when $-\infty<x<-\sqrt{3}$ and $0<x<\sqrt{3}$; e) $y=0$ when $x=1$, $y>0$ when $-\infty<x<-1$ and $1<x<+\infty$, $y<0$ when $0<x<1$ **39.** a) $x=\frac{1}{2}(y-3)$ $(-\infty<y<+\infty)$; b) $x=\sqrt{y+1}$ and $x=-\sqrt{y+1}$ $(-1\leqslant y<+\infty)$;

c) $x=\sqrt[3]{1-y^3}$ $(-\infty<y<+\infty)$; d) $x=2\cdot 10^y$ $(-\infty<y<+\infty)$; e) $x=\frac{1}{3}\tan y\left(-\frac{\pi}{2}<y<\frac{\pi}{2}\right)$. **40.** $x=y$ when $-\infty<y\leqslant 0$; $x=\sqrt{y}$ when $0<y<+\infty$. **41.** a) $y=u^{10}$, $u=2x-5$; b) $y=2^u$, $u=\cos x$; c) $y=\log u$, $u=\tan v$, $v=\frac{x}{2}$; d) $y=\arcsin u$, $u=3^v$, $v=-x^2$. **42.** a) $y=\sin^2 x$; b) $y=\arctan\sqrt{\log x}$; c) $y=2(x^2-1)$ if $|x|\leqslant 1$, and $y=0$ if $|x|>1$. **43.** a) $y=-\cos x^2$, $\sqrt{\pi}\leqslant|x|\leqslant\sqrt{2\pi}$; b) $y=\log(10-10^x)$, $-\infty<x<1$; c) $y=\frac{x}{3}$ when $-\infty<x<0$ and $y=x$ when $0\leqslant x<+\infty$. **46. Hint.** See Appendix VI, Fig. 1. **51. Hint.** Completing the square in the quadratic trinomial we will have $y=y_0+a(x-x_0)^2$ where $x_0=-b/2a$ and $y_0=(4ac-b^2)/4a$. Whence the desired graph is a parabola $y=ax^2$ displaced along the x-axis by x_0 and along the y-axis by y_0. **53. Hint.** See Appendix VI, Fig. 2. **58. Hint.** See Appendix VI, Fig. 3. **61. Hint.** The graph is a hyperbola $y=\frac{m}{x}$, shifted along the x-axis by x_0 and along the y-axis by y_0. **62. Hint.** Taking the integral part, we have $y=\frac{2}{3}-\frac{13}{9}\Big/\left(x+\frac{2}{3}\right)$ (Cf. 61*). **65. Hint.** See Appendix VI, Fig. 4. **67. Hint.** See Appendix VI, Fig. 5. **71. Hint.** See Appendix VI, Fig. 6. **72. Hint.** See Appendix VI, Fig. 7. **73. Hint.** See Appendix VI, Fig. 8. **75. Hint.** See Appendix VI, Fig. 19 **78. Hint.** See Appendix VI, Fig. 23. **80. Hint.** See Appendix VI, Fig. 9. **81. Hint.** See Appendix VI, Fig. 9. **82. Hint.** See Appendix VI, Fig. 10 **83. Hint.** See Appendix VI, Fig. 10. **84. Hint.** See Appendix VI, Fig 11. **85. Hint.** See Appendix VI, Fig. 11. **87. Hint.** The period of the function is $T=2\pi/n$. **89. Hint.** The desired graph is the sine curve $y=5\sin 2x$ with amplitude 5 and period π displaced rightwards along the x-axis by the quantity $1\frac{1}{2}$. **90. Hint.** Putting $a=A\cos\varphi$ and $b=-A\sin\varphi$, we will have $y=A\sin(x-\varphi)$ where $A=\sqrt{a^2+b^2}$ and $\varphi=\arctan\left(-\frac{b}{a}\right)$. In our case, $A=10$, $\varphi=0.927$. **92. Hint.** $\cos^2 x=\frac{1}{2}(1+\cos 2x)$. **93. Hint.** The desired graph is the sum of the graphs $y_1=x$ and $y_2=\sin x$. **94. Hint.** The desired graph is the product of the graphs $y_1=x$ and $y_2=\sin x$. **99. Hint.** The function is even For $x>0$ we determine the points at which 1) $y=0$; 2) $y=1$; and 3) $y=-1$. When $x\to+\infty$, $y\to 1$. **101. Hint.** See Appendix VI, Fig. 14. **102. Hint.** See Appendix VI, Fig. 15. **103. Hint.** See Appendix VI, Fig. 17. **104. Hint.** See Appendix VI, Fig. 17. **105. Hint.** See Appendix VI, Fig. 18. **107. Hint.** See Appendix VI, Fig. 18. **118. Hint.** See Appendix VI, Fig. 12. **119. Hint.** See Appendix VI, Fig. 12. **120. Hint.** See Appendix VI, Fig. 13. **121. Hint.** See Appendix VI, Fig. 13. **132. Hint.** See Appendix VI, Fig. 30. **133. Hint.** See Appendix VI, Fig. 32. **134. Hint.** See Appendix VI, Fig. 31. **138. Hint.** See Appendix VI, Fig. 33. **139. Hint.** See Appendix VI, Fig. 28. **140. Hint.** See Appendix VI, Fig. 25. **141. Hint.**

Form a table of values:

t	0	1	2	3	...	-1	-2	-3
x	0	1	8	27	..	-1	-8	-27
y	0	1	4	9	..	1	4	9

Constructing the points (x, y) obtained, we get the desired curve (see Appendix VI, Fig. 7). (Here, the parameter t cannot be laid off geometrically!) **142.** See Appendix VI, Fig. 19. **143.** See Appendix VI, Fig. 27. **144.** See Appendix VI, Fig. 29. **145.** See Appendix VI, Fig. 22 **150.** See Appendix VI, Fig. 28. **151. Hint.** Solving the equation for y, we get $y=\pm\sqrt{25-x^2}$. It is now easy to construct the desired curve from the points. **153.** See Appendix VI, Fig. 21. **156.** See Appendix VI, Fig. 27. It is sufficient to construct the points (x, y) corresponding to the abscissas $x=0,\ \pm\frac{a}{2},\ \pm a$. **157. Hint.** Solving the equation for x, we have $x=10\log y-y$ $^{(*)}$. Whence we get the points (x, y) of the sought-for curve, assigning to the ordinate y arbitrary values $(y>0)$ and calculating the abscissa x from the formula $^{(*)}$ Bear in mind that $\log y\to-\infty$ as $y\to 0$. **159. Hint.** Passing to polar coordinates $r=\sqrt{x^2+y^2}$ and $\tan\varphi=\frac{y}{x}$, we will have $r=e^{\varphi}$ (see Appendix VI, Fig 32) **160. Hint.** Passing to polar coordinates $x=r\cos\varphi$, and $y=r\sin\varphi$, we will have $r=\frac{3\sin\varphi\cos\varphi}{\cos^3\varphi+\sin^3\varphi}$ (see Appendix VI, Fig. 32) **161.** $F=32+1,\ 8C$ **162.** $y=0.6x(10-x)$; $y_{\max}=15$ when $x=5$. **163.** $y=\frac{ab}{2}\sin x$; $y_{\max}=\frac{ab}{2}$ when $x=\frac{\pi}{2}$. **164.** a) $x_1=\frac{1}{2}$, $x_2=2$; b) $x=0.68$; c) $x_1=1.37$, $x_2=10$; d) $x=0.40$; e) $x=1.50$; f) $x=0.86$. **165.** a) $x_1=2$, $y_1=5$; $x_2=5$, $y_2=2$; b) $x_1=-3$, $y_1=-2$; $x_2=-2$, $y_2=-3$; $x_3=2, y_3=3; x_4=3$, $y_4=2$; c) $x_1=2$, $y_1=2$; $x_2\approx 3.1$, $y_2\approx -2.5$; d) $x_1\approx -3.6$, $y_1\approx -3.1$; $x_2\approx -2.7$, $y_2\approx 2.9$; $x_3\approx 2.9$, $y_3\approx 1.8$; $x_4\approx 3.4$, $y_4\approx -1.6$; e) $x_1=\frac{\pi}{4}$, $y_1=\frac{\sqrt{2}}{2}$; $x_2=\frac{5\pi}{4}$, $y_2=-\frac{\sqrt{2}}{2}$. **166.** $n>\frac{1}{\sqrt{\varepsilon}}$. a) $n\geqslant 4$; b) $n>10$; c) $n\geqslant 32$. **167.** $n>\frac{1}{\varepsilon}-1=N$. a) $N=9$; b) $N=99$; c) $N=999$. **168.** $\delta=\frac{\varepsilon}{5}$ $(\varepsilon<1)$. a) 0.02; b) 0.002; c) 0.0002. **169.** a) $\log x<-N$ when $0<x<\delta(N)$; b) $2^x>N$ when $x>X(N)$; c) $|f(x)|>N$ when $|x|>X(N)$. **170.** a) 0; b) 1; c) 2; d) $\frac{7}{30}$. **171.** $\frac{1}{2}$. **172.** 1. **173.** $-\frac{3}{2}$. **174.** 1. **175.** 3. **176.** 1. **177.** $\frac{3}{4}$. **178.** $\frac{1}{3}$. **Hint.** Use the formula $1^2+2^2+\ldots+n^2=\frac{1}{6}n(n+1)(2n+1)$. **179.** 0. **180.** 0. **181.** 1. **182.** 0. **183.** ∞. **184.** 0. **185.** 72. **186.** 2. **187.** 2. **188.** ∞. **189.** 0. **190.** 1. **191.** 0. **192.** ∞. **193.** -2. **194.** ∞. **195.** $\frac{1}{2}$ **196.** $\frac{a-1}{3a^2}$. **197.** $3x^2$. **198.** -1. **199.** $\frac{1}{2}$.

200. 3. 201. $\frac{4}{3}$. 202. $\frac{1}{9}$. 203. $-\frac{1}{56}$. 204. 12. 205. $\frac{3}{2}$. 206. $-\frac{1}{3}$. 207. 1. 208. $\frac{1}{2\sqrt{x}}$. 209. $\frac{1}{3\sqrt[3]{x^2}}$. 210. $-\frac{1}{3}$. 211. 0. 212. $\frac{a}{2}$. 213. $-\frac{5}{2}$. 214. $\frac{1}{2}$. 215. 0. 216. a) $\frac{1}{2}\sin 2$; b) 0. 217. 3. 218. $\frac{5}{2}$. 219. $\frac{1}{3}$. 220. π. 221. $\frac{1}{2}$. 222. $\cos\alpha$. 223. $-\sin\alpha$. 224. π. 225. $\cos x$. 226. $-\frac{1}{\sqrt{2}}$. 227. a) 0; b) 1. 228. $\frac{2}{\pi}$. 229. $\frac{1}{2}$. 230. 0. 231. $-\frac{1}{\sqrt{3}}$. 232. $\frac{1}{2}(n^2-m^2)$. 233. $\frac{1}{2}$. 234. 1. 235. $\frac{2}{3}$. 236. $\frac{2}{\pi}$. 237. $-\frac{1}{4}$. 238. π. 239. $\frac{1}{4}$. 240. 1. 241. 1. 242. $\frac{1}{4}$. 243. 0 244. $\frac{3}{2}$. 245. 0. 246. e^{-1}. 247. e^2. 248. e^{-1}. 249. e^{-4}. 250. e^x. 251. e. 252. a) 1. **Solution.** $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x}} = \lim\limits_{x\to 0}[1-(1-\cos x)]^{\frac{1}{x}} =$

$$=\lim_{x\to 0}\left(1-2\sin^2\frac{x}{2}\right)^{\frac{1}{x}}=\lim_{x\to 0}\left[\left(1-2\sin^2\frac{x}{2}\right)^{-\frac{1}{2\sin^2\frac{x}{2}}}\right]^{-\frac{2\sin^2\frac{x}{2}}{x}}=$$

$$=e^{\lim\limits_{x\to 0}\left(-\frac{2\sin^2\frac{x}{2}}{x}\right)}.$$

Since $\lim\limits_{x\to 0}\left(-\frac{2\sin^2\frac{x}{2}}{x}\right)=-2\lim\limits_{x\to 0}\left[\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\frac{x^2}{4x}\right]=-2\cdot 1\cdot\lim\limits_{x\to 0}\frac{x}{4}=0$, it follows that $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x}}=e^0=1$. b) $\frac{1}{\sqrt{e}}$. **Solution.** As in the preceding case (see a), $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}}=e^{\lim\limits_{x\to 0}\left(\frac{-2\sin^2\frac{x}{2}}{x^2}\right)}$. Since $\lim\limits_{x\to 0}\left(\frac{-2\sin^2\frac{x}{2}}{x^2}\right)=$

$=-2\lim\limits_{x\to 0}\left[\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\frac{x^2}{4x^2}\right]=-\frac{1}{2}$, it follows that $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}}=e^{-\frac{1}{2}}=$ $=\frac{1}{\sqrt{e}}$. 253. $\ln 2$. 254. $10\log e$. 255. 1. 256. 1. 257. $-\frac{1}{2}$. 258. 1. **Hint.** Put $e^x-1=\alpha$, where $\alpha\to 0$. 259. $\ln a$. **Hint.** Utilize the identity $a=e^{\ln a}$. 260. $\ln a$ **Hint.** Put $\frac{1}{n}=\alpha$, where $\alpha\to 0$ (see Example 259) 261. $a-b$. 262. 1. 263. a) 1; b) $\frac{1}{2}$. 264. a) -1; b) 1. 265. a) -1; b) 1. 266. a) 1; b) 0. 267. a) 0; b) 1. 268. a) -1; b) 1. 269. a) -1; b) 1. 270. a) $-\infty$; b) $+\infty$.

271. Solution. If $x \neq k\pi$ $(k=0, \pm 1, \pm 2, \ldots)$, then $\cos^2 x < 1$ and $y=0$; but if $x=k\pi$, then $\cos^2 x=1$ and $y=1$. **272.** $y=x$ when $0 \leqslant x < 1$; $y=\frac{1}{2}$ when $x=1$; $y=0$ when $x>1$ **273.** $y=|x|$. **274.** $y=-\frac{\pi}{2}$ when $x<0$; $y=0$ when $x=0$; $y=\frac{\pi}{2}$ when $x>0$. **275.** $y=1$ when $0 \leqslant x \leqslant 1$; $y=x$ when $1<x<+\infty$. **276.** $\frac{61}{450}$. **277.** $x_1 \to -\frac{c}{b}$; $x_2 \to \infty$. **278.** π. **279.** $2\pi R$. **280** $\frac{e}{e-1}$. **281.** $1\frac{1}{3}$. **282.** $\frac{\sqrt{e^{\pi}+1}}{e^{\frac{\pi}{2}}-1}$. **284.** $\lim\limits_{n\to\infty} AC_n=\frac{l}{3}$. **285.** $\frac{ab}{2}$. **286.** $k=1$, $b=0$; the straight line $y=x$ is the asymptote of the curve $y=\frac{x^3+1}{x^2+1}$. **287.** $Q_t^{(n)}=Q_0\left(1+\frac{kt}{n}\right)^n$, where k is the proportionality factor (law of compound interest); $Q_t=Q_0e^{kt}$. **288.** $|x|>\frac{1}{\varepsilon}$, a) $|x|>10$; b) $|x|>100$; c) $|x|>1000$. **289.** $|x-1|<\frac{\varepsilon}{2}$ when $0<\varepsilon<1$; a) $|x-1|<0.05$; b) $|x-1|<0.005$; c) $|x-1|<0.0005$ **290.** $|x-2|<\frac{1}{N}=\delta$; a) $\delta=0.1$; b) $\delta=0.01$; c) $\delta=0.001$. **291.** a) Second, b) third. $\frac{1}{2}$, $\frac{3}{2}$. **292.** a) 1; b) 2; c) 3. **293** a) 1; b) $\frac{1}{4}$; c) $\frac{2}{3}$; d) 2; e) 3. **295.** No **296.** 15. **297.** -1. **298.** -1. **299.** 3. **300.** a) 1.03 (1.0296); b) 0.985 (0.9849); c) 3.167 (3.1623) **Hint.** $\sqrt{10}=\sqrt{9+1}=3\sqrt{1+\frac{1}{9}}$; d) 10.954 (10.954). **301.** 1) 0.98 (0.9804); 2) 1.03 (1.0309); 3) 0.0095 (0.00952); 4) 3.875 (3.8730); 5) 1.12 (1.125); 6) 0.72 (0.7480); 7) 0.043 (0.04139). **303.** a) 2; b) 4; c) $\frac{1}{2}$; d) $\frac{2}{3}$. **307.** **Hint.** If $x>0$, then when $|\Delta x|<x$ we have $|\sqrt{x+\Delta x}-\sqrt{x}|=$ $=|\Delta x|/(\sqrt{x+\Delta x}+\sqrt{x}) \leqslant |\Delta x|/\sqrt{x}$. **309.** **Hint.** Take advantage of the inequality $|\cos(x+\Delta x)-\cos x| \leqslant |\Delta x|$ **310.** a) $x \neq \frac{\pi}{2}+k\pi$, where k is an integer; b) $x \neq k\pi$, where k is an integer **311.** **Hint.** Take advantage of the inequality $||x+\Delta x|-|x|| \leqslant |\Delta x|$ **313.** $A=4$. **314.** $f(0)=1$. **315.** No **316.** a) $f(0)=n$; b) $f(0)=\frac{1}{2}$; c) $f(0)=2$; d) $f(0)=2$; e) $f(0)=0$; f) $f(0)=1$. **317.** $x=2$ is a discontinuity of the second kind. **318.** $x=-1$ is a removable discontinuity. **319.** $x=-2$ is a discontinuity of the second kind; $x=2$ is a removable discontinuity **320.** $x=0$ is a discontinuity of the first kind. **321.** a) $x=0$ is a discontinuity of the second kind; b) $x=0$ is a removable discontinuity. **322.** $x=0$ is a removable discontinuity, $x=k\pi$ $(k=\pm 1, \pm 2, \ldots)$ are infinite discontinuities **323.** $x=2\pi k \pm \frac{\pi}{2}$ $(k=0, \pm 1, \pm 2, \ldots)$ are infinite discontinuities. **324.** $x=k\pi$ $(k=0, \pm 1, \pm 2, \ldots)$ are infinite discontinuities. **325.** $x=0$ is a discontinuity of the first kind. **326.** $x=-1$ is a removable discontinuity; $x=1$ is a point of discontinuity of the first kind. **327.** $x=-1$ is a discon-

tinuity of the second kind. **328.** $x=0$ is a removable discontinuity. **329.** $x=1$ is a discontinuity of the first kind. **330.** $x=3$ is a discontinuity of the first kind. **332.** $x=1$ is a discontinuity of the first kind. **333.** The function is continuous. **334.** a) $x=0$ is a discontinuity of the first kind; b) the function is continuous; c) $x=k\pi$ (k is integral) are discontinuities of the first kind. **335.** a) $x=k$ (k is integral) are discontinuities of the first kind; b) $x=k$ ($k \neq 0$ is integral) are points of discontinuity of the first kind. **337.** No, since the function $y=E(x)$ is discontinuous at $x=1$. **338.** 1.53. **339. Hint.** Show that when x_0 is sufficiently large, we have $P(-x_0)P(x_0)<0$.

Chapter II

341. a) 3; b) 0.21; c) $2h+h^2$. **342.** a) 0.1; b) -3; c) $\sqrt[3]{a+h}-\sqrt[3]{a}$. **344.** a) 624; 1560; b) 0.01; 100; c) -1; 0.000011. **345.** a) $a\Delta x$; b) $3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3$; $3x^2+3x\,\Delta x+(\Delta x)^2$; c) $-\frac{2x\,\Delta x+(\Delta x)^2}{x^2(x+\Delta x)^2}$; $-\frac{2x+\Delta x}{x^2(x+\Delta x)^2}$; d) $\sqrt{x+\Delta x}-\sqrt{x}$; $\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}$; e) $2^x(2^{\Delta x}-1)$; $\frac{2^x(2^{\Delta x}-1)}{\Delta x}$; f) $\ln\frac{x+\Delta x}{x}$; $\frac{1}{\Delta x}\ln\left(1+\frac{\Delta x}{x}\right)$. **346.** a) -1; b) 0.1; c) $-h$; 0. **347.** 21. **348.** 15 cm/sec. **349.** 7.5. **350.** $\frac{f(x+\Delta x)-f(x)}{\Delta x}$. **351.** $f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$.
352. a) $\frac{\Delta\varphi}{\Delta t}$; b) $\frac{d\varphi}{dt}=\lim_{\Delta t\to 0}\frac{\Delta\varphi}{\Delta t}$, where φ is the angle of turn at time t. **353.** a) $\frac{\Delta T}{\Delta t}$; b) $\frac{dT}{dt}=\lim_{\Delta t\to 0}\frac{\Delta T}{\Delta t}$, where T is the temperature at time t. **354.** $\frac{dQ}{dt}=\lim_{\Delta t\to 0}\frac{\Delta Q}{\Delta t}$, where Q is the quantity of substance at time t. **355.** a) $\frac{\Delta m}{\Delta x}$; b) $\lim_{\Delta x\to 0}\frac{\Delta m}{\Delta x}$ **356.** a) $-\frac{1}{6}\approx-0.16$; b) $-\frac{5}{21}\approx-0.238$; c) $-\frac{50}{201}\approx-0.249$; $y'_{x=2}=-0.25$. **357.** $\sec^2 x$. **Solution.**

$$y'=\lim_{\Delta x\to 0}\frac{\tan(x+\Delta x)-\tan x}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x\cos x\cos(x+\Delta x)}=\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}\times$$
$$\times\lim_{\Delta x\to 0}\frac{1}{\cos x\cos(x+\Delta x)}=\frac{1}{\cos^2 x}=\sec^2 x.$$

358. a) $3x^2$; b) $-\frac{2}{x^3}$; c) $\frac{1}{2\sqrt{x}}$; d) $\frac{-1}{\sin^2 x}$. **359.** $\frac{1}{12}$ **Solution.**

$$f'(8)=\lim_{\Delta x\to 0}\frac{f(8+\Delta x)-f(8)}{\Delta x}=$$
$$=\lim_{\Delta x\to 0}\frac{\sqrt[3]{8+\Delta x}-\sqrt[3]{8}}{\Delta x}=\lim_{\Delta x\to 0}\frac{8+\Delta x-8}{\Delta x\left[\sqrt[3]{(8+\Delta x)^2}+\sqrt[3]{(8+\Delta x)8}+\sqrt[3]{8^2}\right]}=$$
$$=\lim_{\Delta x\to 0}\frac{1}{\sqrt[3]{(8+\Delta x)^2}+2\sqrt[3]{8+\Delta x}+4}=\frac{1}{12}.$$

360. $f'(0)=-8$, $f'(1)=0$, $f'(2)=0$. **361.** $x_1=0$, $x_2=3$. **Hint.** For the given function the equation $f'(x)=f(x)$ has the form $3x^2=x^3$. **362.** 30m/sec. **363.** 1, 2. **364.** -1. **365.** $f'(x_0)=\frac{-1}{x_0^2}$. **366.** -1, 2, $\tan\varphi=3$. **Hint.** Use the results of Example 3 and Problem 365. **367. Solution.** a) $f'(0)=\lim_{\Delta x\to 0}\frac{\sqrt[3]{(\Delta x)^2}}{\Delta x}=\lim_{\Delta x\to 0}\frac{1}{\sqrt[3]{\Delta x}}=\pm\infty$;

b) $f'(1) = \lim\limits_{\Delta x \to 0} \frac{\sqrt[5]{1+\Delta x} - 1}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{1}{\sqrt[5]{(\Delta x)^4}} = \infty$; c) $f'_-\left(\frac{2k+1}{2}\pi\right) =$

$= \lim\limits_{\Delta x \to -0} \frac{\left|\cos\left(\frac{2k+1}{2}\pi + \Delta x\right)\right|}{\Delta x} = \lim\limits_{\Delta x \to -0} \frac{|\sin \Delta x|}{\Delta x} = -1$; $f'_+\left(\frac{2k+1}{2}\pi\right) =$

$= \lim\limits_{\Delta x \to +0} \frac{|\sin \Delta x|}{\Delta x} = 1$. 368. $5x^4 - 12x^2 + 2$. 369. $-\frac{1}{3} + 2x - 2x^3$. 370. $2ax + b$.

371. $-\frac{15x^2}{a}$. 372. $mat^{m-1} + b(m+n)t^{m+n-1}$. 373. $\frac{6ax^5}{\sqrt{a^2+b^2}}$. 374. $-\frac{\pi}{x^2}$.

375. $2x^{-\frac{1}{3}} - 5x^{\frac{3}{2}} - 3x^{-4}$. 376. $\frac{8}{3}x^{\frac{5}{3}}$. **Hint.** $y = x^2 x^{\frac{2}{3}} = x^{\frac{8}{3}}$. 377. $\frac{4b}{3x^2\sqrt[3]{x}} -$

$-\frac{2a}{3x\sqrt[3]{x^2}}$. 378. $\frac{bc-ad}{(c+dx)^2}$. 379. $\frac{-2x^2-6x+25}{(x^2-5x+5)^2}$. 380. $\frac{1-4x}{x^2(2x-1)^2}$.

381. $\frac{1}{\sqrt{z}(1-\sqrt{z})^2}$. 382. $5\cos x - 3\sin x$. 383. $\frac{4}{\sin^2 2x}$. 384. $\frac{-2}{(\sin x - \cos x)^2}$.

385. $t^2 \sin t$. 386. $y' = 0$. 387. $\cot x - \frac{x}{\sin^2 x}$. 388. $\arcsin x + \frac{x}{\sqrt{1-x^2}}$.

389. $x \arctan x$. 390. $x^6 e^x (x+7)$. 391. xe^x. 392. $e^x \frac{x-2}{x^3}$. 393. $\frac{5x^4 - x^5}{e^x}$.

394. $e^x(\cos x - \sin x)$. 395. $x^2 e^x$. 396. $e^x\left(\arcsin x + \frac{1}{\sqrt{1-x^2}}\right)$. 397. $\frac{x(2\ln x - 1)}{\ln^2 x}$.

398. $3x^2 \ln x$. 399. $\frac{2}{x} + \frac{\ln x}{x^2} - \frac{2}{x^2}$. 400. $\frac{2\ln x}{x \ln 10} - \frac{1}{x}$. 401. $\sinh x + x\cosh x$.

402. $\frac{2x\cosh x - x^2 \sinh x}{\cosh^2 x}$. 403. $-\tanh^2 x$. 404. $\frac{-3(x\ln x + \sinh x \cosh x)}{x \ln^2 x \cdot \sinh^2 x}$.

405. $\frac{-2x^2}{1-x^4}$. 406. $\frac{1}{\sqrt{1-x^2}} \operatorname{arc\,sinh} x + \frac{1}{\sqrt{1+x^2}} \arcsin x$.

407. $\frac{x - \sqrt{x^2-1}\operatorname{arc\,cosh} x}{x^2\sqrt{x^2-1}}$. 408. $\frac{1+2x \operatorname{arc\,tanh} x}{(1-x^2)^2}$. 410. $\frac{3a}{c}\left(\frac{ax+b}{c}\right)^2$.

411. $12ab + 18b^2 y$. 412. $16x(3+2x^2)^3$. 413. $\frac{x^2-1}{(2x-1)^3}$. 414. $\frac{-x}{\sqrt{1-x^2}}$.

415. $\frac{bx^2}{\sqrt[3]{(a+bx^3)^2}}$. 416. $-\sqrt{\sqrt[3]{\frac{a^2}{x^2}} - 1}$. 418. $\frac{1 - \tan^2 x + \tan^4 x}{\cos^2 x}$.

419. $\frac{-1}{2\sin^2 x\sqrt{\cot x}}$. 420. $2 - 15\cos^2 x \sin x$. 421. $\frac{-16\cos 2t}{\sin^3 2t}$. **Hint.** $x = \sin^{-2} t +$
$+ \cos^{-2} t$. 422. $\frac{\sin x}{(1-3\cos x)^3}$. 423. $\frac{\sin^3 x}{\cos^4 x}$. 424. $\frac{3\cos x + 2\sin x}{2\sqrt{15\sin x - 10\cos x}}$.

425. $\frac{2\cos x}{3\sqrt[3]{\sin x}} + \frac{3\sin x}{\cos^4 x}$. 426. $\frac{1}{2\sqrt{1-x^2}\sqrt{1+\arcsin x}}$.

427. $\frac{1}{2(1+x^2)\sqrt{\arctan x}} - \frac{3(\arcsin x)^2}{\sqrt{1-x^2}}$. 428. $\frac{-1}{(1+x^2)(\arctan x)^2}$.

429. $\dfrac{e^x+xe^x+1}{2\sqrt{xe^x+x}}$. **430.** $\dfrac{2e^x-2^x\ln 2}{3\sqrt[3]{(2e^x-2^x+1)^2}}+\dfrac{5\ln^4 x}{x}$. **432.** $(2x-5)\times$ $\times\cos(x^5-5x+1)-\dfrac{a}{x^2\cos^2\dfrac{a}{x}}$. **433.** $-\alpha\sin(\alpha x+\beta)$. **434.** $\sin(2t+\varphi)$.

435. $-2\dfrac{\cos x}{\sin^3 x}$. **436.** $\dfrac{-1}{\sin^2\dfrac{x}{a}}$. **437.** $x\cos 2x^2\sin 3x^3$. **438.** **Solution.** $\dfrac{1}{\sqrt{1-(2x)^2}}(2x)'=\dfrac{2}{\sqrt{1-4x^2}}$. **439.** $\dfrac{-2}{x\sqrt{x^4-1}}$. **440.** $\dfrac{-1}{2\sqrt{x-x^2}}$. **441.** $\dfrac{-1}{1+x^2}$.

442. $\dfrac{-1}{1+x^2}$. **443.** $-10xe^{-x^2}$. **444.** $-2x5^{-x^2}\ln 5$. **445.** $2x10^{2x}(1+x\ln 10)$.

446. $\sin 2^t+2^t t\cos 2^t\ln 2$. **447.** $\dfrac{-e^x}{\sqrt{1-e^{2x}}}$. **448.** $\dfrac{2}{2x+7}$. **449.** $\cot x\log e$.

450. $\dfrac{-2x}{1-x^2}$. **451.** $\dfrac{2\ln x}{x}-\dfrac{1}{x\ln x}$. **452.** $\dfrac{(e^x+5\cos x)\sqrt{1-x^2}-4}{(e^x+5\sin x-4\arcsin x)\sqrt{1-x^2}}$.

453. $\dfrac{1}{(1+\ln^2 x)x}+\dfrac{1}{(1+x^2)\arctan x}$. **454.** $\dfrac{1}{2x\sqrt{\ln x+1}}+\dfrac{1}{2(\sqrt{x}+x)}$.

455. Solution. $y'=(\sin^3 5x)'\cos^2\dfrac{x}{3}+\sin^3 5x\left(\cos^2\dfrac{x}{3}\right)'=3\sin^2 5x\cos 5x\,5\cos^2\dfrac{x}{3}+$ $+\sin^3 5x\,2\cos\dfrac{x}{3}\left(-\sin\dfrac{x}{3}\right)\dfrac{1}{3}=15\sin^2 5x\cos 5x\cos^2\dfrac{x}{3}-\dfrac{2}{3}\sin^3 5x\cos\dfrac{x}{3}\sin\dfrac{x}{3}$.

456. $\dfrac{4x+3}{(x-2)^3}$ **457.** $\dfrac{x^2+4x-6}{(x-3)^5}$. **458.** $\dfrac{x^7}{(1-x^2)^5}$. **459.** $\dfrac{x-1}{x^2\sqrt{2x^2-2x+1}}$.

460. $\dfrac{1}{\sqrt{(a^2+x^2)^3}}$. **461.** $\dfrac{x^2}{\sqrt{(1+x^2)^5}}$. **462.** $\dfrac{(1+\sqrt{x})^2}{\sqrt[3]{x}}$. **463.** $x^5\sqrt[3]{(1+x^3)^2}$.

464. $\dfrac{1}{\sqrt[4]{(x-1)^3(x+2)^5}}$. **465.** $4x^3(a-2x^3)(a-5x^3)$.

466. $\dfrac{2abmnx^{n-1}(a+bx^n)^{m-1}}{(a-bx^n)^{m+1}}$. **467.** $\dfrac{x^3-1}{(x+2)^6}$. **468.** $\dfrac{a-3x}{2\sqrt{a-x}}$.

469. $\dfrac{3x^2+2(a+b+c)x+ab+bc+ac}{2\sqrt{(x+a)(x+b)(x+c)}}$. **470.** $\dfrac{1+2\sqrt{y}}{6\sqrt{y}\sqrt[3]{(y+\sqrt{y})^2}}$.

471. $2(7t+4)\sqrt[3]{3t+2}$. **472.** $\dfrac{y-a}{\sqrt{(2ay-y^2)^3}}$. **473.** $\dfrac{1}{\sqrt{e^x+1}}$. **474.** $\sin^3 x\cos^2 x$.

475. $\dfrac{1}{\sin^4 x\cos^4 x}$. **476.** $10\tan 5x\sec^2 5x$. **477.** $x\cos x^2$. **478.** $3t^2\sin 2t^3$.

479. $3\cos x\cos 2x$. **480.** $\tan^4 x$. **481.** $\dfrac{\cos 2x}{\sin^4 x}$. **482.** $\dfrac{(\alpha-\beta)\sin 2x}{2\sqrt{\alpha\sin^2 x+\beta\cos^2 x}}$. **483.** 0.

484. $\dfrac{1}{2}\dfrac{\arcsin x(2\arccos x-\arcsin x)}{\sqrt{1-x^2}}$. **485.** $\dfrac{2}{x\sqrt{2x^2-1}}$. **486.** $\dfrac{1}{1+x^2}$.

487. $\dfrac{x\arccos x-\sqrt{1-x^2}}{(1-x^2)^{3/2}}$. **488.** $\dfrac{1}{\sqrt{a-bx^2}}$. **489.** $\sqrt{\dfrac{a-x}{a+x}}$. **490.** $2\sqrt{a^2-x^2}$.

491. $\dfrac{-x}{\sqrt{2x-x^2}}$. **492.** $\arcsin\sqrt{x}$. **493.** $\dfrac{5}{\sqrt{1-25x^2}\arcsin 5x}$.

494. $\frac{1}{x\sqrt{1-\ln^2 x}}$. **495.** $\frac{\sin\alpha}{1-2x\cos\alpha+x^2}$. **496.** $\frac{1}{5+4\sin x}$.
497. $4x\sqrt{\frac{x}{b-x}}$. **498.** $\frac{\sin^2 x}{1+\cos^2 x}$. **499.** $\frac{a}{2}\sqrt{e^{ax}}$. **500.** $\sin 2x e^{\sin^2 x}$.
501. $2m^2p\,(2ma^{mx}+b)^{p-1}a^{mx}\ln a$. **502.** $e^{\alpha t}(\alpha\cos\beta t-\beta\sin\beta t)$. **503.** $e^{\alpha x}\sin\beta x$.
504. $e^{-x}\cos 3x$. **505.** $x^{n-1}a^{-x^2}(n-2x^2\ln a)$. **506.** $-\frac{1}{2}y\tan x\,(1+\sqrt{\cos x}\ln a)$.
507. $\frac{3^{\cot\frac{1}{x}}\ln 3}{\left(x\sin\frac{1}{x}\right)^2}$. **508.** $\frac{2ax+b}{ax^2+bx+c}$. **509.** $\frac{1}{\sqrt{a^2+x^2}}$. **510.** $\frac{\sqrt{x}}{1+\sqrt{x}}$.
511. $\frac{1}{\sqrt{2ax+x^2}}$. **512.** $\frac{-2}{x\ln^3 x}$. **513.** $-\frac{1}{x^2}\tan\frac{x-1}{x}$. **514.** $\frac{2x+11}{x^2-x-2}$. **Hint.** $y=5\ln(x-2)-3\ln(x+1)$. **515.** $\frac{3x^2-16x+19}{(x-1)(x-2)(x-3)}$. **516.** $\frac{1}{\sin^3 x\cos x}$.
517. $\sqrt{x^2-a^2}$. **518.** $\frac{-6x^2}{(3-2x^3)\ln(3-2x^3)}$. **519.** $\frac{15a\ln^2(ax+b)}{ax+b}$.
520. $\frac{2}{\sqrt{x^2+a^2}}$. **521.** $\frac{mx+n}{x^2-a^2}$. **522.** $\sqrt{2}\sin\ln x$. **523.** $\frac{1}{\sin^3 x}$.
524. $\frac{\sqrt{1+x^2}}{x}$. **525.** $\frac{x+1}{x^3-1}$. **526.** $\frac{3}{\sqrt{1-9x^2}}[2^{\arcsin 3x}\ln 2+2(1-\arccos 3x)]$.
527. $\left(3^{\frac{\sin ax}{\cos bx}}\ln 3+\frac{\sin^2 ax}{\cos^2 bx}\right)\frac{a\cos ax\cos bx+b\sin ax\sin bx}{\cos^2 bx}$. **528.** $\frac{1}{1+2\sin x}$.
529. $\frac{1}{x(1+\ln^2 x)}$. **530.** $\frac{1}{\sqrt{1-x^2}\arcsin x}+\frac{\ln x}{x}+\frac{1}{x\sqrt{1-\ln^2 x}}$.
531. $-\frac{1}{x(1+\ln^2 x)}$. **532.** $\frac{x^2}{x^4+x^2-2}$. **533.** $\frac{2}{\cos x\sqrt{\sin x}}$. **534.** $\frac{x^2-3x}{x^4-1}$.
535. $\frac{1}{1+x^3}$. **536.** $\frac{\arcsin x}{(1-x^2)^{3/2}}$. **537.** $6\sinh^2 2x\cdot\cosh 2x$. **538.** $e^{\alpha x}(\alpha\cosh\beta x+\beta\sinh\beta x)$: **539.** $6\tanh^2 2x\,(1-\tanh^2 2x)$ **540.** $2\coth 2x$. **541.** $\frac{2x}{\sqrt{a^4+x^4}}$.
542. $\frac{1}{x\sqrt{\ln^2 x-1}}$ **543.** $\frac{1}{\cos 2x}$. **544.** $\frac{-1}{\sin x}$. **545.** $\frac{2}{1-x^2}$. **546.** $x\,\mathrm{arc\,tanh}\,x$
547. $x\,\mathrm{arc\,sinh}\,x$. **548.** a) $y'=1$ when $x>0$; $y'=-1$ when $x<0$; $y'(0)$ does not exist; b) $y'=|2x|$. **549.** $y'=\frac{1}{x}$. **550.** $f'(x)=\begin{cases}-1 \text{ when } x\leqslant 0,\\ -e^{-x} \text{ when } x>0.\end{cases}$
552. $\frac{1}{2}+\frac{\sqrt{3}}{3}$. **553.** 6π. **554.** a) $f'_-(0)=-1$, $f'_+(0)=1$; b) $f'_-(0)=\frac{2}{a}$, $f'_+(0)=\frac{-2}{a}$; c) $f'_-(0)=1$, $f'_+(0)=0$; d) $f'_-(0)=f'_+(0)=0$, e) $f'_-(0)$ and $f'_+(0)$ do not exist. **555.** $1-x$. **556.** $2+\frac{x-3}{4}$. **557.** -1. **558.** 0 **561.** **Solution.** We have $y'=e^{-x}(1-x)$. Since $e^{-x}=\frac{y}{x}$, it follows that $y'=\frac{y}{x}(1-x)$ or $xy'=y(1-x)$ **566.** $(1+2x)(1+3x)+2(1+x)(1+3x)+3(x+1)(1+2x)$.
567. $-\frac{(x+2)(5x^2+19x+20)}{(x+1)^4(x+3)^5}$. **568.** $\frac{x^2-4x+2}{2\sqrt{x(x-1)(x-2)^3}}$

569. $\frac{3x^2+5}{3(x^2+1)}\sqrt[3]{\frac{x^2}{x^2+1}}$. **570.** $\frac{(x-2)^9(x^2-7x+1)}{(x-1)(x-2)(x-3)\sqrt{(x-1)^5(x-3)^4}}$.

571. $-\frac{5x^2+x-24}{3(x-1)^{1/2}(x+2)^{5/3}(x+3)^{5/2}}$. **572.** $x^x(1+\ln x)$. **573.** $x^{x^2+1}(1+2\ln x)$.

574. $\sqrt[x]{x}\,\frac{1-\ln x}{x^2}$. **575.** $x^{\sqrt{x}-\frac{1}{2}}\left(1+\frac{1}{2}\ln x\right)$. **576.** $x^{x^x}x^x\left(\frac{1}{x}+\ln x+\ln^2 x\right)$.

577. $x^{\sin x}\left(\frac{\sin x}{x}+\cos x\ln x\right)$. **578.** $(\cos x)^{\sin x}(\cos x\ln\cos x-\sin x\tan x)$.

579. $\left(1+\frac{1}{x}\right)^x\left[\ln\left(1+\frac{1}{x}\right)+\frac{1}{1+x}\right]$. **580.** $(\arctan x)^x\times$ $\times\left[\ln\arctan x+\frac{x}{(1+x^2)\arctan x}\right]$. **581.** a) $x'_y=\frac{1}{3(1+x^2)}$; b) $x'_y=\frac{2}{2-\cos x}$; c) $x'_y=\frac{10}{1+5e^{\frac{x}{2}}}$. **582.** $\frac{3}{2}t^2$. **583.** $\frac{t-1}{t+1}$. **584.** $\frac{-2t}{1-t^2}$.

585. $\frac{t(2-t^3)}{1-2t^3}$. **586.** $\frac{2}{3\sqrt[6]{t}}$. **587.** $\frac{t+1}{t(t^2+1)}$. **588.** $\tan t$. **589.** $-\frac{b}{a}$.

590. $-\frac{b}{a}\tan t$. **591.** $-\tan 3t$. **592.** $y'_x=\begin{cases}-1 \text{ when } t<0,\\ 1 \text{ when } t>0.\end{cases}$ **593.** $-2e^{3t}$.

594. $\tan t$. **596.** 1. **597.** ∞. **599.** No. **600.** Yes, since the equality is an identity. **601.** $\frac{2}{5}$. **602.** $-\frac{b^2x}{a^2y}$. **603.** $-\frac{x^2}{y^2}$. **604.** $-\frac{x(3x+2y)}{x^2+2y}$. **605.** $-\sqrt{\frac{y}{x}}$.

606. $-\sqrt[3]{\frac{y}{x}}$. **607.** $\frac{2y^2}{3(x^2-y^2)+2xy}=\frac{1-y^3}{1+3xy^2+4y^3}$. **608.** $\frac{10}{10-3\cos y}$.

609. -1. **610.** $\frac{y\cos^2 y}{1-x\cos^2 y}$. **611.** $\frac{y}{x}\,\frac{1-x^2-y^2}{1+x^2+y^2}$. **612.** $(x+y)^2$. **613.** $y'=$ $=\frac{1}{e^y-1}=\frac{1}{x+y-1}$. **614.** $\frac{y}{x}+e^{\frac{y}{x}}$. **615.** $\frac{y}{x-y}$. **616.** $\frac{x+y}{x-y}$.

617. $\frac{cy+x\sqrt{x^2+y^2}}{cx-y\sqrt{x^2+y^2}}$. **618.** $\frac{x\ln y-y}{y\ln x-x}\,\frac{y}{x}$. **620.** a) 0; b) $\frac{1}{2}$; c) 0. **622.** 45°; $\arctan 2\approx 63°26'$. **623.** 45°. **624.** $\arctan\frac{2}{e}\approx 36°21'$. **625.** (0, 20); (1, 15); (−2, −12). **626.** (1, −3). **627.** $y=x^2-x+1$. **628.** $k=\frac{-1}{11}$. **629.** $\left(\frac{1}{8}, -\frac{1}{16}\right)$.

631. $y-5=0$; $x+2=0$. **632.** $x-1=0$; $y=0$. **633.** a) $y=2x$; $y=-\frac{1}{2}x$; b) $x-2y-1=0$; $2x+y-2=0$; c) $6x+2y-\pi=0$; $2x-6y+3\pi=0$; d) $y=x-1$; $y=1-x$; e) $2x+y-3=0$; $x-2y+1=0$ for the point (1, 1); $2x-y+3=0$; $x+2y-1=0$ for the point (−1, 1). **634.** $7x-10y+6=0$, $10x+7y-34=0$. **635.** $y=0$; $(\pi+4)x+(\pi-4)y-\frac{\pi^2\sqrt{2}}{4}=0$. **636.** $5x+6y-13=0$, $6x-5y+21=0$. **637.** $x+y-2=0$. **638.** At the point (1, 0): $y=2x-2$; $y=\frac{1-x}{2}$; at the point (2, 0): $y=-x+2$; $y=x-2$; at the point (3, 0): $y=2x-6$; $y=\frac{3-x}{2}$. **639.** $14x-13y+12=0$; $13x+14y-41=0$.

640. **Hint.** The equation of the tangent is $\frac{x}{2x_0}+\frac{y}{2y_0}=1$. Hence, the tangent crosses the x-axis at the point $A(2x_0, 0)$ and the y-axis at $B(0, 2y_0)$. Finding the midpoint of AB, we get the point (x_0, y_0). **643.** 40° 36′. **644.** The parabolas are tangent at the point (0, 0) and intersect at an angle $\arctan\frac{1}{7}\approx 8°\,8'$ at the point (1, 1). **647.** $S_t=S_n=2$; $t=n=2\sqrt{2}$. **648.** $\frac{1}{\ln 2}$. **652.** $T=2a\sin\frac{t}{2}\tan\frac{t}{2}$; $N=2a\sin\frac{t}{2}$; $S_t=2a\sin^2\frac{t}{2}\tan\frac{t}{2}$; $S_n=a\sin t$. **653.** $\arctan\frac{1}{k}$. **654.** $\frac{\pi}{2}+2\varphi$. **655.** $S_t=4\pi^2 a$; $S_n=a$; $t=2\pi a\sqrt{1+4\pi^2}$; $n=a\sqrt{1+4\pi^2}$; $\tan\mu=2\pi$. **656.** $S_t=a$; $S_n=\frac{a}{\varphi_0{}^2}$; $t=\sqrt{a^2+\varrho_0^2}$; $n=\frac{\varrho_0}{a}\sqrt{a^2+\varrho_0^2}$; $\tan\mu=-\varphi_0$. **657.** 3 cm/sec; 0; —9 cm/sec **658.** 15 cm/sec. **659.** $-\frac{3}{2}$ m/sec. **660.** The equation of the trajectory is $y=x\tan\alpha-\frac{g}{2v_0^2\cos^2\alpha}x^2$. The range is $\frac{v_0^2\sin 2\alpha}{g}$. The velocity, $\sqrt{v_2^0-2v_0gt\sin\alpha+g^2t^2}$; the slope of the velocity vector is $\frac{v_0\sin\alpha-gt}{v_0\cos\alpha}$. **Hint.** To determine the trajectory, eliminate the parameter t from the given system. The range is the abscissa of the point A (Fig. 17). The projections of velocity on the axes are $\frac{dx}{dt}$ and $\frac{dy}{dt}$ The magnitude of the velocity is $\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$; the velocity vector is directed along the tangent to the trajectory **661.** Diminishes with the velocity 0.4 **62.** $\left(\frac{9}{8}, \frac{9}{2}\right)$. **663.** The diagonal increases at a rate of ~ 3.8 cm/sec, the area, at a rate of 40 cm²/sec **664.** The surface area increases at a rate of 0 2π m²/sec, the volume, at a rate of 0.05 π m³/sec. **665.** $\frac{\pi}{3}$ cm/sec **666.** The mass of the rod is 360 g, the density at M is $5x$ g/cm, the density at A is 0, the density at B is 60 g/cm. **667.** $56x^6+210x^4$. **668.** $e^{x^2}(4x^2+2)$. **669.** $2\cos 2x$ **670.** $\frac{2(1-x^2)}{3(1+x^2)^2}$. **671.** $\frac{-x}{\sqrt{(a^2+x^2)^3}}$ **672.** $2\arctan x+\frac{2x}{1+x^2}$. **673.** $\frac{2}{1-x^2}+\frac{2x\arcsin x}{(1-x^2)^{3/2}}$. **674.** $\frac{1}{a}\cosh\frac{x}{a}$. **679.** $y'''=6$. **680.** $f'''(3)=4320$ **681.** $y^{\mathrm{V}}=\frac{24}{(x+1)^5}$. **682.** $y^{\mathrm{VI}}=-64\sin 2x$ **684.** 0; 1; 2; 2. **685.** The velocity is $v=5$; 4 997; 4.7. The acceleration, $a=0$; —0.006; —0.6. **686.** The law of motion of the point M_1 is $x=a\cos\omega t$; the velocity at time t is $-a\omega\sin\omega t$; the acceleration at time t is $-a\omega^2\cos\omega t$. Initial velocity, 0; initial acceleration: $-a\omega^2$; velocity when $x=0$ is $-a\omega$; acceleration when $x=0$ is 0. The maximum absolute value of velocity is $a\omega$; the maximum absolute value of acceleration is $a\omega^2$. **687.** $y^{(n)}=n!a^n$. **688.** a) $n!(1-x)^{-(n+1)}$, b) $(-1)^{n+1}\frac{1\cdot 3\ldots(2n-3)}{2^n x^{n-\frac{1}{2}}}$. **689.** a) $\sin\left(x+n\frac{\pi}{2}\right)$; b) $2^n\cos\left(2x+n\frac{\pi}{2}\right)$;

c) $(-3)^n e^{-3x}$; d) $(-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$; e) $\frac{(-1)^{n+1}n!}{(1+x)^{n+1}}$; f) $\frac{2n!}{(1-x)^{n+1}}$; g) $2^{n-1}\sin\left[2x+(n-1)\frac{\pi}{2}\right]$; h) $\frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$. **690.** a) $x\cdot e^x+ne^x$; b) $2^{n-1}e^{-2x}\left[2(-1)^n x^2+2n(-1)^{n-1}x+\frac{n(n-1)}{2}(-1)^{n-2}\right]$; c) $(1-x^2)\times$ $\times\cos\left(x+\frac{n\pi}{2}\right)-2nx\cos\left(x+\frac{(n-1)\pi}{2}\right)-n(n-1)\cos\left(x+\frac{(n-2)\pi}{2}\right)$; d) $\frac{(-1)^{n-1}\cdot 1\cdot 3\ldots(2n-3)}{2^n x^{\frac{2n+1}{2}}}[x-(2n-1)]$; e) $\frac{(-1)^n 6(n-4)!}{x^{n-3}}$ for $n\geqslant 4$.

691 $y^{(n)}(0)=(n-1)!$ **692.** a) $9t^3$; b) $2t^2+2$; c) $-\sqrt{1-t^2}$. **693.** a) $\frac{-1}{a\sin^3 t}$; b) $\frac{1}{3a\cos^4 t\sin t}$; c) $\frac{-1}{4a\sin^4\frac{t}{2}}$; d) $\frac{-1}{at\sin^3 t}$. **694.** a) 0; b) $2e^{3at}$. **695.** a) $(1+t^2)\times$ $\times(1+3t^2)$; b) $t\frac{1+t}{(1-t)^3}$. **696.** $\frac{-2e^{-t}}{(\cos t+\sin t)^3}$. **697.** $\left(\frac{d^2y}{dx^2}\right)_{t=0}=1$.
699. $\frac{2\cot^4 t}{\sin t}$. **700.** $\frac{4e^{2t}(2\sin t-\cos t)}{(\sin t+\cos t)^5}$. **701.** $-6e^{3t}(1+3t+t^2)$. **702.** $m^n t^m$.
703. $\frac{d^2x}{dy^2}=\frac{-f''(x)}{[f'(x)]^3}$; $\frac{d^3x}{dy^3}=\frac{3[f''(x)]^2-f'(x)f'''(x)}{[f'(x)]^5}$. **705.** $-\frac{p^2}{y^3}$. **706.** $-\frac{b^4}{a^2y^3}$.
707. $-\frac{2y^2+2}{y^5}$. **708.** $\frac{d^2y}{dx^2}=\frac{y}{(1-y)^3}$; $\frac{d^2x}{dy^2}=\frac{1}{y^2}$. **709.** $\frac{111}{256}$. **710.** $-\frac{1}{16}$.
711. a) $\frac{1}{3}$; b) $-\frac{3a^2x}{y^5}$. **712.** $\Delta y=0.009001$; $dy=0.009$. **713.** $d(1-x^3)=1$ when $x=1$ and $\Delta x=-\frac{1}{3}$. **714.** $dS=2x\,\Delta x$, $\Delta S=2x\,\Delta x+(\Delta x)^2$. **717.** For $x=0$.
718. No. **719.** $dy=-\frac{\pi}{72}\approx-0.0436$. **720.** $dy=\frac{1}{2700}\approx 0.00037$.
721. $dy=\frac{\pi}{45}\approx 0.0698$. **722.** $\frac{-m\,dx}{x^{m+1}}$. **723.** $\frac{dx}{(1-x)^2}$. **724.** $\frac{dx}{\sqrt{a^2-x^2}}$.
725. $\frac{a\,dx}{x^2+a^2}$. **726.** $-2xe^{-x^2}dx$. **727.** $\ln x\,dx$. **728.** $\frac{-2dx}{1-x^2}$. **729.** $-\frac{1+\cos\varphi}{\sin^2\varphi}d\varphi$.
730. $-\frac{e^t dt}{1+e^{2t}}$. **732.** $-\frac{10x+8y}{7x+5y}dx$. **733.** $\frac{-ye^{-\frac{x}{y}}dx}{y^2-xe^{-\frac{x}{y}}}=\frac{y}{x-y}dx$. **734** $\frac{x+y}{x-y}dx$.
735. $\frac{12}{11}dx$. **737.** a) 0.485; b) 0.965; c) 1.2; d) -0.045; e) $\frac{\pi}{4}+0.025\approx 0.81$.
738. 565 cm³. **739.** $\sqrt{5}\approx 2.25$; $\sqrt{17}\approx 4.13$; $\sqrt{70}\approx 8.38$; $\sqrt{640}\approx 25.3$.
740. $\sqrt[3]{10}\approx 2.16$; $\sqrt[3]{70}\approx 4.13$; $\sqrt[3]{200}\approx 5.85$. **741.** a) 5; b) 1.1; c) 0.93; d) 0.9. **742.** 1.0019. **743.** 0.57. **744.** 2.03. **748.** $\frac{-(dx)^2}{(1-x^2)^{3/2}}$. **749.** $\frac{-x(dx)^2}{(1-x^2)^{3/2}}$.
750. $\left(-\sin x\ln x+\frac{2\cos x}{x}-\frac{\sin x}{x^2}\right)(dx)^2$. **751.** $\frac{2\ln x-3}{x^3}(dx)^3$. **752.** $-e^{-x}\times$ $\times(x^2-6x+6)(dx)^3$. **753.** $\frac{384(dx)^4}{(2-x)^5}$. **754.** $3\cdot 2^n\sin\left(2x+5+\frac{n\pi}{2}\right)(dx)^n$.

755. $e^{x\cos\alpha}\sin(x\sin\alpha+n\alpha)\cdot(dx)^n$. 757. No, since $f'(2)$ does not exist. 758. No. The point $x=\frac{\pi}{2}$ is a discontinuity of the function. 762. $\xi=0$. 763. (2, 4). 765. a) $\xi=\frac{14}{9}$; b) $\xi=\frac{\pi}{4}$. 768. $\ln x=(x-1)-\frac{1}{2}(x-1)^2+\frac{2(x-1)^3}{3!\,\xi^3}$, where $\xi=1+\theta(x-1)$, $0<\theta<1$. 769. $\sin x=x-\frac{x^3}{3!}+\frac{x^4}{4!}\sin\xi_1$, where $\xi_1=\theta_1 x$, $0<\theta_1<1$; $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^6}{6!}\sin\xi_2$, where $\xi_2=\theta_2 x$, $0<\theta_2<1$. 770. $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^{n-1}}{(n-1)!}+\frac{x^n}{n!}e^{\xi}$, where $\xi=\theta x$, $0<\theta<1$. 772. Error: a) $\frac{1}{16}\frac{x^3}{(1+\xi)^{5/2}}$; b) $\frac{5}{81}\frac{x^3}{(1+\xi)^{8/3}}$; in both cases $\xi=\theta x$; $0<\theta<1$. 773. The error is less than $\frac{3}{5!}=\frac{1}{40}$. 775. **Solution.** We have $\sqrt{\frac{a+x}{a-x}}=\left(1+\frac{x}{a}\right)^{\frac{1}{2}}\left(1-\frac{x}{a}\right)^{-\frac{1}{2}}$. Expanding both factors in powers of x, we get: $\left(1+\frac{x}{a}\right)^{\frac{1}{2}}\approx 1+\frac{1}{2}\frac{x}{a}-\frac{1}{8}\frac{x^2}{a^2}$; $\left(1-\frac{x}{a}\right)^{-\frac{1}{2}}\approx 1+\frac{1}{2}\frac{x}{a}+\frac{3}{8}\frac{x^2}{a^2}$. Multiplying, we will have: $\sqrt{\frac{a+x}{a-x}}\approx 1+\frac{x}{a}+\frac{x^2}{2a^2}$. Then, expanding $e^{\frac{x}{a}}$ in powers of $\frac{x}{a}$, we get the same polynomial $e^{\frac{x}{a}}\approx 1+\frac{x}{a}+\frac{x^2}{2a^2}$. 777. $-\frac{1}{3}$. 778. ∞ 779. 1 780. 3. 781. $\frac{1}{2}$ 782. 5. 783. ∞. 784. 0. 785. $\frac{\pi^2}{2}$. 786. 1. 788. $\frac{2}{\pi}$. 789. 1. 790. 0. 791. a. 792. ∞ for $n>1$; a for $n=1$; 0 for $n<1$. 793. 0. 795. $\frac{1}{5}$. 796. $\frac{1}{12}$ 797. -1. 799. 1. 800. e^3. 801. 1. 802. 1 803. 1 804. $\frac{1}{e}$. 805. $\frac{1}{e}$. 806. $\frac{1}{e}$. 807. 1. 808. 1. 810. **Hint.** Find $\lim\limits_{\alpha\to 0}\frac{S}{\frac{2}{3}bh}$, where $S=\frac{R^2}{2}(\alpha-\sin\alpha)$ is the exact expression for the area of the segment (R is the radius of the corresponding circle).

Chapter III

811. $(-\infty, -2)$, increases; $(-2, \infty)$, decreases. 812. $(-\infty, 2)$, decreases; $(2, \infty)$, increases. 813. $(-\infty, \infty)$, increases. 814. $(-\infty, 0)$ and $(2, \infty)$, increases; $(0, 2)$, decreases 815. $(-\infty, 2)$ and $(2, \infty)$, decreases. 816. $(-\infty, 1)$, increases; $(1, \infty)$, decreases. 817. $(-\infty, -2)$, $(-2, 8)$ and $(8, \infty)$, decreases. 818. $(0, 1)$, decreases; $(1, \infty)$, increases. 819. $(-\infty, -1)$ and $(1, \infty)$, increases; $(-1, 1)$, decreases 820. $(-\infty, \infty)$, increases 821. $\left(0, \frac{1}{e}\right)$, decreases; $\left(\frac{1}{e}, \infty\right)$, increases. 822. $(-2, 0)$, increases. 823. $(-\infty, 2)$, decreases;

(2, ∞), increases. **824.** $(-\infty, a)$ and (a, ∞), decreases. **825.** $(-\infty, 0)$ and (0, 1), decreases; $(1, \infty)$, increases **827.** $y_{max}=\frac{9}{4}$ when $x=\frac{1}{2}$. **828.** No extremum. **830.** $y_{min}=0$ when $x=0$; $y_{min}=0$ when $x=12$; $y_{max}=1296$ when $x=6$. **831.** $y_{min}\approx-0.76$ when $x\approx0.23$; $y_{max}=0$ when $x=1$; $y_{min}\approx-0.05$ when $x\approx1.43$. No extremum when $x=2$. **832.** No extremum. **833.** $y_{max}=-2$ when $x=0$; $y_{min}=2$ when $x=2$ **834.** $y_{max}=\frac{9}{16}$ when $x=3.2$. **835.** $y_{max}=-3\sqrt{3}$ when $x=-\frac{2}{\sqrt{3}}$; $y_{min}=3\sqrt{3}$ when $x=\frac{2}{\sqrt{3}}$ **836.** $y_{max}=\sqrt{2}$ when $x=0$ **837.** $y_{max}=-\sqrt{3}$ when $x=-2\sqrt{3}$; $y_{min}=\sqrt{3}$ when $x=2\sqrt{3}$. 838. $y_{min}=0$ when $x=\pm1$; $y_{max}=1$ when $x=0$ **839.** $y_{min}=-\frac{3}{2}\sqrt{3}$ when $x=\left(k-\frac{1}{6}\right)\pi$; $y_{max}=\frac{3}{2}\sqrt{3}$ when $x=\left(k+\frac{1}{6}\pi\right)$ $(k=0, \pm1, \pm2, \ldots)$. 840. $y_{max}=5$ when $x=12k\pi$; $y_{max}=5\cos\frac{2\pi}{5}$ when $x=12\left(k\pm\frac{2}{5}\right)\pi$; $y_{min}=-5\cos\frac{\pi}{5}$ when $x=12\left(k\pm\frac{1}{5}\right)\pi$; $y_{min}=1$ when $x=6(2k+1)\pi$ $(k=0, \pm1, \pm2, \ldots)$. **841.** $y_{min}=0$ when $x=0$. **842.** $y_{min}=-\frac{1}{e}$ when $x=\frac{1}{e}$. **843.** $y_{max}=\frac{4}{e^2}$ when $x=\frac{1}{e^2}$; $y_{min}=0$ when $x=1$ **844.** $y_{min}=1$ when $x=0$ **845.** $y_{min}=-\frac{1}{e}$ when $x=-1$. **846.** $y_{min}=0$ when $x=0$; $y_{max}=\frac{4}{e^2}$ when $x=2$ **847.** $y_{min}=e$ when $x=1$. **848.** No extremum. **849.** Smallest value is $m=-\frac{1}{2}$ for $x=-1$; greatest value, $M=\frac{1}{2}$ when $x=1$. **850.** $m=0$ when $x=0$ and $x=10$; $M=5$ for $x=5$. **851.** $m=\frac{1}{2}$ when $x=(2k+1)\frac{\pi}{4}$; $M=1$ for $x=\frac{k\pi}{2}$ $(k=0, \pm1, \pm2, \ldots)$. **852.** $m=0$ when $x=1$; $M=\pi$ when $x=-1$. **853.** $m=-1$ when $x=-1$; $M=27$ when $x=3$. **854.** a) $m=-6$ when $x=1$; $M=266$ when $x=5$; b) $m=-1579$ when $x=-10$; $M=3745$ when $x=12$. **856.** $p=-2$, $q=4$. **861.** Each of the terms must be equal to $\frac{a}{2}$ **862.** The rectangle must be a square with side $\frac{l}{4}$. **863.** Isosceles. **864.** The side adjoining the wall must be twice the other side **865.** The side of the cut-out square must be equal to $\frac{a}{6}$. **866.** The altitude must be half the base. **867.** That whose altitude is equal to the diameter of the base **868.** Altitude of the cylinder, $\frac{2R}{\sqrt{3}}$; radius of its base $R\sqrt{\frac{2}{3}}$, where R is the radius of the given sphere. **869.** Altitude of the cylinder, $R\sqrt{2}$ where R is the radius of the given sphere. **870.** Altitude of the cone, $\frac{4}{3}R$

where R is the radius of the given sphere. **871.** Altitude of the cone, $\frac{4}{3}R$, where R is the radius of the given sphere. **872.** Radius of the base of the cone $\frac{3}{2}r$, where r is the radius of the base of the given cylinder. **873.** That whose altitude is twice the diameter of the sphere. **874.** $\varphi=\pi$, that is, the cross-section of the channel is a semicircle. **875.** The central angle of the sector is $2\pi\sqrt{\frac{2}{3}}$. **876.** The altitude of the cylindrical part must be zero; that is, the vessel should be in the shape of a hemisphere. **877.** $h=\left(l^{\frac{2}{3}}-d^{\frac{2}{3}}\right)^{\frac{3}{2}}$. **878.** $\frac{x}{2x_0}+\frac{y}{2y_0}=1$. **879.** The sides of the rectangle are $a\sqrt{2}$ and $b\sqrt{2}$, where a and b are the respective semiaxes of the ellipse. **880.** The coordinates of the vertices of the rectangle which lie on the parabola $\left(\frac{2}{3}a;\ \pm 2\sqrt{\frac{pa}{3}}\right)$. **881.** $\left(\pm\frac{1}{\sqrt{3}},\ \frac{3}{4}\right)$. **882.** The angle is equal to the greatest of the numbers $\arccos\frac{1}{k}$ and $\arctan\frac{h}{d}$. **883.** $AM=a\frac{\sqrt[3]{p}}{\sqrt[3]{p}+\sqrt[3]{q}}$. **884.** $\frac{r}{\sqrt{2}}$. **885.** a) $x=y=\frac{d}{\sqrt{2}}$; b) $x=\frac{d}{\sqrt{3}}$; $y=d\sqrt{\frac{2}{3}}$. **886.** $x=\sqrt{\frac{2aQ}{q}}$; $P_{\min}=\sqrt{2aqQ}$. **887.** $\sqrt{Mm}$. **Hint.** For a completely elastic impact of two spheres, the velocity imparted to the stationary sphere of mass m_1 after impact with a sphere of mass m_2 moving with velocity v is equal to $\frac{2m_2v}{m_1+m_2}$. **888.** $n=\sqrt{\frac{NR}{r}}$ (if this number is not an integer or is not a divisor of N, we take the closest integer which is a divisor of N). Since the internal resistance of the battery is $\frac{n^2r}{N}$, the physical meaning of the solution obtained is as follows: the internal resistance of the battery must be as close as possible to the external resistance. **889.** $y=\frac{2}{3}h$. **891.** $(-\infty,\ 2)$, concave down; $(2,\ \infty)$, concave up; $M(2,\ 12)$, point of inflection. **892.** $(-\infty,\ \infty)$, concave up. **893.** $(-\infty,\ -3)$, concave down, $(-3,\ \infty)$, concave up; no points of inflection. **894.** $(-\infty,\ -6)$ and $(0,\ 6)$, concave up; $(-6,\ 0)$ and $(6,\ \infty)$, concave down; points of inflection $M_1\left(-6,\ -\frac{9}{2}\right)$, $O(0,\ 0)$, $M_2\left(6,\ \frac{9}{2}\right)$. **895.** $(-\infty,\ -\sqrt{3})$ and $(0,\ \sqrt{3})$, concave up; $(-\sqrt{3},\ 0)$ and $(\sqrt{3},\ \infty)$, concave down; points of inflection $M_{1,2}(\pm\sqrt{3},\ 0)$ and $O(0,\ 0)$. **896.** $\left((4k+1)\frac{\pi}{2},\ (4k+3)\frac{\pi}{2}\right)$, concave up; $\left((4k+3)\frac{\pi}{2},\ (4k+5)\frac{\pi}{2}\right)$, concave down ($k=0,\ \pm 1,\ \pm 2,\ \ldots$); points of inflection, $\left((2k+1)\frac{\pi}{2},\ 0\right)$. **897.** $(2k\pi,\ (2k+1)\pi)$, concave up; $((2k-1)\pi,\ 2k\pi)$, concave down ($k=0,\ \pm 1,\ \pm 2,\ \ldots$); the abscissas of the points of inflection are equal to $x=k\pi$. **898.** $\left(0,\ \frac{1}{\sqrt{e^3}}\right)$, concave down; $\left(\frac{1}{\sqrt{e^3}}, \infty\right)$, concave up; $M\left(\frac{1}{\sqrt{e^3}}, -\frac{3}{2e^3}\right)$ is a point of inflection. **899.** $(-\infty, 0)$, concave up; $(0, \infty)$, concave down; $O(0, 0)$ is a point of inflection. **900.** $(-\infty, -3)$ and $(-1, \infty)$, concave up; $(-3, -1)$, concave down; points of inflection are $M_1\left(-3, \frac{10}{e^3}\right)$ and $M_2\left(-1, \frac{2}{e}\right)$. **901.** $x=2$, $y=0$. **902.** $x=1$, $x=3$, $y=0$ **903.** $x=\pm 2$, $y=1$. **904.** $y=x$. **905.** $y=-x$, left, $y=x$, right. **906.** $y=-1$, left, $y=1$, right **907.** $x=\pm 1$, $y=-x$, left, $y=x$, right **908.** $y=-2$, left, $y=2x-2$, right. **909.** $y=2$ **910.** $x=0$, $y=1$, left, $y=0$, right. **911.** $x=0$, $y=1$. **912.** $y=0$. **913.** $x=-1$. **914.** $y=x-\pi$, left; $y=x+\pi$, right. **915.** $y=a$. **916.** $y_{max}=0$ when $x=0$; $y_{min}=-4$ when $x=2$; point of inflection, $M_1(1, -2)$. **917.** $y_{max}=1$ when $x=\pm\sqrt{3}$; $y_{min}=0$ when $x=0$; points of inflection $M_{1,2}\left(\pm 1, \frac{5}{9}\right)$ **918.** $y_{max}=4$ when $x=-1$; $y_{min}=0$ when $x=1$, point of inflection, $M_1(0, 2)$. **919.** $y_{max}=8$ when $x=-2$, $y_{min}=0$ when $x=2$; point of inflection, $M(0, 4)$. **920.** $y_{min}=-1$ when $x=0$; points of inflection $M_{1,2}(\pm\sqrt{5}, 0)$ and $M_{3,4}\left(\pm 1, -\frac{64}{125}\right)$. **921.** $y_{max}=-2$ when $x=0$; $y_{min}=2$ when $x=2$; asymptotes, $x=1$, $y=x-1$. **922.** Points of inflection $M_{1,2}(\pm 1, \mp 2)$; asymptote $x=0$. **923.** $y_{max}=-4$ when $x=-1$; $y_{min}=4$ when $x=1$; asymptote, $x=0$. **924.** $y_{min}=3$ when $x=1$; point of inflection, $M(-\sqrt[3]{2}, 0)$; asymptote, $x=0$. **925.** $y_{max}=\frac{1}{3}$ when $x=0$, points of inflection, $M_{1,2}\left(\pm 1, \frac{1}{4}\right)$; asymptote, $y=0$ **926.** $y_{max}=-2$ when $x=0$; asymptotes, $x=\pm 2$ and $y=0$. **927.** $y_{min}=-1$ when $x=-1$; $y_{max}=1$ when $x=1$; points of inflection, $O(0, 0)$ and $M_{1,2}\left(\pm 2\sqrt{3}, \pm\frac{\sqrt{3}}{2}\right)$; asymptote, $y=0$ **928.** $y_{max}=1$ when $x=4$; point of inflection, $M\left(5, \frac{8}{9}\right)$; asymptotes, $x=2$ and $y=0$. **929.** Point of inflection, $O(0, 0)$; asymptotes, $x=\pm 2$ and $y=0$. **930.** $y_{max}=-\frac{27}{16}$ when $x=\frac{8}{3}$; asymptotes, $x=0$, $x=4$ and $y=0$ **931.** $y_{max}=-4$ when $x=-1$; $y_{min}=4$ when $x=1$; asymptotes, $x=0$ and $y=3x$ **932.** $A(0, 2)$ and $B(4, 2)$ are end-points; $y_{max}=2\sqrt{2}$ when $x=2$ **933.** $A(-8, -4)$ and $B(8, 4)$ are end-points. Point of inflection, $O(0, 0)$. **934.** End-point, $A(-3, 0)$; $y_{min}=-2$ when $x=-2$. **935.** End-points, $A(-\sqrt{3}, 0)$, $O(0, 0)$ and $B(\sqrt{3}, 0)$; $y_{max}=\sqrt{2}$ when $x=-1$; point of inflection, $M\left(\sqrt{3+2\sqrt{3}}, \sqrt{6\sqrt{1+\frac{2}{\sqrt{3}}}}\right)$. **936.** $y_{max}=1$ when $x=0$, points of inflection, $M_{1,2}(\pm 1, 0)$. **937.** Points of inflection, $M_1(0, 1)$ and $M_2(1, 0)$; asymptote, $y=-x$. **938.** $y_{max}=0$ when $x=-1$; $y_{min}=-1$ (when $x=0$) **939.** $y_{max}=2$ when $x=0$; points of inflection, $M_{1,2}(\pm 1, \sqrt[3]{2})$; asymptote, $y=0$. **940.** $y_{min}=-4$ when $x=-4$; $y_{max}=4$ when $x=4$; point of inflection, $O(0, 0)$; asymptote, $y=0$. **941.** $y_{min}=\sqrt[3]{4}$ when $x=2$, $y_{min}=\sqrt[3]{4}$ when $x=4$; $y_{max}=2$ when $x=3$. **942.** $y_{min}=2$ when $x=0$; asymptote, $x=\pm 2$. **943.** Asymptotes, $x=\pm 2$ and $y=0$. **944.** $y_{min}=\frac{\sqrt{3}}{\sqrt[3]{2}}$ when $x=\sqrt{3}$;

$y_{max}=-\frac{\sqrt{3}}{\sqrt[3]{2}}$ when $x=-3$; points of inflection, $M_1\left(-3,\ -\frac{3}{2}\right)$, $O(0,\ 0)$ and $M_2\left(3,\ \frac{3}{2}\right)$; asymptotes, $x=\pm 1$ **945.** $y_{min}=\frac{3}{\sqrt[3]{2}}$ when $x=6$; point of inflection, $M\left(12,\ \frac{12}{\sqrt[3]{100}}\right)$; asymptote, $x=2$ **946.** $y_{max}=\frac{1}{e}$ when $x=1$; point of inflection, $M\left(2,\ \frac{2}{e^2}\right)$; asymptote, $y=0$. **947.** Points of inflection, $M_1\left(-3a,\ \frac{10a}{e^3}\right)$ and $M_2\left(-a,\ \frac{2a}{e}\right)$; asymptote, $y=0$. **948.** $y_{max}=e^2$ when $x=4$; points of inflection, $M_{1,2}\left(\frac{8\pm 2\sqrt{2}}{2},\ e^{\frac{3}{2}}\right)$; asymptote, $y=0$. **949.** $y_{max}=2$ when $x=0$; points of inflection, $M_{1,2}\left(\pm 1,\ \frac{3}{e}\right)$. **950.** $y_{max}=1$ when $x=\pm 1$; $y_{min}=0$ when $x=0$. **951.** $y_{max}=0.74$ when $x=e^2\approx 7.39$; point of inflection, $M(e^{8/3}\approx 14.39,\ 0.70)$; asymptotes, $x=0$ and $y=0$. **952.** $y_{min}=-\frac{a^2}{4e}$ when $x=\frac{a}{\sqrt{e}}$, point of inflection, $M\left(\frac{a}{\sqrt{e^3}},\ -\frac{3a^2}{4e^2}\right)$. **953.** $y_{min}=e$ when $x=e$; point of inflection, $M\left(e^2,\ \frac{e^2}{2}\right)$; asymptote, $x=1$; $y\to 0$ when $x\to 0$. **954.** $y_{max}=\frac{4}{e^2}\approx 0.54$ when $x=\frac{1}{e^2}-1\approx -0.86$; $y_{min}=0$ when $x=0$; point of inflection, $M\left(\frac{1}{e}-1\approx -0.63;\ \frac{1}{e}\approx 0.37\right)$; $y\to 0$ as $x\to -1+0$ (limiting end-point). **955.** $y_{min}=1$ when $x=\pm\sqrt{2}$; points of inflection, $M_{1,2}(\pm 1.89,\ 1.33)$; asymptotes, $x=\pm 1$. **956.** Asymptote, $y=0$. **957.** Asymptotes, $y=0$ (when $x\to +\infty$) and $y=-x$ (as $x\to -\infty$). **958.** Asymptotes, $x=-\frac{1}{e}$, $x=0$, $y=1$; the function is not defined on the interval $\left[-\frac{1}{e},\ 0\right]$. **959.** Periodic function with period 2π. $y_{min}=-\sqrt{2}$ when $x=\frac{5}{4}\pi+2k\pi$; $y_{max}=\sqrt{2}$ when $x=\frac{\pi}{4}+2k\pi$ ($k=0,\ \pm 1,\ \pm 2,\ \ldots$); points of inflection, $M_k\left(\frac{3}{4}\pi+k\pi,\ 0\right)$. **960.** Periodic function with period 2π. $y_{min}=-\frac{3}{4}\sqrt{3}$ when $x=\frac{5}{3}\pi+2k\pi$; $y_{max}=\frac{3}{4}\sqrt{3}$ when $x=\frac{\pi}{3}+2k\pi$ ($k=0,\ \pm 1,\ \pm 2,\ \ldots$); points of inflection, $M_k(k\pi,\ 0)$ and $N_k\left(\arccos\left(-\frac{1}{4}\right)+2k\pi,\ \frac{3}{16}\sqrt{15}\right)$. **961.** Periodic function with period 2π. On the interval $[-\pi,\ \pi]$, $y_{max}=\frac{1}{4}$ when $x=\pm\frac{\pi}{3}$; $y_{min}=-2$ when $x=\pm\pi$; $y_{min}=0$ when $x=0$; points of inflection, $M_{1,2}(\pm 0.57,\ 0.13)$ and $M_{3,4}(\pm 2.20,\ -0.95)$. **962.** Odd periodic function with period 2π. On interval $[0,\ 2\pi]$, $y_{max}=1$ when $x=0$; $y_{min}=0.71$, when $x=\frac{\pi}{4}$; $y_{max}=1$ when

$x=\frac{\pi}{2}$; $y_{min}=-1$ when $x=\pi$; $y_{max}=-0.71$ when $x=\frac{5}{4}\pi$; $y_{min}=-1$ when $x=\frac{3}{2}\pi$; $y_{max}=1$ when $x=2\pi$; points of inflection, $M_1(0.36,\ 0.86)$; $M_2(1.21,\ 0.86)$; $M_3(2.36,\ 0)$; $M_4(3.51,\ -0.86)$; $M_5(4.35,\ -0.86)$; $M_6(5.50,\ 0)$. **963.** Periodic function with period 2π. $y_{min}=\frac{\sqrt{2}}{2}$ when $x=\frac{\pi}{4}+2k\pi$; $y_{max}=-\frac{\sqrt{2}}{2}$ when $x=-\frac{3}{4}\pi+2k\pi$ ($k=0,\ \pm 1,\ \pm 2,\ \ldots$); asymptotes, $x=\frac{3}{4}\pi+k\pi$ **964.** Periodic function with period π; points of inflection, $M_k\left(\frac{\pi}{4}+k\pi,\ \frac{\sqrt{2}}{2}\right)$ ($k=0,\ \pm 1,\ \pm 2,\ \ldots$); asymptotes, $x=\frac{3}{4}\pi+k\pi$. **965.** Even periodic function with period 2π On the interval $[0,\ \pi$ $y_{max}=\frac{4}{3\sqrt{3}}$ when $x=\arccos\frac{1}{\sqrt{3}}$; $y_{max}=0$ when $x=\pi$; $y_{min}=-\frac{4}{3\sqrt{3}}$ when $x=\arccos\left(-\frac{1}{\sqrt{3}}\right)$; $y_{min}=0$ when $x=0$; points of inflection, $M_1\left(\frac{\pi}{2},\ 0\right)$; $M_2\left(\arcsin\frac{\sqrt{2}}{3},\ \frac{4\sqrt{7}}{27}\right)$; $M_3\left(\pi-\arcsin\frac{\sqrt{2}}{3},\ -\frac{4\sqrt{7}}{27}\right)$. **966.** Even periodic function with period 2π. On the interval $[0,\ \pi]$ $y_{max}=1$ when $x=0$; $y_{max}=\frac{2}{3\sqrt{6}}$ when $x=\arccos\left(-\frac{1}{\sqrt{6}}\right)$; $y_{min}=-\frac{2}{3\sqrt{6}}$ when $x=\arccos\frac{1}{\sqrt{6}}$; $y_{min}=-1$ when $x=\pi$; points of inflection, $M_1\left(\frac{\pi}{2},\ 0\right)$; $M_2\left(\arccos\sqrt{\frac{13}{18}},\ \frac{4}{9}\sqrt{\frac{13}{18}}\right)$; $M_3\left(\arccos\left(-\sqrt{\frac{13}{18}}\right),\ -\frac{4}{9}\sqrt{\frac{13}{18}}\right)$. **967.** Odd function. Points of inflection, $M_k(k\pi,\ k\pi)$ ($k=0,\ \pm 1,\ \pm 2,\ \ldots$). **968.** Even function. End-points, $A_{1,2}(\pm 2\ 83,\ -1\ 57)$ $y_{max}=1\ 57$ when $x=0$ (cusp); points of inflection, $M_{1,2}(\pm 1.54,\ -0.34)$. **969.** Odd function. Limiting points of graph $(-1,\ -\infty)$ and $(1,\ +\infty)$. Point of inflection, $O(0,\ 0)$; asymptotes, $x=\pm 1$. **970.** Odd function. $y_{max}=\frac{\pi}{2}-1+2k\pi$ when $x=\frac{\pi}{4}+k\pi$; $y_{min}=\frac{3}{2}\pi+1+2k\pi$ when $x=\frac{3}{4}\pi+k\pi$; points of inflection, $M_k(k\pi,\ 2k\pi)$; asymptotes, $x=\frac{2k+1}{2}\pi$ ($k=0,\ \pm 1,\ \pm 2,\ \ldots$). **971.** Even function. $y_{min}=0$ when $x=0$; asymptotes, $y=-\frac{\pi}{2}x-1$ (as $x\to-\infty$) and $y=\frac{\pi}{2}x-1$ (as $x\to+\infty$). **972.** $y_{min}=0$ when $x=0$ (node); asymptote, $y=1$. **973.** $y_{min}=1+\frac{\pi}{2}$ when $x=1$; $y_{max}=\frac{3\pi}{2}-1$ when $x=-1$; point of inflection (centre of symmetry) $(0,\ \pi)$; asymptotes, $y=x+2\pi$ (left) and $y=x$ (right). **974.** Odd function. $y_{min}=1.285$ when $x=1$; $y_{max}=1.856$ when $x=-1$; point of inflection, $M\left(0,\ \frac{\pi}{2}\right)$; asymptotes, $y=\frac{x}{2}+\pi$ (when $x\to-\infty$) and $y=\frac{x}{2}$ (as $x\to+\infty$). **975.** Asymptotes, $x=0$ and $y=x-\ln 2$.

976. $y_{\min}=1.32$ when $x=\pm 1$; asymptote, $x=0$. **977.** Periodic function with period 2π. $y_{\min}=\frac{1}{6}$ when $x=\frac{3}{2}\pi+2k\pi$; $y_{\max}=e$ when $x=\frac{\pi}{2}+2k\pi$ $(k=0, \pm 1, \pm 2, \ldots)$; points of inflection, $M_k\left(\arcsin\frac{\sqrt{5}-1}{2}+2k\pi,\ e^{\frac{\sqrt{5}-1}{2}}\right)$ and $N_k\left(-\arcsin\frac{\sqrt{5}-1}{2}+(2k+1)\pi,\ e^{\frac{\sqrt{5}+1}{2}}\right)$. **978.** End-points, $A(0, 1)$ and $B(1, 4.81)$. Point of inflection, $M(0.28, 1.74)$. **979.** Points of inflection, $M(0.5, 1.59)$; asymptotes, $y=0.21$ (as $x\to-\infty$) and $y=4.81$ (as $x\to+\infty$). **980.** The domain of definition of the function is the set of intervals $(2k\pi, 2k\pi+\pi)$, where $k=0, \pm 1, \pm 2, \ldots$ Periodic function with period 2π. $y_{\max}=0$ when $x=\frac{\pi}{2}+2k\pi$ $(k=0, \pm 1, \pm 2, \ldots)$; asymptotes, $x=k\pi$. **981.** The domain of definition is the set of intervals $\left[\left(2k-\frac{1}{2}\right)\pi, \left(2k+\frac{1}{2}\right)\pi\right]$, where k is an integer. Periodic function with period 2π. Points of inflection, $M_k(2k\pi, 0)$ $(k=0, \pm 1, \pm 2, \ldots)$; asymptotes, $x=\pm\frac{\pi}{2}+2k\pi$. **982.** Domain of definition, $x>0$; monotonic increasing function; asymptote, $x=0$. **983.** Domain of definition, $|x-2k\pi|<\frac{\pi}{2}$ $(k=0, \pm 1, \pm 2, \ldots)$. Periodic function with period 2π $y_{\min}=1$ when $x=2k\pi$ $(k=0, \pm 1, \pm 2, \ldots)$; asymptotes, $x=\frac{\pi}{2}+k\pi$. **984.** Asymptote, $y=1.57$; $y\to-1.57$ as $x\to 0$ (limiting end-point). **985.** End-points, $A_{1,2}(\pm 1.31, 1.57)$; $y_{\min}=0$ when $x=0$. **986.** $y_{\min}=\left(\frac{1}{e}\right)^{\frac{1}{e}}\approx 0.69$ when $x=\frac{1}{e}\approx 0.37$; $y\to 1$ as $x\to+0$ **987.** Limiting end-point, $A(+0, 0)$; $y_{\max}=e^{\frac{1}{e}}\approx 1.44$ when $x=e\approx 2.72$; asymptote, $y=1$; point of inflection, $M_1(0.58, 0.12)$ and $M_2(4.35, 1.40)$. **988.** $x_{\min}=-1$ when $t=1$ $(y=3)$; $y_{\min}=-1$ when $t=-1$ $(x=3)$ **989.** To obtain the graph it is sufficient to vary t from 0 to 2π. $x_{\min}=-a$ when $t=\pi$ $(y=0)$; $x_{\max}=a$ when $t=0$ $(y=0)$; $y_{\min}=-a$ (cusp) when $t=+\frac{3\pi}{2}$ $(x=0)$; $y_{\max}=+a$ (cusp) when $t=\frac{\pi}{2}$ $(x=0)$; points of inflection when $t=\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ $\left(x=\pm\frac{a}{2\sqrt{2}}, y=\pm\frac{a}{\sqrt{2}}\right)$.

990. $x_{\min}=-\frac{1}{e}$ when $t=-1$ $(y=-e)$; $y_{\max}=\frac{1}{e}$ when $t=1$ $(x=e)$; points of inflection when $t=-\sqrt{2}$, i.e., $\left(-\frac{\sqrt{2}}{e^{\sqrt{2}}},\ -\sqrt{2}e^{\sqrt{2}}\right)$ and when $t=\sqrt{2}$, i.e., $\left(\sqrt{2}\,e^{\sqrt{2}},\ \frac{\sqrt{2}}{e^{\sqrt{2}}}\right)$; asymptotes, $x=0$ and $y=0$. **991.** $x_{\min}=1$ and $y_{\min}=1$ when $t=0$ (cusp); asymptote, $y=2x$ when $t\to+\infty$. **992.** $y_{\min}=0$ when $t=0$.

993. $ds=\frac{a}{y}dx$, $\cos\alpha=\frac{y}{a}$; $\sin\alpha=-\frac{x}{a}$. **994.** $ds=\frac{1}{a}\sqrt{\frac{a^4-c^2x^2}{a^2-x^2}}\,dx$; $\cos\alpha=\frac{a\sqrt{a^2-x^2}}{\sqrt{a^4-c^2x^2}}$; $\sin\alpha=-\frac{bx}{\sqrt{a^2-c^2x^2}}$, where $c=\sqrt{a^2-b^2}$. **995.** $ds=\frac{1}{y}\sqrt{p^2+y^2}\,dx$; $\cos\alpha=\frac{y}{\sqrt{p^2+y^2}}$; $\sin\alpha=\frac{p}{\sqrt{p^2+y^2}}$. **996.** $ds=\sqrt[3]{\frac{a}{x}}\,dx$; $\cos\alpha=\sqrt[3]{\frac{x}{a}}$; $\sin\alpha=-\sqrt[3]{\frac{y}{a}}$. **997.** $ds=\cosh\frac{x}{a}dx$; $\cos\alpha=\frac{1}{\cosh\frac{x}{a}}$; $\sin\alpha=\tanh\frac{x}{a}$. **998.** $ds=2a\sin\frac{t}{2}dt$; $\cos\alpha=\sin\frac{t}{2}$; $\sin\alpha=\cos\frac{t}{2}$. **999.** $ds=3a\sin t\cos t\,dt$; $\cos\alpha=-\cos t$; $\sin\alpha=\sin t$. **1000.** $ds=a\sqrt{1+\varphi^2}\,d\varphi$; $\cos\beta=\frac{1}{\sqrt{1+\varphi^2}}$. **1001.** $ds=\frac{a}{\varphi^2}\sqrt{1+\varphi^2}\,d\varphi$; $\cos\beta=-\frac{1}{\sqrt{1+\varphi^2}}$. **1002.** $ds=\frac{a}{\cos^3\frac{\varphi}{2}}d\varphi$; $\sin\beta=\cos\frac{\varphi}{2}$. **1003.** $ds=a\cos\frac{\varphi}{2}d\varphi$; $\sin\beta=\cos\frac{\varphi}{2}$. **1004.** $ds=r\sqrt{1+(\ln a)^2}\,d\varphi$; $\sin\beta=\frac{1}{\sqrt{1+(\ln\alpha)^2}}$. **1005.** $ds=\frac{a^2}{r}d\varphi$; $\sin\beta=\cos 2\varphi$. **1006.** $K=36$. **1007.** $K=\frac{1}{3\sqrt{2}}$. **1008.** $K_A=\frac{a}{b^2}$; $K_B=\frac{b}{a^2}$. **1009.** $K=\frac{6}{13\sqrt{13}}$. **1010.** $K=\frac{3}{a\sqrt{2}}$ at both vertices. **1011.** $\left(\frac{9}{8},\ 3\right)$ and $\left(\frac{9}{8},\ -3\right)$. **1012.** $\left(-\frac{\ln 2}{2},\ \frac{\sqrt{2}}{2}\right)$. **1013.** $R=\left|\frac{(1+9x^4)^{3/2}}{6x}\right|$. **1014.** $R=\frac{(b^4x^2+a^4y^2)^{3/2}}{a^4b^4}$. **1015.** $R=\left|\frac{(y^2+1)^2}{4y}\right|$. **1016.** $R=\left|\frac{3}{2}a\sin 2t\right|$. **1017.** $R=|at|$. **1018.** $R=|r\sqrt{1+k^2}|$. **1019.** $R=\left|\frac{4}{3}a\cos\frac{\varphi}{2}\right|$. **1020.** $R_{\text{least}}=|p|$. **1022.** (2,2). **1023.** $\left(-\frac{11}{2}a,\ \frac{16}{3}a\right)$. **1024.** $(x-3)^2+\left(y-\frac{3}{2}\right)^2=\frac{1}{4}$. **1025.** $(x+2)^2+(y-3)^2=8$. **1026.** $pY^2=\frac{8}{27}(X-p)^3$ (semicubical parabola). **1027.** $(aX)^{\frac{2}{3}}+(bY)^{\frac{2}{3}}=c^{\frac{4}{3}}$, where $c^2=a^2-b^2$.

Chapter IV

In the answers of this section the arbitrary additive constant C is omitted for the sake of brevity. **1031.** $\frac{5}{7}a^2x^7$. **1032.** $2x^3+4x^2+3x$. **1033.** $\frac{x^4}{4}+\frac{(a+b)x^3}{3}+\frac{abx^2}{2}$. **1034.** $a^2x+\frac{abx^4}{2}+\frac{b^2x^7}{7}$. **1035.** $\frac{2x}{3}\sqrt{2px}$. **1036.** $\frac{nx^{\frac{n-1}{n}}}{n-1}$.

1037. $\sqrt[n]{nx}$. **1038.** $a^2x-\frac{9}{5}a^{\frac{4}{3}}x^{\frac{5}{3}}+\frac{9}{7}a^{\frac{2}{3}}x^{\frac{7}{3}}-\frac{x^3}{3}$. **1039.** $\frac{2x^2\sqrt{x}}{5}+x$.
1040. $\frac{3x^4\sqrt[3]{x}}{13}-\frac{3x^2\sqrt[3]{x}}{7}-6\sqrt[3]{x}$. **1041.** $\frac{2x^{2m}\sqrt{x}}{4m+1}-\frac{4x^{m+n}\sqrt{x}}{2m+2n+1}+\frac{2x^{2n}\sqrt{x}}{4n+1}$.
1042. $2a\sqrt{ax}-4ax+4x\sqrt{ax}-2x^2+\frac{2x^3}{5\sqrt{ax}}$ **1043.** $\frac{1}{\sqrt{7}}\operatorname{arc\,tan}\frac{x}{\sqrt{7}}$.
1044. $\frac{1}{2\sqrt{10}}\ln\left|\frac{x-\sqrt{10}}{x+\sqrt{10}}\right|$. **1045.** $\ln(x+\sqrt{4+x^2})$. **1046.** $\operatorname{arc\,sin}\frac{x}{2\sqrt{2}}$.
1047. $\operatorname{arc\,sin}\frac{x}{\sqrt{2}}-\ln(x+\sqrt{x^2+2})$. **1048*.** a) $\tan x-x$. **Hint.** Put $\tan^2 x=\sec^2 x-1$; b) $x-\tanh x$. **Hint.** Put $\tanh^2 x=1-\frac{1}{\cosh^2 x}$. **1049.** a) $-\cot x-x$; b) $x-\coth x$. **1050.** $\frac{(3e)^x}{\ln 3+1}$. **1051.** $a\ln\left|\frac{c}{a-x}\right|$. **Solution.** $\int\frac{a}{a-x}dx=-a\int\frac{d(a-x)}{a-x}=-a\ln|a-x|+a\ln c=a\ln\left|\frac{c}{a-x}\right|$. **1052.** $x+\ln|2x+1|$.
Solution. Dividing the numerator by the denominator, we get $\frac{2x+3}{2x+1}=1+\frac{2}{2x+1}$. Whence $\int\frac{2x+3}{2x+1}dx=\int dx+\int\frac{2\,dx}{2x+1}=x+\int\frac{d(2x+1)}{2x+1}=x+\ln|2x+1|$. **1053.** $-\frac{3}{2}x+\frac{11}{4}\ln|3+2x|$. **1054.** $\frac{x}{b}-\frac{a}{b^2}\ln|a+bx|$.
1055. $\frac{a}{\alpha}x+\frac{b\alpha-a\beta}{\alpha^2}\ln|\alpha x+\beta|$. **1056.** $\frac{x^2}{2}+x+2\ln|x-1|$. **1057.** $\frac{x^2}{2}+2x+\ln|x+3|$. **1058.** $\frac{x^4}{4}+\frac{x^3}{3}+x^2+2x+3\ln|x-1|$. **1059.** $a^2x+2ab\ln|x-a|-\frac{b^2}{x-a}$. **1060.** $\ln|x+1|+\frac{1}{x+1}$. **Hint.** $\int\frac{x\,dx}{(x+1)^2}=\int\frac{(x+1)-1}{(x+1)^2}dx=\int\frac{dx}{x+1}-\int\frac{dx}{(x+1)^2}$. **1061.** $-2b\sqrt{1-y}$. **1062.** $-\frac{2}{3b}\sqrt{(a-bx)^3}$.
1063. $\sqrt{x^2+1}$. **Solution.** $\int\frac{x\,dx}{\sqrt{x^2+1}}=\frac{1}{2}\int\frac{d(x^2+1)}{\sqrt{x^2+1}}=\sqrt{x^2+1}$. **1064.** $2\sqrt{x}+\frac{\ln^2 x}{2}$. **1065.** $\frac{1}{\sqrt{15}}\operatorname{arc\,tan}x\sqrt{\frac{3}{5}}$. **1066.** $\frac{1}{4\sqrt{14}}\ln\left|\frac{x\sqrt{7}-2\sqrt{2}}{x\sqrt{7}+2\sqrt{2}}\right|$.
1067. $\frac{1}{2\sqrt{a^2-b^2}}\ln\left|\frac{\sqrt{a+b}+x\sqrt{a-b}}{\sqrt{a+b}-x\sqrt{a-b}}\right|$. **1068.** $x-\sqrt{2}\operatorname{arc\,tan}\frac{x}{\sqrt{2}}$.
1069. $-\left(\frac{x^2}{2}+\frac{a^2}{2}\ln|a^2-x^2|\right)$ **1070.** $x-\frac{5}{2}\ln(x^2+4)+\arctan\frac{x}{2}$.
1071. $\frac{1}{2\sqrt{2}}\ln(2\sqrt{2}x+\sqrt{7+8x^2})$. **1072.** $\frac{1}{\sqrt{5}}\operatorname{arc\,sin}x\sqrt{\frac{5}{7}}$.
1073. $\frac{1}{3}\ln|3x^2-2|-\frac{5}{2\sqrt{6}}\ln\left|\frac{x\sqrt{3}-\sqrt{2}}{x\sqrt{3}+\sqrt{2}}\right|$. **1074.** $\frac{3}{\sqrt{35}}\operatorname{arc\,tan}\sqrt{\frac{5}{7}}x-$

$-\frac{1}{5}\ln(5x^2+7)$. 1075. $\frac{3}{5}\sqrt{5x^2+1}+\frac{1}{\sqrt{5}}\ln(x\sqrt{5}+\sqrt{5x^2+1})$. 1076. $\sqrt{x^2-4}+$ $+3\ln|x+\sqrt{x^2-4}|$. 1077. $\frac{1}{2}\ln|x^2-5|$. 1078. $\frac{1}{4}\ln(2x^2+3)$. 1079. $\frac{1}{2a}\ln(a^2x^2+b^2)+\frac{1}{a}\operatorname{arc\,tan}\frac{ax}{b}$. 1080. $\frac{1}{2}\operatorname{arc\,sin}\frac{x^2}{a^2}$. 1081. $\frac{1}{3}\operatorname{arc\,tan}x^3$. 1082. $\frac{1}{3}\ln|x^3+\sqrt{x^6-1}|$. 1083. $\frac{2}{3}\sqrt{(\operatorname{arc\,sin}x)^3}$. 1084. $\frac{\left(\operatorname{arc\,tan}\frac{x}{2}\right)^2}{4}$. 1085. $\frac{1}{8}\ln(1+4x^2)-\frac{\sqrt{(\operatorname{arc\,tan}2x)^3}}{3}$ 1086. $2\sqrt{\ln(x+\sqrt{1+x^2})}$. 1087. $-\frac{a}{m}e^{-mx}$. 1088. $-\frac{1}{3\ln 4}4^{2-3x}$. 1089. e^t+e^{-t}. 1090. $\frac{a}{2}e^{\frac{2x}{a}}+2x-$ $-\frac{a}{2}e^{-\frac{2x}{a}}$. 1091. $\frac{1}{\ln a-\ln b}\left(\frac{a^x}{b^x}-\frac{b^x}{a^x}\right)-2x$. 1092. $\frac{2}{3\ln a}\sqrt{a^{3x}}+\frac{2}{\ln a\sqrt{a^x}}$. 1093. $-\frac{1}{2e^{x^2+1}}$. 1094. $\frac{1}{2\ln 7}7^{x^2}$. 1095. $-e^{\frac{1}{x}}$. 1096. $\frac{2}{\ln 5}5^{\sqrt{x}}$. 1097. $\ln|e^x-1|$. 1098. $-\frac{2}{3b}\sqrt{(a-be^x)^3}$ 1099. $\frac{3a}{4}(e^{\frac{x}{a}}+1)^{\frac{4}{3}}$. 1100. $\frac{x}{3}-$ $-\frac{1}{3\ln 2}\ln(2^x+3)$ Hint. $\frac{1}{2^x+3}=\frac{1}{3}\left(1-\frac{2^x}{2^x+3}\right)$. 1101. $\frac{1}{\ln a}\operatorname{arc\,tan}(a^x)$. 1102. $-\frac{1}{2b}\ln\left|\frac{1+e^{-bx}}{1-e^{-bx}}\right|$. 1103. $\operatorname{arc\,sin}e^t$ 1104. $-\frac{1}{b}\cos(a+bx)$. 1105. $\sqrt{2}\sin\frac{x}{\sqrt{2}}$ 1106. $x-\frac{1}{2a}\cos 2ax$. 1107. $2\sin\sqrt{x}$. 1108. $-\ln 10\times$ $\times\cos(\log x)$ 1109. $\frac{x}{2}-\frac{\sin 2x}{4}$. Hint. Put $\sin^2 x=\frac{1}{2}(1-\cos 2x)$. 1110. $\frac{x}{2}+$ $+\frac{\sin 2x}{4}$ Hint. See hint in 1109 1111. $\frac{1}{a}\tan(ax+b)$. 1112. $-\frac{\cot ax}{a}-x$. 1113. $a\ln\left|\tan\frac{x}{2a}\right|$. 1114. $\frac{1}{15}\ln\left|\tan\left(\frac{5x}{2}+\frac{\pi}{8}\right)\right|$. 1115. $\frac{1}{a}\ln\left|\tan\frac{ax+b}{2}\right|$. 1116. $\frac{1}{2}\tan(x^2)$. 1117. $\frac{1}{2}\cos(1-x^2)$. 1118. $x-\frac{1}{\sqrt{2}}\cot x\sqrt{2}-$ $-\sqrt{2}\ln\left|\tan\frac{x\sqrt{2}}{2}\right|$. 1119. $-\ln|\cos x|$. 1120. $\ln|\sin x|$. 1121. $(a-b)\times$ $\times\ln\left|\sin\frac{x}{a-b}\right|$. 1122. $5\ln\left|\sin\frac{x}{5}\right|$. 1123. $-2\ln|\cos\sqrt{x}|$. 1124. $\frac{1}{2}\ln\times$ $\times|\sin(x^2+1)|$. 1125. $\ln|\tan x|$. 1126. $\frac{a}{2}\sin^2\frac{x}{a}$. 1127. $\frac{\sin^4 6x}{24}$. 1128. $-\frac{1}{4a\sin^4 ax}$. 1129. $-\frac{1}{3}\ln(3+\cos 3x)$. 1130. $-\frac{1}{2}\sqrt{\cos 2x}$. 1131. $-\frac{2}{9}\sqrt{(1+3\cos^2 x)^3}$. 1132. $\frac{3}{4}\tan^4\frac{x}{3}$. 1133. $\frac{2}{3}\sqrt{\tan^3 x}$. 1134. $-\frac{3\cot^{\frac{5}{3}}x}{5}$. 1135. $\frac{1}{3}\left(\tan 3x+\frac{1}{\cos 3x}\right)$. 1136. $\frac{1}{a}\left(\ln\left|\tan\frac{ax}{2}\right|+2\sin ax\right)$.

1137. $\frac{1}{3a}\ln|b-a\cot 3x|$. 1138. $\frac{2}{5}\cosh 5x-\frac{3}{5}\sinh 5x$. 1139. $-\frac{x}{2}+\frac{1}{4}\sinh 2x$. 1140. $\ln\left|\tanh\frac{x}{2}\right|$. 1141. $2\,\text{arc tan}\,e^x$. 1142. $\ln|\tanh x|$. 1143. $\ln\cosh x$. 1144. $\ln|\sinh x|$. 1145. $-\frac{5}{12}\sqrt[5]{(5-x^2)^6}$. 1146. $\frac{1}{4}\ln|x^4-4x+1|$. 1147. $\frac{1}{4\sqrt{5}}\times\text{arc tan}\frac{x^4}{\sqrt{5}}$. 1148. $-\frac{1}{2}e^{-x^2}$. 1149. $\sqrt{\frac{3}{2}}\,\text{arc tan}\left(x\sqrt{\frac{3}{2}}\right)-\frac{1}{\sqrt{3}}\ln(x\sqrt{3}+\sqrt{2+3x^2})$. 1150. $\frac{x^3}{3}-\frac{x^2}{2}+x-2\ln|x+1|$. 1151. $-\frac{2}{\sqrt{e^x}}$. 1152. $\ln|x+\cos x|$. 1153. $\frac{1}{3}\left(\ln|\sec 3x+\tan 3x|+\frac{1}{\sin 3x}\right)$. 1154. $-\frac{1}{\ln x}$. 1155. $\ln|\tan x+\sqrt{\tan^2 x-2}|$. 1156. $\sqrt{2}\,\text{arc tan}\,(x\sqrt{2})-\frac{1}{4(2x^2+1)}$. 1157. $\frac{a^{\sin x}}{\ln a}$. 1158. $\sqrt[3]{\frac{(x^3+1)^2}{2}}$. 1159. $\frac{1}{2}\,\text{arc sin}\,(x^2)$. 1160. $\frac{1}{a}\tan ax-x$. 1161. $\frac{x}{2}-\frac{\sin x}{2}$. 1162. $\text{arc sin}\frac{\tan x}{2}$. 1163. $a\ln\left|\tan\left(\frac{x}{2a}+\frac{\pi}{4}\right)\right|$. 1164. $\frac{3}{4}\sqrt[3]{(1+\ln x)^4}$. 1165. $-2\ln|\cos\sqrt{x-1}|$. 1166. $\frac{1}{2}\ln\left|\tan\frac{x^2}{2}\right|$. 1167. $e^{\text{arc tan}\,x}+\frac{\ln^2(1+x^2)}{4}+\text{arc tan}\,x$. 1168. $-\ln|\sin x+\cos x|$. 1169. $\sqrt{2}\ln\left|\tan\frac{x}{2\sqrt{2}}\right|-2x-\sqrt{2}\cos\frac{x}{\sqrt{2}}$. 1170. $x+\frac{1}{\sqrt{2}}\ln\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right|$. 1171. $\ln|x|+2\,\text{arc tan}\,x$. 1172. $e^{\sin^2 x}$. 1173. $\frac{5}{\sqrt{3}}\,\text{arc sin}\frac{x\sqrt{3}}{2}+\sqrt{4-3x^2}$. 1174. $x-\ln(1+e^x)$. 1175. $\frac{1}{\sqrt{a^2-b^2}}\,\text{arc tan}\,x\sqrt{\frac{a-b}{a+b}}$. 1176. $\ln(e^x+\sqrt{e^{2x}-2})$. 1177. $\frac{1}{a}\ln|\tan ax|$. 1178. $-\frac{T}{2\pi}\cos\left(\frac{2\pi t}{T}+\varphi_0\right)$. 1179. $\frac{1}{4}\ln\left|\frac{2+\ln x}{2-\ln x}\right|$. 1180. $-\frac{\left(\text{arc cos}\frac{x}{2}\right)^2}{2}$. 1181. $-e^{-\tan x}$. 1182. $\frac{1}{2}\,\text{arc sin}\left(\frac{\sin^2 x}{\sqrt{2}}\right)$. 1183. $-2\cot 2x$. 1184. $\frac{(\text{arc sin}\,x)^2}{2}-\sqrt{1-x^2}$. 1185. $\ln(\sec x+\sqrt{\sec^2 x+1})$. 1186. $\frac{1}{4\sqrt{5}}\ln\left|\frac{\sqrt{5}+\sin 2x}{\sqrt{5}-\sin 2x}\right|$. 1187. $\frac{1}{\sqrt{2}}\,\text{arc tan}\left(\frac{\tan x}{\sqrt{2}}\right)$. **Hint.** $\int\frac{dx}{1+\cos^2 x}=\int\frac{dx}{\sin^2 x+2\cos^2 x}=\int\frac{\frac{dx}{\cos^2 x}}{\tan^2 x+2}$. 1188. $\frac{2}{3}\sqrt{[\ln(x+\sqrt{1+x^2})]^3}$. 1189. $\frac{1}{3}\sinh(x^3+3)$. 1190. $\frac{1}{\ln 3}3^{\tanh x}$. 1191. a) $\frac{1}{\sqrt{2}}\,\text{arc cos}\frac{\sqrt{2}}{x}$ when $x>\sqrt{2}$; b) $-\ln(1+e^{-x})$;

c) $\frac{1}{80}(5x^2-3)^8$; d) $\frac{2}{3}\sqrt{(x+1)^3}-2\sqrt{x+1}$; e) $\ln(\sin x+\sqrt{1+\sin^2 x})$.
1192. $\frac{1}{4}\left[\frac{(2x+5)^{12}}{12}-\frac{5(2x+5)^{11}}{11}\right]$. 1193. $2\left(\frac{\sqrt{x^3}}{3}-\frac{x}{2}+2\sqrt{x}-2\ln|1+\sqrt{x}|\right)$.
1194. $\ln\left|\frac{\sqrt{2x+1}-1}{\sqrt{2x+1}+1}\right|$. 1195. $2\arctan\sqrt{e^x-1}$. 1196. $\ln x-\ln 2\ln|\ln x+2\ln 2|$. 1197. $\frac{(\arcsin x)^3}{3}$. 1198. $\frac{2}{3}(e^x-2)\sqrt{e^x+1}$. 1199. $\frac{2}{5}(\cos^2 x-5)\times\sqrt{\cos x}$. 1200. $\ln\left|\frac{1}{1+\sqrt{x^2+1}}\right|$. Hint. Put $x=\frac{1}{t}$. 1201. $-\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\arcsin x$. 1202. $-\frac{x^2}{3}\sqrt{2-x^2}-\frac{4}{3}\sqrt{2-x^2}$. 1203. $\sqrt{x^2-a^2}-a\arccos\frac{a}{x}$. 1204. $\arccos\frac{1}{x}$, if $x>0$, and $\arccos\left(-\frac{1}{x}\right)$ if $x<0$ *) Hint. Put $x=\frac{1}{t}$. 1205. $\sqrt{x^2+1}-\ln\left|\frac{1+\sqrt{x^2+1}}{x}\right|$. 1206. $-\frac{\sqrt{4-x^2}}{4x}$. Note. The substitution $x=\frac{1}{z}$ may be used in place of the trigonometric substitution.
1207. $\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\arcsin x$. 1208. $2\arcsin\sqrt{x}$. 1210. $\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}|$. 1211. $x\ln x-x$. 1212. $x\arctan x-\frac{1}{2}\ln(1+x^2)$.
1213. $x\arcsin x+\sqrt{1-x^2}$. 1214. $\sin x-x\cos x$. 1215. $\frac{x\sin 3x}{3}+\frac{\cos 3x}{9}$.
1216. $-\frac{x+1}{e^x}$. 1217. $-\frac{x\ln 2+1}{2^x\ln^2 2}$. 1218. $\frac{e^{3x}}{27}(9x^2-6x+2)$. Solution. In place of repeated integration by parts we can use the following method of undetermined coefficients:

$$\int x^2e^{3x}dx=(Ax^2+Bx+C)e^{3x}$$

or, after differentiation,

$$x^2e^{3x}=(Ax^2+Bx+C)\,3e^{3x}+(2Ax+B)e^{3x}.$$

Cancelling out e^{3x} and equating the coefficients of identical powers of x, we get:

$$1=3A;\ 0=3B+2A;\ 0=3C+B,$$

whence $A=\frac{1}{3}$; $B=-\frac{2}{9}$; $C=\frac{2}{27}$. In the general form, $\int P_n(x)\,e^{ax}\,dx=Q_n(x)\,e^{ax}$, where $P_n(x)$ is the given polynomial of degree n and $Q_n(x)$ is a polynomial of degree n with undetermined coefficients 1219. $-e^{-x}(x^2+5)$. Hint. See Problem 1218*. 1220. $-3e^{-\frac{x}{3}}(x^3+9x^2+54x+162)$. Hint. See

*) Henceforward, in similar cases we shall sometimes give an answer that is good for only a part of the domain of the integrand.

Problem 1218*. **1221.** $-\frac{x\cos 2x}{4}+\frac{\sin 2x}{8}$. **1222.** $\frac{2x^2+10x+11}{4}\sin 2x+$ $+\frac{2x+5}{4}\cos 2x$ **Hint.** It is also advisable to apply the method of undetermined coefficients in the form

$$\int P_n(x)\cos\beta x\,dx=Q_n(x)\cos\beta x+R_n(x)\sin\beta x,$$

where $P_n(x)$ is the given polynomial of degree n, and $Q_n(x)$ and $R_n(x)$ are polynomials of degree n with undetermined coefficients (see Problem 1218*). **1223.** $\frac{x^3}{3}\ln x-\frac{x^3}{9}$. **1224.** $x\ln^2 x-2x\ln x+2x$. **1225.** $-\frac{\ln x}{2x^2}-\frac{1}{4x^2}$. **1226.** $2\sqrt{x}\ln x-4\sqrt{x}$. **1227.** $\frac{x^2+1}{2}\arctan x-\frac{x}{2}$. **1228.** $\frac{x^2}{2}\arcsin x-\frac{1}{4}\times$ $\times\arcsin x+\frac{x}{4}\sqrt{1-x^2}$. **1229.** $x\ln(x+\sqrt{1+x^2})-\sqrt{1+x^2}$. **1230.** $-x\cot x+$ $+\ln|\sin x|$. **1231.** $-\frac{x}{\sin x}+\ln\left|\tan\frac{x}{2}\right|$. **1232.** $\frac{e^x(\sin x-\cos x)}{2}$. **1233.** $\frac{3^x(\sin x+\cos x\ln 3)}{1+(\ln 3)^2}$. **1234.** $\frac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}$. **1235.** $\frac{x}{2}[\sin(\ln x)-$ $-\cos(\ln x)]$. **1236.** $-\frac{e^{-x^2}}{2}(x^2+1)$. **1237.** $2e^{\sqrt{x}}(\sqrt{x}-1)$. **1238.** $\left(\frac{x^3}{3}-x^2+\right.$ $\left.+3x\right)\ln x-\frac{x^3}{9}+\frac{x^2}{2}-3x$. **1239.** $\frac{x^2-1}{2}\ln\frac{1-x}{1+x}-x$. **1240.** $-\frac{\ln^2 x}{x}-\frac{2\ln x}{x}-\frac{2}{x}$. **1241.** $[\ln(\ln x)-1]\cdot\ln x$. **1242.** $\frac{x^3}{3}\arctan 3x-\frac{x^2}{18}+\frac{1}{162}\ln(9x^2+1)$. **1243.** $\frac{1+x^2}{2}\times$ $\times(\arctan x)^2-x\arctan x+\frac{1}{2}\ln(1+x^2)$. **1244.** $x(\arcsin x)^2+2\sqrt{1-x^2}\times$ $\times\arcsin x-2x$. **1245.** $-\frac{\arcsin x}{x}+\ln\left|\frac{x}{1+\sqrt{1-x^2}}\right|$. **1246.** $-2\sqrt{1-x}\times$ $\times\arcsin\sqrt{x}+2\sqrt{x}$. **1247.** $\frac{x\tan 2x}{2}+\frac{\ln|\cos 2x|}{4}-\frac{x^2}{2}$. **1248.** $\frac{e^{-x}}{2}\times$ $\times\left(\frac{\cos 2x-2\sin 2x}{5}-1\right)$. **1249.** $\frac{x}{2}+\frac{x\cos(2\ln x)+2x\sin(2\ln x)}{10}$. **1250.** $-\frac{x}{2(x^2+1)}+\frac{1}{2}\arctan x$. **Solution.** Putting $u=x$ and $dv=\frac{x\,dx}{(x^2+1)^2}$, we get $du=dx$ and $v=-\frac{1}{2(x^2+1)}$. Whence $\int\frac{x^2\,dx}{(x^2+1)^2}=-\frac{x}{2(x^2+1)}+$ $+\int\frac{dx}{2(x^2+1)}=-\frac{x}{2(x^2+1)}+\frac{1}{2}\arctan x+C$. **1251.** $\frac{1}{2a^2}\left(\frac{1}{a}\arctan\frac{x}{a}+\right.$ $\left.+\frac{x}{x^2+a^2}\right)$. **Hint.** Utilize the identity $1\equiv\frac{1}{a^2}[(x^2+a^2)-x^2]$. **1252.** $\frac{x}{2}\times$ $\times\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}$. **Solution.** Put $u=\sqrt{a^2-x^2}$ and $dv=dx$; whence $du=-\frac{x\,dx}{\sqrt{a^2-x^2}}$ and $v=x$; we have $\int\sqrt{a^2-x^2}\,dx=x\sqrt{a^2-x^2}-\int\frac{-x^2\,dx}{\sqrt{a^2-x^2}}=$ $=x\sqrt{a^2-x^2}-\int\frac{(a^2-x^2)-a^2}{\sqrt{a^2-x^2}}dx=x\sqrt{a^2-x^2}-\int\sqrt{a^2-x^2}\,dx+a^2\int\frac{dx}{\sqrt{a^2-x^2}}$.

Consequently, $2\int \sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}$. **1253.** $\frac{x}{2}\sqrt{A+x^2}+\frac{A}{2}\ln|x+\sqrt{A+x^2}|$. **Hint.** See Problem 1252*. **1254.** $-\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\arcsin\frac{x}{3}$. **Hint.** See Problem 1252*. **1255.** $\frac{1}{2}\arctan\frac{x+1}{2}$. **1256.** $\frac{1}{2}\times\ln\left|\frac{x}{x+2}\right|$. **1257.** $\frac{2}{\sqrt{11}}\arctan\frac{6x-1}{\sqrt{11}}$. **1258.** $\frac{1}{2}\ln(x^2-7x+13)+\frac{7}{\sqrt{3}}\times\arctan\frac{2x-7}{\sqrt{3}}$. **1259.** $\frac{3}{2}\ln(x^2-4x+5)+4\arctan(x-2)$. **1260.** $x-\frac{5}{2}\ln(x^2+3x+4)+\frac{9}{\sqrt{7}}\arctan\frac{2x+3}{\sqrt{7}}$. **1261.** $x+3\ln(x^2-6x+10)+8\arctan(x-3)$.

1262. $\frac{1}{\sqrt{2}}\arcsin\frac{4x-3}{5}$. **1263.** $\arcsin(2x-1)$. **1264.** $\ln\left|x+\frac{p}{2}+\sqrt{x^2+px+q}\right|$.

1265. $3\sqrt{x^2-4x+5}$. **1266.** $-2\sqrt{1-x-x^2}-9\arcsin\frac{2x+1}{\sqrt{5}}$.

1267. $\frac{1}{5}\sqrt{5x^2-2x+1}+\frac{1}{5\sqrt{5}}\ln\left(x\sqrt{5}-\frac{1}{\sqrt{5}}+\sqrt{5x^2-2x+1}\right)$.

1268. $\ln\left|\frac{x}{1+\sqrt{1-x^2}}\right|$. **1269.** $-\arcsin\frac{2-x}{x\sqrt{5}}$. **1270.** $\arcsin\frac{2-x}{(1-x)\sqrt{2}}\ (x>\sqrt{2})$.

1271. $-\arcsin\frac{1}{x+1}$. **1272.** $\frac{x+1}{2}\sqrt{x^2+2x+5}+2\ln(x+1+\sqrt{x^2+2x+5})$.

1273. $\frac{2x-1}{4}\sqrt{x-x^2}+\frac{1}{8}\arcsin(2x-1)$. **1274.** $\frac{2x+1}{4}\sqrt{2-x-x^2}+\frac{9}{8}\arcsin\frac{2x+1}{3}$. **1275.** $\frac{1}{4}\ln\left|\frac{x^2-3}{x^2-1}\right|$. **1276.** $-\frac{1}{\sqrt{3}}\arctan\frac{3-\sin x}{\sqrt{3}}$.

1277. $\ln\left(e^x+\frac{1}{2}+\sqrt{1+e^x+e^{2x}}\right)$ **1278.** $-\ln|\cos x+2+\sqrt{\cos^2 x+4\cos x+1}|$.

1279. $-\sqrt{1-4\ln x-\ln^2 x}-2\arcsin\frac{2+\ln x}{\sqrt{5}}$. **1280.** $\frac{1}{a-b}\ln\left|\frac{x+b}{x+a}\right|$.

1281. $x+3\ln|x-3|-3\ln|x-2|$ **1282.** $\frac{1}{12}\ln\left|\frac{(x-1)(x+3)^3}{(x+2)^4}\right|$.

1283. $\ln\left|\frac{(x-1)^4(x-4)^5}{(x+3)^7}\right|$. **1284.** $5x+\ln\left|\frac{x^{\frac{1}{2}}(x-4)^{\frac{161}{6}}}{(x-1)^{\frac{7}{3}}}\right|$. **1285.** $\frac{1}{1+x}+\ln\left|\frac{x}{x+1}\right|$.

1286. $\frac{1}{4}x+\frac{1}{16}\ln\left|\frac{x^{16}}{(2x-1)^7(2x+1)^9}\right|$ **1287.** $\frac{x^2}{2}-\frac{11}{(x-2)^2}-\frac{8}{x-2}$.

1288. $-\frac{9}{2(x-3)}-\frac{1}{2(x+1)}$. **1289.** $\frac{8}{49(x-5)}-\frac{27}{49(x+2)}+\frac{30}{343}\ln\left|\frac{x-5}{x+2}\right|$.

1290. $-\frac{1}{2(x^2-3x+2)^2}$. **1291.** $x+\ln\left|\frac{x}{\sqrt{x^2+1}}\right|$ **1292.** $x+\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\arctan x$. **1293.** $\frac{1}{52}\ln|x-3|-\frac{1}{20}\ln|x-1|+\frac{1}{65}\ln(x^2+4x+5)+\frac{7}{130}\times$

$\times \arctan(x+2)$. **1294.** $\frac{1}{6}\ln\frac{(x+1)^2}{x^2-x+1}+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$. **1295.** $\frac{1}{4\sqrt{2}}\times$ $\times\ln\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\frac{\sqrt{2}}{4}\arctan\frac{x\sqrt{2}}{1-x^2}$. **1296.** $\frac{1}{4}\ln\frac{x^2+x+1}{x^2-x+1}+\frac{1}{2\sqrt{3}}\times$ $\times\arctan\frac{x^2-1}{x\sqrt{3}}$. **1297.** $\frac{x}{2(1+x^2)}+\frac{\arctan x}{2}$. **1298.** $\frac{2x-1}{2(x^2+2x+2)}+$ $+\arctan(x+1)$. **1299.** $\ln|x+1|+\frac{x+2}{3(x^2+x+1)}+\frac{5}{3\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}-$ $-\frac{1}{2}\ln(x^2+x+1)$. **1300.** $\frac{3x-17}{2(x^2-4x+5)}+\frac{1}{2}\ln(x^2-4x+5)+\frac{15}{2}\arctan(x-2)$.
1301. $\frac{-x^2+x}{4(x+1)(x^2+1)}+\frac{1}{2}\ln|x+1|-\frac{1}{4}\ln(x^2+1)+\frac{1}{4}\arctan x$.
1302. $-\frac{3}{8}\arctan x-\frac{x}{4(x^4-1)}+\frac{3}{16}\ln\left|\frac{x-1}{x+1}\right|$. **1303.** $\frac{15x^5+40x^3+33x}{48(1+x^2)^3}+$ $+\frac{15}{48}\arctan x$. **1304.** $x-\frac{x-1}{x^2-2x+2}+2\ln(x^2-2x+2)+3\arctan(x-1)$.
1305. $\frac{1}{21}(8\ln|x^3+8|-\ln|x^3+1|)$. **1306.** $\frac{1}{2}\ln|x^4-1|-$ $-\frac{1}{4}\ln|x^8+x^4-1|-\frac{1}{2\sqrt{5}}\ln\left|\frac{2x^4+1-\sqrt{5}}{2x^4+1+\sqrt{5}}\right|$. **1307.** $-\frac{13}{2(x-4)^2}+\frac{3}{x-4}+$ $+2\ln\left|\frac{x-4}{x-2}\right|$. **1308.** $\frac{1}{3}\left(2\ln\left|\frac{x^3+1}{x^3}\right|-\frac{1}{x^3}-\frac{1}{x^3+1}\right)$. **1309.** $\frac{1}{x-1}+$ $+\ln\left|\frac{x-2}{x-1}\right|$. **1310.** $\ln|x|-\frac{1}{7}\ln|x^7+1|$. **Hint.** Put $1=(x^7+1)-x^7$.
1311. $\ln|x|-\frac{1}{5}\ln|x^5+1|+\frac{1}{5(x^5+1)}$. **1312.** $\frac{1}{3}\arctan(x+1)-\frac{1}{6}\arctan\times$ $\times\frac{x+1}{2}$. **1313.** $-\frac{1}{9(x-1)^9}-\frac{1}{4(x-1)^8}-\frac{1}{7(x-1)^7}$. **1314.** $-\frac{1}{5x^5}+\frac{1}{3x^3}-\frac{1}{x}-$ $-\arctan x$. **1315.** $2\sqrt{x-1}\left[\frac{(x-1)^3}{7}+\frac{3(x-1)^2}{5}+x\right]$. **1316.** $\frac{3}{10a^2}\times$ $\times\left[2\sqrt[3]{(ax+b)^5}-5b\sqrt[3]{(ax+b)^2}\right]$ **1317.** $2\arctan\sqrt{x+1}$. **1318.** $6\sqrt[6]{x}-$ $-3\sqrt[3]{x}+2\sqrt{x}-6\ln(1+\sqrt[6]{x})$. **1319.** $\frac{6}{7}x\sqrt[6]{x}-\frac{6}{5}\sqrt[6]{x^5}-\frac{3}{2}\sqrt[3]{x^2}+$ $+2\sqrt{x}-3\sqrt[3]{x}-6\sqrt[6]{x}-3\ln|1+\sqrt[3]{x}|+6\arctan\sqrt[6]{x}$.
1320. $\ln\left|\frac{(\sqrt{x+1}-1)^2}{x+2+\sqrt{x+1}}\right|-\frac{2}{\sqrt{3}}\arctan\frac{2\sqrt{x+1}+1}{\sqrt{3}}$. **1321.** $2\sqrt{x}-2\sqrt{2}\times$ $\times\arctan\sqrt{\frac{x}{2}}$. **1322.** $-2\arctan\sqrt{1-x}$. **1323.** $\frac{\sqrt{x^2-1}}{2}(x-2)+\frac{1}{2}\ln|x+$ $+\sqrt{x^2-1}|$. **1324.** $\frac{1}{3}\ln\frac{z^2+z+1}{(z-1)^2}+\frac{2}{\sqrt{3}}\arctan\frac{2z+1}{\sqrt{3}}+\frac{2z}{z^3-1}$, where $z=\sqrt[3]{\frac{x+1}{x-1}}$. **1325.** $-\frac{\sqrt{2x+3}}{x}$. **1326.** $\frac{2x+3}{8}\sqrt{x^2-x+1}+\frac{1}{16}\ln(2x-1+$

$+2\sqrt{x^2-x+1})$. **1327.** $-\frac{8+4x^2+3x^4}{15}\sqrt{1-x^2}$. **1328.** $\left(\frac{5}{16}x-\frac{5}{24}x^3+\frac{1}{6}x^5\right)\times$ $\times\sqrt{1+x^2}-\frac{5}{16}\ln(x+\sqrt{1+x^2})$. **1329.** $\left(\frac{1}{4x^4}+\frac{3}{8x^2}\right)\sqrt{x^2-1}-\frac{3}{8}\arcsin\frac{1}{x}$. **1330.** $\frac{1}{2(x+1)^2}\sqrt{x^2+2x}-\frac{1}{2}\arcsin\frac{1}{x+1}$. **1331.** $\frac{2x-1}{4}\sqrt{x^2-x+1}+\frac{19}{8}\ln\times$ $\times(2x-1+2\sqrt{x^2-x+1})$. **1332.** $\frac{1}{2}\frac{1+x^2}{\sqrt{1+2x^2}}$. **1333.** $\frac{1}{4}\ln\frac{\sqrt[4]{x^{-4}+1}+1}{\sqrt[4]{x^{-4}+1}-1}-$ $-\frac{1}{2}\arctan\sqrt[4]{x^{-4}+1}$. **1334.** $\frac{(2x^3-1)\sqrt{1+x^3}}{3x^3}$. **1335.** $\frac{1}{10}\ln\frac{(z-1)^2}{z^2+z+1}+$ $+\frac{\sqrt{3}}{5}\arctan\frac{2z+1}{\sqrt{3}}$, where $z=\sqrt[3]{1+x^5}$. **1336.** $-\frac{1}{8}\frac{4+3x^3}{x(2+x^3)^{2/3}}$. **1337.** $-2\sqrt[3]{(x^{-\frac{3}{4}}+1)^2}$. **1338.** $\sin x-\frac{1}{3}\sin^3 x$. **1339.** $-\cos x+\frac{2}{3}\cos^3 x-$ $-\frac{1}{5}\cos^5 x$. **1340.** $\frac{\sin^3 x}{3}-\frac{\sin^5 x}{5}$. **1341.** $\frac{1}{4}\cos^8\frac{x}{2}-\frac{1}{3}\cos^6\frac{x}{2}$. **1342.** $\frac{\sin^2 x}{2}-$ $-\frac{1}{2\sin^2 x}-2\ln|\sin x|$. **1343.** $\frac{3x}{8}-\frac{\sin 2x}{4}+\frac{\sin 4x}{32}$. **1344.** $\frac{x}{8}-\frac{\sin 4x}{32}$. **1345.** $\frac{x}{16}-\frac{\sin 4x}{64}+\frac{\sin^3 2x}{48}$. **1346.** $\frac{5}{16}x+\frac{1}{12}\sin 6x+\frac{1}{64}\sin 12x+$ $+\frac{1}{144}\sin^3 6x$. **1347.** $-\cot x-\frac{\cot^3 x}{3}$. **1348.** $\tan x+\frac{2}{3}\tan^3 x+\frac{1}{5}\tan^5 x$. **1349.** $-\frac{\cot^3 x}{3}-\frac{\cot^5 x}{5}$. **1350.** $\tan x+\frac{\tan^3 x}{3}-2\cot 2x$. **1351.** $\frac{1}{2}\tan^2 x+$ $+3\ln|\tan x|-\frac{3}{2\tan^2 x}-\frac{1}{4\tan^4 x}$. **1352.** $\frac{1}{\cos^2\frac{x}{2}}+2\ln\left|\tan\frac{x}{2}\right|$. **1353.** $\frac{\sqrt{2}}{2}\times$ $\times\left[\ln\left|\tan\frac{x}{2}\right|+\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|\right]$. **1354.** $\frac{-\cos x}{4\sin^4 x}-\frac{3\cos x}{8\sin^2 x}+\frac{3}{8}\ln\left|\tan\frac{x}{2}\right|$. **1355.** $\frac{\sin 4x}{16\cos^4 4x}+\frac{3\sin 4x}{32\cos^2 4x}+\frac{3}{32}\ln\left|\tan\left(2x+\frac{\pi}{4}\right)\right|$. **1356.** $\frac{1}{5}\tan 5x-x$. **1357.** $-\frac{\cot^2 x}{2}-\ln|\sin x|$. **1358.** $-\frac{1}{3}\cot^3 x+\cot x+x$. **1359.** $\frac{3}{2}\tan^2\frac{x}{3}+$ $+\tan^3\frac{x}{3}-3\tan\frac{x}{3}+3\ln\left|\cos\frac{x}{3}\right|+x$. **1360.** $\frac{x^2}{4}-\frac{\sin 2x^2}{8}$. **1361.** $-\frac{\cot^3 x}{3}$. **1362.** $-\frac{3}{4}\sqrt[3]{\cos^4 x}+\frac{3}{5}\sqrt[3]{\cos^{10} x}-\frac{3}{16}\sqrt[3]{\cos^{16} x}$. **1363.** $2\sqrt{\tan x}$. **1364.** $\frac{1}{2\sqrt{2}}\times$ $\times\ln\frac{z^2+z\sqrt{2}+1}{z^2-z\sqrt{2}+1}-\frac{1}{\sqrt{2}}\arctan\frac{z\sqrt{2}}{z^2-1}$, where $z=\sqrt{\tan x}$. **1365.** $-\frac{\cos 8x}{16}+$ $+\frac{\cos 2x}{4}$. **1366.** $-\frac{\sin 25x}{50}+\frac{\sin 5x}{10}$. **1367.** $\frac{3}{5}\sin\frac{5x}{6}+3\sin\frac{x}{6}$. **1368.** $\frac{3}{2}\cos\frac{x}{3}-$ $-\frac{1}{2}\cos x$. **1369.** $\frac{\sin 2ax}{4a}+\frac{x\cos 2b}{2}$. **1370.** $\frac{t\cos\varphi}{2}-\frac{\sin(2\omega t+\varphi)}{4\omega}$. **1371.** $\frac{\sin x}{2}+$

$+\frac{\sin 5x}{20}+\frac{\sin 7x}{28}$. **1372.** $\frac{1}{24}\cos 6x-\frac{1}{16}\cos 4x-\frac{1}{8}\cos 2x$. **1373.** $\frac{1}{4}\ln\left|\frac{\tan\frac{x}{2}-2}{\tan\frac{x}{2}+2}\right|$.

1374. $\frac{1}{\sqrt{2}}\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right|$. **1375.** $x-\tan\frac{x}{2}$. **1376.** $-x+\tan x+\sec x$.

1377. $\ln\left|\frac{\tan\frac{x}{2}-5}{\tan\frac{x}{2}-3}\right|$. **1378.** $\arctan\left(1+\tan\frac{x}{2}\right)$. **1379.** $\frac{12}{13}x-\frac{5}{13}\ln|2\sin x+$ $+3\cos x|$ **Solution.** We put $3\sin x+2\cos x\equiv\alpha(2\sin x+3\cos x)+$ $+\beta(2\sin x+3\cos x)'$ Whence $2\alpha-3\beta=3$, $3\alpha+2\beta=2$ and, consequently, $\alpha=\frac{12}{13}$, $\beta=-\frac{5}{13}$. We have $\int\frac{3\sin x+2\cos x}{2\sin x+3\cos x}dx=\frac{12}{13}\int dx-\frac{5}{13}\times$ $\times\int\frac{(2\sin x+3\cos x)'}{2\sin x+3\cos x}dx=\frac{12}{13}x-\frac{5}{13}\ln|2\sin x+3\cos x|$. **1380.** $-\ln|\cos x-\sin x|$.
1381. $\frac{1}{2}\arctan\left(\frac{\tan x}{2}\right)$ **Hint.** Divide the numerator and denominator of the fraction by $\cos^2 x$ **1382.** $\frac{1}{\sqrt{15}}\arctan\left(\frac{\sqrt{3}\tan x}{\sqrt{5}}\right)$. **Hint.** See Problem 1381.
1383. $\frac{1}{\sqrt{3}}\ln\left|\frac{2\tan x+3-\sqrt{13}}{2\tan x+3+\sqrt{13}}\right|$. **Hint.** See Problem 1381. **1384.** $\frac{1}{5}\ln\times$ $\times\left|\frac{\tan x-5}{\tan x}\right|$ **Hint.** See Problem 1381. **1385.** $-\frac{1}{2(1-\cos x)^2}$. **1386.** $\ln(1+\sin^2 x)$.
1387. $\frac{1}{2\sqrt{2}}\ln\frac{\sqrt{2}+\sin 2x}{\sqrt{2}-\sin 2x}$. **1388.** $\frac{1}{4}\ln\frac{5-\sin x}{1-\sin x}$. **1389.** $\frac{2}{\sqrt{3}}\arctan\times$ $\times\frac{2\tan\frac{x}{2}-1}{\sqrt{3}}-\frac{1}{\sqrt{2}}\arctan\frac{3\tan\frac{x}{2}-1}{2\sqrt{2}}$. **Hint.** Use the identity $\frac{1}{(2-\sin x)(3-\sin x)}=\frac{1}{2-\sin x}-\frac{1}{3-\sin x}$. **1390.** $-x+2\ln\left|\frac{\tan\frac{x}{2}}{\tan\frac{x}{2}+1}\right|$. **Hint.** Use the identity $\frac{1-\sin x+\cos x}{1+\sin x-\cos x}\equiv-1+\frac{2}{1+\sin x-\cos x}$. **1391.** $\frac{\cosh^3 x}{3}-\cosh x$.
1392. $\frac{3x}{8}+\frac{\sinh 2x}{4}+\frac{\sinh 4x}{32}$. **1393.** $\frac{\sinh^4 x}{4}$. **1394.** $-\frac{x}{8}+\frac{\sinh 4x}{32}$.
1395. $\ln\left|\tanh\frac{x}{2}\right|+\frac{1}{\cosh x}$. **1396.** $-2\coth 2x$. **1397.** $\ln(\cosh x)-\frac{\tanh^2 x}{2}$.
1398. $x-\coth x-\frac{\coth^3 x}{3}$. **1399.** $\arctan(\tanh x)$. **1400.** $\frac{2}{\sqrt{5}}\arctan\left(\frac{3\tanh\frac{x}{2}+2}{\sqrt{5}}\right)$ $\left[\text{or } \frac{2}{\sqrt{5}}\arctan(e^x\sqrt{5})\right]$. **1401.** $\frac{\sinh^2 x}{2}-\frac{\sinh 2x}{4}-\frac{x}{2}$. **Hint.** Use the identity $\frac{-1}{\sinh x-\cosh x}\equiv(\sinh x+\cosh x)$. **1402.** $\frac{1}{\sqrt{2}}\ln(\sqrt{2}\cosh x+\sqrt{\cosh 2x})$.

1403. $\frac{x+1}{2}\sqrt{3-2x-x^2}+2\arcsin\frac{x+1}{2}$. 1404. $\frac{x}{2}\sqrt{2+x^2}+\ln(x+\sqrt{2+x^2})$.
1405. $\frac{x}{2}\sqrt{9+x^2}-\frac{9}{2}\ln(x+\sqrt{9+x^2})$. 1406. $\frac{x-1}{2}\sqrt{x^2-2x+2}+$
$+\frac{1}{2}\ln(x-1+\sqrt{x^2-2x+2})$ 1407. $\frac{x}{2}\sqrt{x^2-4}-2\ln|x+\sqrt{x^2-4}|$.
1408. $\frac{2x+1}{4}\sqrt{x^2+x}-\frac{1}{8}\ln|2x+1+2\sqrt{x^2+x}|$. 1409. $\frac{x-3}{2}\sqrt{x^2-6x-7}-$
$-8\ln|x-3+\sqrt{x^2-6x-7}|$. 1410. $\frac{1}{64}(2x+1)(8x^2+8x+17)\sqrt{x^2+x+1}+$
$+\frac{27}{128}\ln(2x+1+2\sqrt{x^2+x+1})$. 1411. $2\sqrt{\frac{x-2}{x-1}}$ 1412. $\frac{x-1}{4\sqrt{x^2-2x+5}}$.
1413. $\frac{1}{\sqrt{2}}\arctan\frac{x\sqrt{2}}{\sqrt{1-x^2}}$ 1414. $\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{1+x^2}+x\sqrt{2}}{\sqrt{1+x^2}-x\sqrt{2}}\right|$ 1415. $\frac{e^{2x}}{2}\times$
$\times\left(x^4-2x^3+5x^2-5x+\frac{7}{2}\right)$ 1416. $\frac{1}{6}\left(x^3+\frac{x^2}{2}\sin 6x+\frac{x}{6}\cos 6x-\frac{1}{36}\sin 6x\right)$.
1417. $-\frac{x\cos 3x}{6}+\frac{\sin 3x}{18}+\frac{x\cos x}{2}-\frac{\sin x}{2}$ 1418. $\frac{e^{2x}}{8}(2-\sin 2x-\cos 2x)$.
1419. $\frac{e^x}{2}\left(\frac{2\sin 2x+\cos 2x}{5}-\frac{4\sin 4x+\cos 4x}{17}\right)$. 1420. $\frac{e^x}{2}[x(\sin x+\cos x)-\sin x]$.
1421. $-\frac{x}{2}+\frac{1}{3}\ln|e^x-1|+\frac{1}{6}\ln(e^x+2)$ 1422. $x-\ln(2+e^x+2\sqrt{e^{2x}+x+1})$.
1423. $\frac{1}{3}\left[x^3\ln\frac{1+x}{1-x}+\ln(1-x^2)+x^2\right]$ 1424. $x\ln^2(x+\sqrt{1+x^2})-2\sqrt{1+x^2}\times$
$\times\ln(x+\sqrt{1+x^2})+2x$. 1425. $\left(\frac{x^2}{2}-\frac{9}{100}\right)\arccos(5x-2)-\frac{5x+6}{100}\times$
$\times\sqrt{20x-25x^2-3}$. 1426. $\frac{\sin x\cosh x-\cos x\sinh x}{2}$. 1427. $I_n=\frac{1}{2(n-1)a^2}\times$
$\times\left[\frac{x}{(x^2+a^2)^{n-1}}+(2n-3)I_{n-1}\right]$; $I_2=\frac{1}{2a^2}\left(\frac{x}{x^2+a^2}+\frac{1}{a}\arctan\frac{x}{a}\right)$; $I_3=\frac{1}{4a^2}\times$
$\times\left[\frac{x(3x^2+5a^2)}{2a^2(x^2+a^2)^2}+\frac{3}{2a^3}\arctan\frac{x}{a}\right]$. 1428. $I_n=-\frac{\cos x\sin^{n-1}x}{n}+\frac{n-1}{n}I_{n-2}$;
$I_4=\frac{3x}{8}-\frac{\cos x\sin^3 x}{4}-\frac{3\sin 2x}{16}$; $I_5=-\frac{\cos x\sin^4 x}{5}-\frac{4}{15}\cos x\sin^2 x-\frac{8}{15}\cos x$.
1429. $I_n=\frac{\sin x}{(n-1)\cos^{n-1}x}+\frac{n-2}{n-1}I_{n-2}$; $I_3=\frac{\sin x}{2\cos^2 x}+\frac{1}{2}\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$;
$I_4=\frac{\sin x}{3\cos^3 x}+\frac{2}{3}\tan x$. 1430. $I_n=-x^ne^{-x}+nI_{n-1}$; $I_{10}=-e^{-x}(x^{10}+10x^9+$
$+10\cdot 9x^8+\ldots+10\cdot 9\cdot 8\ldots 2x+10\cdot 9\ldots 1)$. 1431. $\frac{1}{\sqrt{14}}\arctan\frac{\sqrt{2}(x-1)}{\sqrt{7}}$.
1432. $\ln\sqrt{x^2-2x+2}-4\arctan(x-1)$. 1433. $\frac{(x-1)^2}{2}+\frac{1}{4}\ln\left(x^2+x+\frac{1}{2}\right)+$
$+\frac{1}{2}\arctan(2x+1)$. 1434. $\frac{1}{5}\ln\sqrt{\frac{x^2}{x^2+5}}$. 1435. $2\ln\left|\frac{x+3}{x+2}\right|-\frac{1}{x+2}-\frac{1}{x+3}$.
1436. $\frac{1}{2}\left(\ln\left|\frac{x+1}{\sqrt{x^2+1}}\right|-\frac{1}{x+1}\right)$. 1437. $\frac{1}{4}\left(\frac{x}{x^2+2}+\frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}\right)$.

1438. $\frac{1}{4}\left(\frac{2x}{1-x^2}+\ln\left|\frac{x+1}{x-1}\right|\right)$. **1439.** $\frac{1}{6}\,\frac{x-2}{(x^2-x+1)^2}+\frac{1}{6}\,\frac{2x-1}{x^2-x+1}+\frac{2}{3\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$. **1440.** $\frac{x(3+2\sqrt{x})}{1-2\sqrt{x}}$. **1441.** $-\frac{1}{x}-\frac{4}{3x\sqrt{x}}-\frac{1}{2x^2}$. **1442.** $\ln\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)$. **1443.** $\sqrt{2x}-\frac{3}{5}\sqrt[6]{(2x)^5}$. **1444.** $-\frac{3}{\sqrt[3]{x+1}}$. **1445.** $\frac{2x-1}{\sqrt{4x^2-2x+1}}$. **1446.** $-2(\sqrt[4]{5-x}-1)^2-4\ln(1+\sqrt[4]{5-x})$. **1477.** $\ln|x+\sqrt{x^2-1}|-\frac{x}{\sqrt{x^2-1}}$. **1448.** $-\frac{1}{2}\sqrt{\frac{1-x^2}{1+x^2}}$. **1449.** $\frac{1}{2}\times\arcsin\frac{x^2+1}{\sqrt{2}}$. **1450.** $\frac{x-1}{\sqrt{x^2+1}}$. **1451.** $\frac{1}{8}\ln\left|\frac{\sqrt{4-x^2}-2}{x}\right|-\frac{1}{8\sqrt{3}}\times\arcsin\frac{2(x+1)}{x+4}$. Hint. $\frac{1}{x^2+4x}=\frac{1}{4}\left(\frac{1}{x}-\frac{1}{x+4}\right)$. **1452.** $\frac{x}{2}\sqrt{x^2-9}-\frac{9}{2}\ln|x+\sqrt{x^2-9}|$. **1453.** $\frac{1}{16}(8x-1)\sqrt{x-4x^2}+\frac{1}{64}\arcsin(8x-1)$. **1454.** $\ln\left|\frac{x}{2x+1+2\sqrt{x^2+x+1}}\right|$. **1455.** $\frac{(x^2+2x+2)\sqrt{x^2+2x+2}}{3}-\frac{(x+1)}{2}\sqrt{x^2+2x+2}-\frac{1}{2}\ln(x+1+\sqrt{x^2+2x+2})$. **1456.** $\frac{\sqrt{x^2-1}}{x}-\frac{\sqrt{(x^2-1)^3}}{3x^3}$. **1457.** $\frac{1}{3}\ln\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right|$. **1458.** $-\frac{1}{3}\ln|z-1|+\frac{1}{6}\ln(z^2+z+1)-\frac{1}{\sqrt{3}}\arctan\frac{2z+1}{\sqrt{3}}$, where $z=\frac{\sqrt[3]{1+x^3}}{x}$. **1459.** $\frac{5}{2}\times\ln(x^2+\sqrt{1+x^4})$. **1460.** $\frac{3x}{8}+\frac{\sin 2x}{4}+\frac{\sin 4x}{32}$. **1461.** $\ln|\tan x|-\cot^2 x-\frac{1}{4}\cot^4 x$. **1462.** $-\cot x-\frac{2\sqrt{(\cot x)^3}}{3}$. **1463.** $\frac{5}{12}(\cos^2 x-6)\sqrt[5]{\cos^2 x}$. **1464.** $-\frac{\cos 5x}{20\sin^4 5x}-\frac{3\cos 5x}{40\sin^2 5x}+\frac{3}{40}\ln\left|\tan\frac{5x}{2}\right|$. **1465.** $\frac{\tan^3 x}{3}+\frac{\tan^5 x}{5}$. **1466.** $\frac{1}{4}\sin 2x$. **1467.** $\tan^2\left(\frac{x}{2}+\frac{\pi}{4}\right)+2\ln\left|\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$. **1468.** $-\frac{1}{\sqrt{3}}\times\arctan\frac{4\tan\frac{x}{2}-1}{\sqrt{3}}$. **1469.** $\frac{1}{\sqrt{10}}\arctan\left(\frac{2\tan x}{\sqrt{10}}\right)$. **1470.** $\arctan(2\tan x+1)$. **1471.** $\frac{1}{2}\ln|\tan x+\sec x|-\frac{1}{2}\operatorname{cosec} x$. **1472.** $\frac{2}{\sqrt{3}}\times\arctan\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right)-\frac{1}{\sqrt{2}}\times\arctan\left(\frac{\tan\frac{x}{2}}{\sqrt{2}}\right)$. **1473.** $\ln|\tan x+2+\sqrt{\tan^2 x+4\tan x+1}|$. **1474.** $\frac{1}{a}\times\ln(\sin ax+\sqrt{a^2+\sin^2 ax})$. **1475.** $\frac{1}{3}x\tan 3x+\frac{1}{9}\ln|\cos 3x|$. **1476.** $\frac{x^2}{4}-\frac{x\sin 2x}{4}-\frac{\cos 2x}{8}$. **1477.** $\frac{e^{2x}}{4}(2x-1)$. **1478** $\frac{1}{3}e^{x^3}$. **1479.** $\frac{x^3}{3}\cdot\ln\sqrt{1-x}-$

$-\frac{1}{6}\ln|x-1|-\frac{x^3}{18}-\frac{x^2}{12}-\frac{x}{6}$. 1480. $\sqrt{1+x^2}\arctan x-\ln(x+\sqrt{1+x^2})$. 1481. $\frac{1}{3}\sin\frac{3x}{2}-\frac{1}{10}\sin\frac{5x}{2}-\frac{1}{2}\sin\frac{x}{2}$. 1482. $-\frac{1}{1+\tan x}$. 1483. $\ln|1+\cot x|-\cot x$. 1481. $\frac{\sinh^2 x}{2}$. 1485. $-2\cosh\sqrt{1-x}$. 1486. $\frac{1}{5}\ln\cosh 2x$. 1487. $-x\coth x+\ln|\sinh x|$. 1488. $\frac{1}{2e^x}-\frac{x}{4}+\frac{1}{4}\ln|e^x-2|$. 1489. $\frac{1}{2}\arctan\frac{e^x-3}{2}$. 1490. $\frac{4}{7}\sqrt[4]{(e^x+1)^7}-\frac{4}{3}\sqrt[4]{(e^x+1)^3}$. 1491. $\frac{1}{\ln 4}\ln\frac{1+2^x}{1-2^x}$. 1492. $-\frac{10^{-2x}}{2\ln 10}\times\left(x^2-1+\frac{x}{\ln 10}+\frac{1}{2\ln^2 10}\right)$. 1493. $2\sqrt{e^x+1}+\ln\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}$. 1494. $\ln\left|\frac{x}{\sqrt{1+x^2}}\right|-\frac{\arctan x}{x}$. 1495. $\frac{1}{4}\left(x^4\arcsin\frac{1}{x}+\frac{x^2+2}{3}\sqrt{x^2-1}\right)$. 1496. $\frac{x}{2}(\cos\ln x+\sin\ln x)$. 1497. $\frac{1}{5}\left(-x^2\cos 5x+\frac{2}{5}x\sin 5x+3x\cos 5x+\frac{2}{25}\cos 5x-\frac{3}{5}\sin 5x\right)$. 1498. $\frac{1}{2}\left[(x^2-2)\arctan(2x+3)+\frac{3}{4}\ln(2x^2+6x+5)-\frac{x}{2}\right]$. 1499. $\frac{1}{2}\sqrt{x-x^2}+\left(x-\frac{1}{2}\right)\arcsin\sqrt{x}$. 1500. $\frac{x|x|}{2}$.

Chapter V

1501. $b-a$. 1502. $v_0T-g\frac{T^2}{2}$. 1503. 3. 1504. $\frac{2^{10}-1}{\ln 2}$. 1505. 156. **Hint.** Divide the interval from $x=1$ to $x=5$ on the x-axis into subintervals so that the abscissas of the points of division should form a geometric progression: $x_0=1$, $x_1=x_0q$, $x_2=x_0q^2$, ..., $x_n=x_0q^n$. 1506. $\ln\frac{b}{a}$. **Hint.** See Problem 1505. 1507. $1-\cos x$. **Hint.** Utilize the formula $\sin\alpha+\sin 2\alpha+\ldots+\sin n\alpha=\frac{1}{2\sin\frac{\alpha}{2}}\left[\cos\frac{\alpha}{2}-\cos\left(n+\frac{1}{2}\right)\alpha\right]$. 1508. 1) $\frac{dI}{da}=-\frac{1}{\ln a}$; 2) $\frac{dI}{db}=\frac{1}{\ln b}$. 1509. $\ln x$. 1510. $-\sqrt{1+x^4}$. 1511. $2xe^{-x^4}-e^{-x^2}$. 1512. $\frac{\cos x}{2\sqrt{x}}+\frac{1}{x^2}\cos\frac{1}{x^2}$. 1513. $x=n\pi$ ($n=1, 2, 3, \ldots$). 1514. $\ln 2$. 1515. $-\frac{3}{8}$. 1516. $e^x-e^{-x}=2\sinh x$. 1517. $\sin x$. 1518. $\frac{1}{2}$. **Solution.** The sum $s_n=\frac{1}{n^2}+\frac{2}{n^2}+\ldots+\frac{n-1}{n^2}=\frac{1}{n}\left(\frac{1}{n}+\frac{2}{n}+\ldots+\frac{n-1}{n}\right)$ may be regarded as the integral sum of the function $f(x)=x$ on the interval [0,1]. Therefore, $\lim_{n\to\infty}s_n=\int_0^1 x\,dx=\frac{1}{2}$. 1519. $\ln 2$. **Solution.** The sum $s_n=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n+n}=\frac{1}{n}\left(\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\ldots+\frac{1}{1+\frac{n}{n}}\right)$ may be regarded as the integral sum of the function $f(x)=\frac{1}{1+x}$ on the interval [0,1] where the division points have the form $x_k=1+\frac{k}{n}$ $(k=1, 2, \ldots, n)$. Therefore, $\lim\limits_{n\to\infty} s_n=\int\limits_0^1 \frac{dx}{1+x}=\ln 2$.

1520. $\frac{1}{p+1}$. **1521.** $\frac{7}{3}$. **1522.** $\frac{100}{3}=33\frac{1}{3}$. **1523.** $\frac{7}{4}$. **1524.** $\frac{16}{3}$. **1525.** $-\frac{2}{3}$. **1526.** $\frac{1}{2}\ln\frac{2}{3}$. **1527.** $\ln\frac{9}{8}$. **1528.** $35\frac{1}{15}-32\ln 3$. **1529.** $\arctan 3-\arctan 2=$ $=\arctan\frac{1}{7}$. **1530.** $\ln\frac{4}{3}$. **1531.** $\frac{\pi}{16}$. **1532.** $1-\frac{1}{\sqrt{3}}$. **1533.** $\frac{\pi}{4}$. **1534.** $\frac{\pi}{2}$. **1535.** $\frac{1}{3}\ln\frac{1+\sqrt{5}}{2}$. **1536.** $\frac{\pi}{8}+\frac{1}{4}$. **1537.** $\frac{2}{3}$. **1538.** $\ln 2$. **1539.** $1-\cos 1$. **1540.** 0. **1541.** $\frac{8}{9\sqrt{3}}+\frac{\pi}{6}$. **1542.** $\arctan e-\frac{\pi}{4}$. **1543.** $\sinh 1=\frac{1}{2}\left(e-\frac{1}{e}\right)$. **1544.** $\tanh(\ln 3)-\tanh(\ln 2)=\frac{1}{5}$. **1545.** $-\frac{\pi}{2}+\frac{1}{4}\sinh 2\pi$. **1546.** 2. **1547.** Diverges. **1548.** $\frac{1}{1-p}$, if $p<1$; diverges, if $p\geqslant 1$. **1549.** Diverges. **1550.** $\frac{\pi}{2}$. **1551.** Diverges. **1552.** 1. **1553.** $\frac{1}{p-1}$, if $p>1$; diverges, if $p\leqslant 1$. **1554.** π. **1555.** $\frac{\pi}{\sqrt{5}}$. **1556.** Diverges. **1557.** Diverges. **1558.** $\frac{1}{\ln 2}$ **1559.** Diverges. **1560.** $\frac{1}{\ln a}$. **1561.** Diverges. **1562.** $\frac{1}{k}$ **1563.** $\frac{\pi^2}{8}$. **1564.** $\frac{1}{3}+\frac{1}{4}\ln 3$ **1565.** $\frac{2\pi}{3\sqrt{3}}$. **1566.** Diverges **1567.** Converges **1568.** Diverges **1569.** Converges. **1570.** Converges. **1571.** Converges. **1572.** Diverges **1573.** Converges. **1574.** **Hint.** $B(p, q)=$ $=\int\limits_0^{1/2} f(x)\,dx+\int\limits_{1/2}^1 f(x)\,dx$, where $f(x)=x^{p-1}(1-x)^{q-1}$; since $\lim\limits_{x\to 0} f(x)\,x^{1-p}=1$ and $\lim\limits_{x\to 1}(1-x)^{1-q} f(x)=1$, both integrals converge when $1-p<1$ and $1-q<1$, that is, when $p>0$ and $q>0$. **1575.** **Hint.** $\Gamma(p)=\int\limits_0^1 f(x)\,dx+\int\limits_1^\infty f(x)\,dx$, where $f(x)=x^{p-1}e^{-x}$. The first integral converges when $p>0$, the second when p is arbitrary. **1576.** No. **1577.** $2\sqrt{2}\int\limits_1^2 \sqrt{t}\,dt$. **1578.** $\int\limits_{\pi/6}^{\pi/2}\frac{dt}{\sqrt{1+\sin^2 t}}$. **1579.** $\int\limits_{\ln 2}^{\ln 3} dt$. **1580.** $\int\limits_0^\infty \frac{f(\arctan t)}{1+t^2}\,dt$. **1581.** $x=(b-a)\,t+a$. **1582.** $4-2\ln 3$. **1583.** $8-\frac{9}{2\sqrt{3}}\pi$. **1584.** $2-\frac{\pi}{2}$. **1585.** $\frac{\pi}{\sqrt{5}}$ **1586.** $\frac{\pi}{2\sqrt{1+a^2}}$. **1587.** $1-\frac{\pi}{4}$. **1588.** $\sqrt{3}-\frac{\pi}{3}$. **1589.** $4-\pi$. **1590.** $\frac{1}{5}\ln 112$. **1591.** $\ln\frac{7+2\sqrt{7}}{9}$. **1592.** $\frac{1}{2}+\frac{\pi}{4}$. **1593.** $\frac{\pi a^2}{8}$

1594. $\frac{\pi}{2}$. 1599. $\frac{\pi}{2}-1$. 1600. 1. 1601. $\frac{e^2+3}{8}$. 1602. $\frac{1}{2}(e^{\pi}+1)$. 1603. 1. 1604. $\frac{a}{a^2+b^2}$. 1605. $\frac{b}{a^2+b^2}$. 1606. **Solution.** $\Gamma(p+1)=\int\limits_0^{\infty} x^p e^{-x}\,dx$. Applying the formula of integration by parts, we put $x^p=u$, $e^{-x}dx=dv$. Whence

$$du=px^{p-1}\,dx,\quad v=-e^{-x}$$

and

$$\Gamma(p+1)=[-x^pe^{-x}]_0^{\infty}+p\int\limits_0^{\infty} x^{p-1}e^{-x}\,dx=p\Gamma(p) \qquad (*)$$

If p is a natural number, then, applying formula (*) p times and taking into account that

$$\Gamma(1)=\int\limits_0^{\infty} e^{-x}\,dx=1,$$

we get:

$$\Gamma(p+1)=p!$$

1607. $I_{2k}=\frac{1\cdot3\cdot5\ldots(2k-1)}{2\cdot4\cdot6\ldots2k}\frac{\pi}{2}$, if $n=2k$ is an even number; $I_{2k+1}=$ $=\frac{2\cdot4\cdot6\ldots2k}{1\cdot3\cdot5\ldots(2k+1)}$, if $n=2k+1$ is an odd number

$$I_9=\frac{128}{315};\quad I_{10}=\frac{63\pi}{512}.$$

1608. $\frac{(p-1)!\,(q-1)!}{(p+q-1)!}$. 1609. $\frac{1}{2}\mathrm{B}\left(\frac{m+1}{2},\frac{n+1}{2}\right)$. **Hint.** Put $\sin^2 x=t$. 1610. a) Plus; b) minus; c) plus **Hint.** Sketch the graph of the integrand for values of the argument on the interval of integration 1611. a) First; b) second; c) first. 1612. $\frac{1}{3}$ 1613. a. 1614. $\frac{1}{2}$. 1615. $\frac{3}{8}$. 1616. $2\arcsin\frac{1}{3}$. 1617. $2<I<\sqrt{5}$. 1618. $\frac{2}{9}<I<\frac{2}{7}$. 1619. $\frac{2}{13}\pi<I<\frac{2}{7}\pi$. 1620. $0<I<\frac{\pi^2}{32}$. **Hint.** The integrand increases monotonically. 1621. $\frac{1}{2}<I<\frac{\sqrt{2}}{2}$. 1623. $s=\frac{32}{3}$. 1624. 1. 1625. $\frac{1}{2}$ **Hint.** Take account of the sign of the function. 1626. $4\frac{1}{4}$. 1627. 2. 1628. $\ln 2$. 1629. $m^2\ln 3$. 1630. πa^2. 1631. 12. 1632. $\frac{4}{3}p^2$. 1633. $4\frac{1}{2}$. 1634. $10\frac{2}{3}$. 1635. 4. 1636. $\frac{32}{3}$. 1637. $\frac{\pi}{2}-\frac{1}{3}$. 1638. $e+\frac{1}{e}-2=2(\cosh 1-1)$. 1639. $ab\,[2\sqrt{3}-\ln(2+\sqrt{3})]$. 1640. $\frac{3}{8}\pi a^2$. **Hint.** See Appendix VI, Fig. 27. 1641. $2a^2e^{-1}$. 1642. $\frac{4}{3}a^2$. 1643. 15π. 1644. $\frac{9}{2}\ln 3$. 1645. 1. 1646. $3\pi a^2$. **Hint.** See Appendix VI, Fig. 23. 1647. $a^2\left(2+\frac{\pi}{2}\right)$. **Hint.** See Appendix VI, Fig. 24. 1648. $2\pi+\frac{4}{3}$ and $6\pi-\frac{4}{3}$. 1649. $\frac{16}{3}\pi-\frac{4\sqrt{3}}{3}$ and $\frac{32}{3}\pi+\frac{4\sqrt{3}}{3}$. 1650. $\frac{3}{8}\pi ab$.

1651. $3\pi a^2$. **1652.** $\pi(b^2+2ab)$. **1653.** $6\pi a^2$. **1654.** $\frac{3}{2}a^2$. **Hint.** For the loop, the parameter t varies within the limits $0 \leqslant t \leqslant +\infty$ See Appendix VI, Fig. 22. **1655.** $\frac{3}{2}\pi a^2$. **Hint.** See Appendix VI, Fig. 28. **1656.** $8\pi^3 a^2$. **Hint.** See Appendix VI, Fig. 30. **1657.** $\frac{\pi a^2}{8}$. **1658.** a^2. **1659.** $\frac{\pi a^2}{4}$. **Hint.** See Appendix VI, Fig. 33. **1660.** $\frac{9}{2}\pi$. **1661.** $\frac{14-8\sqrt{2}}{3}a^2$. **1662.** $\frac{\pi p^2}{(1-\varepsilon^2)^{3/2}}$. **1663.** $a^2\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)$. **1664.** $\pi\sqrt{2}$. **Hint.** Pass to polar coordinates. **1665.** $\frac{8}{27}(10\sqrt{10}-1)$. **1666.** $\sqrt{h^2-a^2}$. **Hint.** Utilize the formula $\cosh^2\alpha-\sinh^2\alpha=1$. **1667.** $\sqrt{2}+\ln(1+\sqrt{2})$. **1668.** $\sqrt{1+e^2}-\sqrt{2}+\ln\frac{(\sqrt{1+e^2}-1)(\sqrt{2}+1)}{e}$. **1669.** $1+\frac{1}{2}\ln\frac{3}{2}$. **1670.** $\ln(e+\sqrt{e^2-1})$. **1671.** $\ln(2+\sqrt{3})$ **1672.** $\frac{1}{4}(e^2+1)$. **1673.** $a\ln\frac{a}{b}$. **1674.** $2a\sqrt{3}$. **1675.** $\ln\frac{e^{2b}-1}{e^{2a}-1}+a-b=\ln\frac{\sinh b}{\sinh a}$. **1676.** $\frac{1}{2}aT^2$. **Hint.** See Appendix VI, Fig. 29. **1677.** $\frac{4(a^3-b^3)}{ab}$. **1678.** $16a$. **1679.** $\pi a\sqrt{1+4\pi^2}+\frac{a}{2}\ln(2\pi+\sqrt{1+4\pi^2})$. **1680.** $8a$. **1681.** $2a[\sqrt{2}+\ln(\sqrt{2}+1)]$. **1682.** $\frac{\sqrt{5}}{2}+\ln\frac{3+\sqrt{5}}{2}$. **1683.** $\frac{a\sqrt{1+m^2}}{m}$. **1684.** $\frac{1}{2}[4+\ln 3]$. **1685.** $\frac{\pi a^5}{30}$. **1686.** $\frac{4}{3}\pi ab^2$. **1687.** $\frac{a^3\pi}{4}(e^2+4-e^{-2})$. **1688.** $\frac{3}{8}\pi^2$. **1689.** $v_x=\frac{\pi}{4}$. **1690.** $v_y=\frac{4}{7}\pi$. **1691.** $v_x=\frac{\pi}{2}$; $v_y=2\pi$. **1692.** $\frac{16\pi a^3}{5}$. **1693.** $\frac{32}{15}\pi a^3$. **1694.** $\frac{4}{3}\pi p^3$. **1695.** $\frac{3}{10}\pi$. **1696.** $\frac{\pi a^3}{2}(15-16\ln 2)$. **1697.** $2\pi^2 a^3$. **1698.** $\frac{\pi R^2 H}{2}$. **1699.** $\frac{16}{15}\pi h^2 a$. **1701.** a) $5\pi^2 a^3$; b) $6\pi^3 a^3$; c) $\frac{\pi a^3}{6}(9\pi^2-16)$. **1702.** $\frac{32}{105}\pi a^3$. **1703.** $\frac{8}{3}\pi a^3$. **1704.** $\frac{4}{21}\pi a^3$. **1705.** $\frac{h}{3}\left(AB+\frac{Ab+aB}{2}+ab\right)$. **1706.** $\frac{\pi abh}{3}$. **1707.** $\frac{128}{105}a^3$. **1708.** $\frac{8}{3}\pi a^2 b$. **1709.** $\frac{1}{2}\pi a^2 h$. **1710.** $\frac{16}{3}a^3$. **1711.** $\pi a^2\sqrt{pq}$. **1712.** $\pi abh\left(1+\frac{h^2}{3c^2}\right)$. **1713.** $\frac{4}{3}\pi abc$. **1714.** $\frac{8\pi}{3}[\sqrt{17^3}-1]$; $\frac{16}{3}\pi a^2[5\sqrt{5}-8]$. **1715.** $2\pi[\sqrt{2}+\ln(\sqrt{2}+1)]$. **1716.** $\pi(\sqrt{5}-\sqrt{2})+\pi\ln\frac{2(\sqrt{2}+1)}{\sqrt{5}+1}$. **1717.** $\pi[\sqrt{2}+\ln(1+\sqrt{2})]$. **1718.** $\frac{\pi a^2}{4}(e^2+e^{-2}+4)=\frac{\pi a^2}{2}(2+\sinh 2)$. **1719.** $\frac{12}{5}\pi a^2$. **1720.** $\frac{\pi}{3}(e-1)(e^2+e+4)$. **1721.** $4\pi^2 ab$ **Hint.** Here, $y=b\pm\sqrt{a^2-x^2}$. Taking the plus sign, we get the external surface of a torus; taking the minus sign, we get the internal surface of a torus. **1722.** 1) $2\pi b^2+\frac{2\pi ab}{\varepsilon}\arcsin\varepsilon$; 2) $2\pi a^2+\frac{\pi b^2}{\varepsilon}\ln\frac{1+\varepsilon}{1-\varepsilon}$, where $\varepsilon=\frac{\sqrt{a^2-b^2}}{a}$ (eccentricity of ellipse). **1723.** a) $\frac{64\pi a^2}{3}$; b) $16\pi^2 a^2$; c) $\frac{32}{3}\pi a^2$.

1724. $\frac{128}{5}\pi a^2$. 1725. $2\pi a^2(2-\sqrt{2})$. 1726. $\frac{128}{5}\pi a^2$. 1727. $M_X=\frac{b}{2}\sqrt{a^2+b^2}$; $M_Y=\frac{a}{2}\sqrt{a^2+b^2}$. 1728. $M_a=\frac{ab^2}{2}$; $M_b=\frac{a^2b}{2}$. 1729. $M_X=M_Y=\frac{a^3}{6}$; $\bar{x}=\bar{y}=\frac{a}{3}$. 1730. $M_X=M_Y=\frac{3}{5}a^2$; $\bar{x}=\bar{y}=\frac{2}{5}a$. 1731. $2\pi a^2$. 1732. $\bar{x}=0$; $\bar{y}=\frac{a}{4}\,\frac{2+\sinh 2}{\sinh 1}$. 1733. $\bar{x}=\frac{a\sin\alpha}{\alpha}$; $\bar{y}=0$. 1734. $\bar{x}=\pi a$; $\bar{y}=\frac{4}{3}a$. 1735. $\bar{x}=\frac{4a}{3\pi}$; $\bar{y}=\frac{4b}{3\pi}$. 1736. $\bar{x}=\bar{y}=\frac{9}{20}$. 1737. $\bar{x}=\pi a$; $\bar{y}=\frac{5}{6}a$. 1738. $\left(0, 0, \frac{a}{2}\right)$. **Solution.** Divide the hemisphere into elementary spherical slices of area $d\sigma$ by horizontal planes. We have $d\sigma=2\pi a\,dz$, where dz is the altitude of a slice. Whence $\bar{z}=\frac{2\pi\int_0^a az\,dz}{2\pi a^2}=\frac{a}{2}$. Due to symmetry, $\bar{x}=\bar{y}=0$. 1739. At a distance of $\frac{3}{4}$ altitude from the vertex of the cone. **Solution.** Partition the cone into elements by planes parallel to the base. The mass of an elementary layer (slice) is $dm_i=\gamma\pi\varrho^2\,dz$, where γ is the density, z is the distance of the cutting plane from the vertex of the cone, $\varrho=\frac{r}{h}z$. Whence

$$\bar{z}=\frac{\pi\int_0^h \frac{r^2}{h^2}z^3\,dz}{\frac{1}{3}\pi r^2 h}=\frac{3}{4}h.$$

1740. $\left(0;\ 0;\ +\frac{3}{8}a\right)$. **Solution.** Due to symmetry, $\bar{x}=\bar{y}=0$. To determine $\bar{z}$ we partition the hemisphere into elementary layers (slices) by planes parallel to the horizontal plane. The mass of such an elementary layer $dm=\gamma\pi r^2\,dz$, where γ is the density, z is the distance of the cutting plane from the base of the hemisphere, $r=\sqrt{a^2-z^2}$ is the radius of a cross-section. We have: $\bar{z}=\frac{\pi\int_0^a (a^2-z^2)\,z\,dz}{\frac{2}{3}\pi a^3}=\frac{3}{8}a$. 1741. $I=\pi a^3$. 1742. $I_a=\frac{1}{3}ab^3$; $I_b=\frac{1}{3}a^3b$. 1743. $I=\frac{4}{15}hb^3$. 1744. $I_a=\frac{1}{4}\pi ab^3$; $I_b=\frac{1}{4}\pi a^3b$. 1745. $I=\frac{1}{2}\pi\,(R_2^4-R_1^4)$. **Solution.** We partition the ring into elementary concentric circles. The mass of each such element $dm=\gamma 2\pi r\,dr$ and the moment of inertia $I=2\pi\int_{R_1}^{R_2} r^3\,dr=\frac{1}{2}\pi\,(R_2^4-R_1^4)$; $(\gamma=1)$. 1746. $I=\frac{1}{10}\pi R^4 H\gamma$. **Solution.** We partition the cone into elementary cylindrical tubes parallel to the axis of the cone. The volume of each such elementary tube is $dV=2\pi rh\,dr$, where r is the radius of the tube (the distance to the axis of the cone), $h=H\left(1-\frac{r}{R}\right)$ is the altitude of the tube; then the moment of inertia $I=\gamma\int_0^R 2\pi H\left(1-\frac{r}{R}\right)r^3\,dr=\frac{\gamma\pi R^4H}{10}$, where γ is the density of the cone. **1747.** $I=\frac{2}{5}Ma^2$. **Solution.** We partition the sphere into elementary cylindrical tubes, the axis of which is the given diameter. An elementary volume $dV=2\pi rh\,dr$, where r is the radius of a tube, $h=2a\sqrt{1-\frac{r^2}{a^2}}$ is its altitude. Then the moment of inertia $I=4\pi a\gamma\int_0^a\sqrt{1-\frac{r^2}{a^2}}\,r^3dr=\frac{8}{15}\pi a^5\gamma$, where γ is the density of the sphere, and since the mass $M=\frac{4}{3}\pi a^3\gamma$, it follows that $J=\frac{2}{5}Ma^2$. **1748.** $V=2\pi^2a^2b$; $S=4\pi^2ab$. **1749.** a) $\overline{x}=\overline{y}=\frac{2}{5}a$; b) $\overline{x}=\overline{y}=\frac{9}{10}p$. **1750.** a) $\overline{x}=0$, $\overline{y}=\frac{4}{3}\frac{r}{\pi}$ **Hint.** The coordinate axes are chosen so that the x-axis coincides with the diameter and the origin is the centre of the circle; b) $\overline{x}=\frac{h}{3}$ **Solution.** The volume of the solid — a double cone obtained from rotating a triangle about its base, is equal to $V=\frac{1}{3}\pi bh^2$, where b is the base, h is the altitude of the triangle. By the Guldin theorem, the same volume $V=2\pi\overline{x}\,\frac{1}{2}bh$, where $\overline{x}$ is the distance of the centre of gravity from the base. Whence $\overline{x}=\frac{h}{3}$ **1751.** $v_0t-\frac{gt^2}{2}$. **1752.** $\frac{c^2}{2g}\ln\left(1+\frac{v_0^2}{c^2}\right)$. **1753.** $x=\frac{v_0}{\omega}\sin\omega t$; $v_{av}=\frac{2}{\pi}v_0$ **1754.** $S=10^4\ m$. **1755.** $v=\frac{A}{b}\ln\left(\frac{a}{a-bt}\right)$; $h=\frac{A}{b^2}\times\left[bt_1-(a-bt_1)\ln\frac{a}{a-bt_1}\right]$. **1756.** $A=\frac{\pi\gamma}{2}R^2H^2$ **Hint.** The elementary force (force of gravity) is equal to the weight of water in the volume of a layer of thickness dx, that is, $dF=\gamma\pi R^2\,dx$, where γ is the weight of unit volume of water. Hence, the elementary work of a force $dA=\gamma\pi R^2(H-x)\,dx$, where x is the water level. **1757.** $A=\frac{\pi}{12}\gamma R^2H^2$. **1758.** $A=\frac{\pi\gamma}{4}R^4TM\approx 0.79\cdot10^4=0.79\cdot10^7$ kgm. **1759.** $A=\gamma\pi R^3H$. **1760.** $A=\frac{mgh}{1+\frac{h}{R}}$; $A_\infty=mgR$. **Solution.** The force acting on a mass m is equal to $F=k\frac{mM}{r^2}$, where r is the distance from the centre of the earth. Since for $r=R$ we have $F=mg$, it follows that $kM=gR^2$. The sought-for work will have the form $A=\int_R^{R+h}k\frac{mM}{r^2}\,dr=kmM\left(\frac{1}{R}-\frac{1}{R+h}\right)=\frac{mgh}{1+\frac{h}{R}}$. When $h=\infty$ we have $A_\infty=mgR$. **1761.** $1.8\cdot10^4$ ergs. **Solution.**

The force of interaction of charges is $F=\frac{e_0e_1}{x^2}$ dynes. Consequently, the work performed in moving charge e_1 from point x_1 to x_2 is $A=e_0e_1\int\limits_{x_1}^{x_2}\frac{dx}{x^2}=$ $=e_0e_1\left(\frac{1}{x_1}-\frac{1}{x_2}\right)=1.8\cdot 10^4$ ergs. **1762.** $A=800\ \pi\ln 2$ kgm. **Solution.** For an isothermal process, $pv=p_0v_0$. The work performed in the expansion of a gas from volume v_0 to volume v_1 is $A=\int\limits_{v_0}^{v_1}p\,dv=p_0v_0\ln\frac{v_1}{v_0}$. **1763.** $A\approx 15{,}000$ kgm.

Solution. For an adiabatic process, the Poisson law $pv^k=p_0v_0^k$, where $k\approx 1.4$, holds true. Hence $A=\int\limits_{v_0}^{v_1}\frac{p_0v_0^k}{v^k}dv=\frac{p_0v_0}{k-1}\left[1-\left(\frac{v_0}{v_1}\right)^{k-1}\right]$.

1764. $A=\frac{4}{3}\pi\mu Pa$. **Solution.** If a is the radius of the base of a shaft, then the pressure on unit area of the support $p=\frac{P}{\pi a^2}$. The frictional force of a ring of width dr, at a distance r from the centre, is $\frac{2\mu P}{a^2}r\,dr$. The work performed by frictional forces on a ring in one complete revolution is $dA=\frac{4\pi\mu P}{a^2}r^2dr$. Therefore, the complete work $A=\frac{4\pi\mu P}{a^2}\times\int\limits_0^a r^2\,dr=\frac{4}{3}\pi\mu Pa$.

1765. $\frac{1}{4}MR^2\omega^2$. **Solution.** The kinetic energy of a particle of the disk $dK=\frac{mv^2}{2}=\frac{\varrho r^2\omega^2}{2}d\sigma$, where $d\sigma$ is an element of area, r is the distance of it from the axis of rotation, ϱ is the surface density, $\varrho=\frac{M}{\pi R^2}$. Thus, $dK=\frac{M\omega^2}{2\pi R^2}r^2\,d\sigma$. Whence $K=\frac{M\omega^2}{R^2}\int\limits_0^R r^3\,dr=\frac{MR^2\omega^2}{4}$ **1766.** $K=\frac{3}{20}\times MR^2\omega^2$.

1767. $K=\frac{M}{5}R^2\omega^2=2.3\cdot 10^9$ kgm. **Hint.** The amount of work required is equal to the reserve of kinetic energy. **1768.** $p=\frac{bh^2}{6}$. **1769.** $P=\frac{(a+2b)h^2}{6}\approx 11.3\cdot 10^9\ T$ **1770.** $P=ab\gamma\pi h$. **1771.** $P=\frac{\pi R^2H}{3}$ (the vertical component is directed upwards). **1772.** $533\frac{1}{3}$ gm **1773.** 99.8 cal. **1774.** $M=\frac{hb^2p}{2}$ gf cm. **1775.** $\frac{kMm}{a(a+l)}$ (k is the gravitational constant). **1776.** $\frac{\pi pa^4}{8\mu l}$. **Solution.** $Q=\int\limits_0^a v2\pi r\,dr=\frac{2\pi p}{4\mu l}\int\limits_0^a(a^2-r^2)\,r\,dr=$ $=\frac{\pi p}{2\mu l}\left[\frac{a^2r^2}{2}-\frac{r^4}{4}\right]_0^a=\frac{\pi pa^4}{8\mu l}$. **1777.** $Q=\int\limits_0^{2b}v_a\,dy=\frac{2}{3}p\frac{ab^3}{\mu l}$ **Hint.** Draw the x-axis along the large lower side of the rectangle, and the y-axis, perpendicular to it in the middle. **1778.** **Solution.** $S=\int_{v_1}^{v_2}\frac{1}{a}\,dv$; on the other hand, $\frac{dv}{dt}=a$, whence $dt=\frac{1}{a}\,dv$, and consequently, the acceleration time is $t=\int_{v_1}^{v_2}\frac{dv}{a}=S$.

1779. $M_x=-\int_0^x\frac{Q}{l}(x-t)\,dt+\frac{Q}{2}x=-\frac{Q}{l}\left[xt-\frac{t^2}{2}\right]_0^x+\frac{Q}{2}x=\frac{Qx}{2}\left(1-\frac{x}{l}\right)$.

1780. $M_x=-\int_0^x(x-t)\,kt\,dt+Ax=\frac{kx}{6}(l^2-x^2)$. **1781.** $Q=0.12\,TRI_0^2$ cal. **Hint.** Use the Joule-Lenz law.

Chapter VI

1782. $V=\frac{2}{3}(y^2-x^2)\,x$. **1783.** $S=\frac{2}{3}(x+y)\sqrt{4z^2+3(x-y)^2}$. **1784.** $f\left(\frac{1}{2},\ 3\right)=\frac{5}{3}$; $f(1,\ -1)=-2$. **1785.** $\frac{y^2-x^2}{2xy}$, $\frac{x^2-y^2}{2xy}$, $\frac{y^2-x^2}{2xy}$, $\frac{2xy}{x^2-y^2}$. **1786.** $f(x,\ x^2)=1+x-x^2$. **1787.** $z=\frac{R^4}{1-R^2}$. **1788.** $f(x)=\frac{\sqrt{1+x^2}}{x}$. **Hint.** Represent the given function in the form $f\left(\frac{y}{x}\right)=\sqrt{\left(\frac{x}{y}\right)^2+1}$ and replace $\frac{y}{x}$ by x. **1789.** $f(x,\ y)=\frac{x^2-xy}{2}$. **Solution.** Designate $x+y=u$, $x-y=v$. Then $x=\frac{u+v}{2}$, $y=\frac{u-v}{2}$; $f(u,\ v)=\frac{u+v}{2}\cdot\frac{u-v}{2}+\left(\frac{u-v}{2}\right)^2=\frac{u^2-uv}{2}$. It remains to name the arguments u and v, x and y. **1790.** $f(u)=u^2+2u$; $z=x-1+\sqrt{y}$. **Hint.** In the identity $x=1+f(\sqrt{x}-1)$ put $\sqrt{x}-1=u$; then $x=(u+1)^2$ and, hence, $f(u)=u^2+2u$. **1791.** $f(y)=\sqrt{1+y^2}$; $z=\sqrt{x^2+y^2}$ **Solution.** When $x=1$ we have the identity $\sqrt{1+y^2}=1\cdot f\left(\frac{y}{1}\right)$, i. e., $f(y)=\sqrt{1+y^2}$. Then $f\left(\frac{y}{x}\right)=\sqrt{1+\left(\frac{y}{x}\right)^2}$ and $z=x\sqrt{1+\left(\frac{y}{x}\right)^2}=\sqrt{x^2+y^2}$. **1792.** a) Single circle with centre at origin, including the circle ($x^2+y^2\leqslant 1$); b) bisector of quadrantal angle $y=x$; c) half-plane located above the straight line $x+y=0$ ($x+y>0$); d) strip contained between the straight lines $y=\pm 1$, including these lines ($-1\leqslant y\leqslant 1$); e) a square formed by the segments of the straight lines $x=\pm 1$ and $y=\pm 1$, including its sides ($-1\leqslant x\leqslant 1$, $-1\leqslant y\leqslant 1$); f) part of the plane adjoining the x-axis and contained between the straight lines $y=\pm x$, including these lines and excluding the coordinate origin ($-x\leqslant y\leqslant x$ when $x>0$, $x\leqslant y\leqslant -x$ when $x<0$); g) two strips $x\geqslant 2$, $-2\leqslant y\leqslant 2$ and $x\leqslant -2$, $-2\leqslant y\leqslant 2$; h) the ring contained between the circles $x^2+y^2=a^2$ and $x^2+y^2=2a^2$, including the boundaries; i) strips $2n\pi\leqslant x\leqslant(2n+1)\pi$, $y\geqslant 0$ and $(2n+1)\pi\leqslant x\leqslant(2n+2)\pi$, $y\leqslant 0$, where n is an integer; j) that part of the plane located above the parabola $y=-x^2(x^2+y>0)$; k) the entire xy-plane; l) the entire xy-plane, with the exception of the coordinate origin; m) that part of the plane located above the parabola $y^2=x$ and to the right of the y-axis, including the points of the y-axis and excluding the points of the parabola $(x\geqslant 0,\ y>\sqrt{x})$; n) the entire place except points of the straight lines $x=1$ and $y=0$; o) the family of concentric circles $2\pi k\leqslant x^2+y^2\leqslant \pi(2k+1)$ $(k=0,\ 1,\ 2,\ \ldots)$. **1793.** a) First octant (including boundary); b) First, Third, Sixth and Eighth octants (excluding the boundary); c) a cube bounded by the planes $x=\pm 1$, $y=\pm 1$ and $z=\pm 1$, including its faces; d) a sphere of radius 1 with centre at the origin, including its surface **1794.** a) a plane; the level lines are straight lines parallel to the straight line $x+y=0$; b) a paraboloid of revolution; the level lines are concentric circles with centre at the origin; c) a hyperbolic paraboloid; the level lines are equilateral hyperbolas; d) second-order cone; the level lines are equilateral hyperbolas; e) a parabolic cylinder, the generatrices of which are parallel to the straight line $x+y+1=0$; the level lines are parallel lines; f) the lateral surface of a quadrangular pyramid; the level lines are the outlines of squares; g) level lines are parabolas $y=Cx^2$; h) the level lines are parabolas $y=C\sqrt{x}$; i) the level lines are the circles $C(x^2+y^2)=2x$. **1795.** a) Parabolas $y=C-x^2$ $(C>0)$; b) hyperbolas $xy=C$ $(|C|\leqslant 1)$; c) circles $x^2+y^2=C^2$; d) straight lines $y=ax+C$; e) straight lines $y=Cx\ (x\neq 0)$. **1796.** a) Planes parallel to the plane $x+y+z=0$; b) concentric spheres with centre at origin; c) for $u>0$, one-sheet hyperboloids of revolution about the z-axis; for $u<0$, two-sheet hyperboloids of revolution about the same axis; both families of surfaces are divided by the cone $x^2+y^2-z^2=0$ $(u=0)$. **1797.** a) 0; b) 0;c) 2; d) e^k; e) limit does not exist; f) limit does not exist. **Hint.** In Item(b) pass to polar coordinates In Items (e) and (f), consider the variation of x and y along the straight lines $y=kx$ and show that the given expression may tend to different limits, depending on the choice of k. **1798.** Continuous. **1799.** a) Discontinuity at $x=0$, $y=0$; b) all points of the straight line $x=y$ (line of discontinuity); c) line of discontinuity is the circle $x^2+y^2=1$; d) the lines of discontinuity are the coordinate axes. **1800 Hint.** Putting $y=y_1=\text{const}$, we get the function $\varphi_1(x)=\frac{2xy_1}{x^2+y_1^2}$, which is continuous everywhere, since for $y_1\neq 0$ the denominator $x^2+y_1^2\neq 0$, and when $y_1=0$, $\varphi_1(x)=0$. Similarly, when $x=x_1=\text{const}$, the function $\varphi_2(y)=\frac{2x_1y}{x_1^2+y^2}$ is everywhere continuous. From the set of variables x, y, the function z is discontinuous at the point (0, 0) since there is no $\lim\limits_{\substack{x\to 0\\ y\to 0}} z$. Indeed, passing to polar coordinates $(x=r\cos\varphi, y=r\sin\varphi)$, we get $z=\sin 2\varphi$, whence it is evident that if $x\to 0$ and $y\to 0$ in such manner that $\varphi=\text{const}$ $(0\leqslant\varphi\leqslant 2\pi)$, then $z\to\sin 2\varphi$. Since these limiting values of the function z depend on the direction of φ, it follows that z does not have a limit as $x\to 0$ and $y\to 0$. **1801.** $\frac{\partial z}{\partial x}=3(x^2-ay)$, $\frac{\partial z}{\partial y}=3(y^2-ax)$. **1802.** $\frac{\partial z}{\partial x}=\frac{2y}{(x+y)^2}$, $\frac{\partial z}{\partial y}=-\frac{2x}{(x+y)^2}$. **1803.** $\frac{\partial z}{\partial x}=-\frac{y}{x^2}$, $\frac{\partial z}{\partial y}=\frac{1}{x}$. **1804.** $\frac{\partial z}{\partial x}=\frac{x}{\sqrt{x^2-y^2}}$, $\frac{\partial z}{\partial y}=-\frac{y}{\sqrt{x^2-y^2}}$. **1805.** $\frac{\partial z}{\partial x}=\frac{y^2}{(x^2+y^2)^{3/2}}$, $\frac{\partial z}{\partial y}=-\frac{xy}{(x^2+y^2)^{3/2}}$. **1806.** $\frac{\partial z}{\partial x}=\frac{1}{\sqrt{x^2+y^2}}$, $\frac{\partial z}{\partial y}=\frac{y}{\sqrt{x^2+y^2}\,(x+\sqrt{x^2+y^2})}$. **1807.** $\frac{\partial z}{\partial x}=-\frac{y}{x^2+y^2}$, $\frac{\partial z}{\partial y}=\frac{x}{x^2+y^2}$. **1808.** $\frac{\partial z}{\partial x}=yx^{y-1}$,

$\frac{\partial z}{\partial y}=x^y \ln x$. **1809.** $\frac{\partial z}{\partial x}=-\frac{y}{x^2}e^{\sin\frac{y}{x}}\cos\frac{y}{x}$, $\frac{\partial z}{\partial y}=\frac{1}{x}e^{\sin\frac{y}{x}}\cos\frac{y}{x}$. **1810.** $\frac{\partial z}{\partial x}=\frac{xy^2\sqrt{2x^2-2y^2}}{|y|(x^4-y^4)}$, $\frac{\partial z}{\partial y}=-\frac{yx^2\sqrt{2x^2-2y^2}}{|y|(x^4-y^4)}$. **1811.** $\frac{\partial z}{\partial x}=\frac{1}{\sqrt{y}}\cot\frac{x+a}{\sqrt{y}}$, $\frac{\partial z}{\partial y}=-\frac{x+a}{2y\sqrt{y}}\cot\frac{x+a}{\sqrt{y}}$. **1812.** $\frac{\partial u}{\partial x}=yz(xy)^{z-1}$, $\frac{\partial u}{\partial y}=xz(xy)^{z-1}$, $\frac{\partial u}{\partial z}=(xy)^z\ln(xy)$.
1813. $\frac{\partial u}{\partial x}=yz^{xy}\ln z$, $\frac{\partial u}{\partial y}=xz^{xy}\ln z$, $\frac{\partial u}{\partial z}=xyz^{xy-1}$. **1814.** $f'_x(2, 1)=\frac{1}{2}$, $f'_y(2, 1)=0$. **1815.** $f'_x(1, 2, 0)=1$, $f_y(1, 2, 0)=\frac{1}{2}$, $f^z(1, 2, 0)=\frac{1}{2}$.
1820. $-\frac{x}{(x^2+y^2+z^2)^{3/2}}$. **1821.** r. **1826.** $z=\arctan\frac{y}{x}+\varphi(x)$. **1827.** $z=\frac{x^2}{2}+y^2\ln x+\sin y-\frac{1}{2}$. **1828.** 1) $\tan\alpha=4$, $\tan\beta=\infty$, $\tan\gamma=\frac{1}{4}$; 2) $\tan\alpha=\infty$, $\tan\beta=4$, $\tan\gamma=\frac{1}{4}$. **1829.** $\frac{\partial S}{\partial a}=\frac{1}{2}h$, $\frac{\partial S}{\partial b}=\frac{1}{2}h$, $\frac{\partial S}{\partial h}=\frac{1}{2}(a+b)$. **1830. Hint.** Check to see that the function is equal to zero over the entire x-axis and the entire y-axis, and take advantage of the definition of partial derivatives. Be convinced that $f'_x(0, 0)=f'_y(0, 0)=0$. **1831.** $\Delta f=4\Delta x+\Delta y+2\Delta x^2+2\Delta x\,\Delta y+\Delta x^2\,\Delta y$; $df=4dx+dy$; a) $\Delta f-df=8$; b) $\Delta f-df=0.062$.
1833. $dz=3(x^2-y)\,dx+3(y^2-x)\,dy$. **1834.** $dz=2xy^3dx+3x^2y^2dy$. **1835.** $dz=\frac{4}{(x^2+y^2)^2}(xy^2dx-x^2y\,dy)$. **1836.** $dz=\sin 2x\,dx-\sin 2y\,dy$. **1837.** $dz=y^2x^{y-1}dx+x^y(1+y\ln x)\,dy$. **1838.** $dz=\frac{2}{x^2+y^2}(x\,dx+y\,dy)$. **1839.** $df=\frac{1}{x+y}\left(dx-\frac{x}{y}dy\right)$.
1840. $dz=0$. **1841.** $dz=-\frac{2}{x\sin\frac{2y}{x}}\left(dy-\frac{y}{x}dx\right)$. **1842.** $df(1, 1)=dx-2dy$.
1843. $du=yz\,dx+zx\,dy+xy\,dz$. **1844.** $du=\frac{1}{\sqrt{x^2+y^2+z^2}}(x\,dx+y\,dy+z\,dz)$.
1845. $du=\left(xy+\frac{x}{y}\right)^{z-1}\left[\left(y+\frac{1}{y}\right)z\,dx+\left(1-\frac{1}{y^2}\right)xz\,dy+\left(xy+\frac{x}{y}\right)\ln\left(xy+\frac{x}{y}\right)dz\right]$. **1846.** $du=\frac{z^2}{x^2y^2+z^4}\left(y\,dx+x\,dy-\frac{2xy}{z}dz\right)$. **1847.** $df(3, 4, 5)=\frac{1}{25}(5dz-3dx-4dy)$. **1848.** $dl=0.062$ cm; $\Delta l=0.065$ cm. **1849.** 75 cm^3 (relative to inner dimensions). **1850.** $\frac{1}{8}$ cm. **Hint.** Put the differential of the area of the sector equal to zero and find the differential of the radius from that.
1851. a) 1.00; b) 4.998, c) 0.273. **1853.** Accurate to 4 metres (more exactly, 4.25 m). **1854.** $\pi\frac{\alpha g-\beta l}{g\sqrt{lg}}$. **1855.** $d\alpha=\frac{1}{\varrho}(dy\cos\alpha-dx\sin\alpha)$. **1856.** $\frac{dz}{dt}=\frac{e^t(t\ln t-1)}{t\ln^2 t}$. **1857.** $\frac{du}{dt}=\frac{t}{\sqrt{y}}\cot\frac{x}{\sqrt{y}}\left(6-\frac{x}{2y^2}\right)$. **1858.** $\frac{du}{dt}=2t\ln t\tan t+\frac{(t^2+1)\tan t}{t}+\frac{(t^2+1)\ln t}{\cos^2 t}$. **1859.** $\frac{du}{dt}=0$. **1860.** $\frac{dz}{dx}=(\sin x)^{\cos x}(\cos x\cot x-$

$-\sin x \ln \sin x)$. **1861.** $\frac{\partial z}{\partial x}=-\frac{y}{x^2+y^2}$; $\frac{dz}{dx}=\frac{1}{1+x^2}$. **1862.** $\frac{\partial z}{\partial x}=yx^{y-1}$; $\frac{dz}{dx}=x^y\left[\varphi'(x)\ln x+\frac{y}{x}\right]$. **1863.** $\frac{\partial z}{\partial x}=2xf'_u(u, v)+ye^{xy}f'_v(u, v)$; $\frac{\partial z}{\partial y}=-2yf'_u(u, v)+xe^{xy}f'_v(u, v)$. **1864** $\frac{\partial z}{\partial u}=0$, $\frac{\partial z}{\partial v}=1$. **1865.** $\frac{\partial z}{\partial x}=y\left(1-\frac{1}{x^2}\right)f'\left(xy+\frac{y}{x}\right)$; $\frac{\partial z}{\partial y}=\left(x+\frac{1}{x}\right)f'\left(xy+\frac{y}{x}\right)$. **1867.** $\frac{du}{dx}=f'_x(x, y, z)+\varphi'(x)f'_y(x, y, z)+f'_z(x, y, z)[\psi'_x(x, y)+\psi'_y(x, y)\varphi'(x)]$. **1873.** The perimeter increases at a rate of 2 m/sec, the area increases at a rate of 70 m^2/sec. **1874.** $\frac{1+2t^2+3t^4}{\sqrt{1+t^2+t^4}}$.
1875. $20\sqrt{5-2\sqrt{2}}$ km/hr. **1876.** $-\frac{9\sqrt{3}}{2}$. **1877.** 1. **1878.** $\frac{\sqrt{2}}{2}$. **1879.** $-\frac{\sqrt{3}}{3}$.
1880. $\frac{68}{13}$. **1881.** $\frac{\cos\alpha+\cos\beta+\cos\gamma}{3}$. **1882.** a) (2, 0); b) (0, 0); and (1, 1); c) (7, 2, 1). **1884.** $9i-3j$. **1885.** $\frac{1}{4}(5i-3j)$. **1886.** $6i+3j+2k$. **1887.** $|\operatorname{grad} u|=6$; $\cos\alpha=\frac{2}{3}$, $\cos\beta=-\frac{2}{3}$, $\cos\gamma=\frac{1}{3}$. **1888.** $\cos\varphi=\frac{3}{\sqrt{10}}$. **1889.** $\tan\varphi\approx 8.944$; $\varphi\approx 83°37'$. **1891.** $\frac{\partial^2 z}{\partial x^2}=\frac{abcy^2}{(b^2x^2+a^2y^2)^{3/2}}$; $\frac{\partial^2 z}{\partial x\,\partial y}=-\frac{abcxy}{(b^2x^2+a^2y^2)^{3/2}}$; $\frac{\partial^2 z}{\partial y^2}=\frac{abcx^2}{(b^2x^2+a^2y^2)^{3/2}}$. **1892** $\frac{\partial^2 z}{\partial x^2}=\frac{2(y-x^2)}{(x^2+y)^2}$; $\frac{\partial^2 z}{\partial x\,\partial y}=-\frac{2x}{(x^2+y)^2}$; $\frac{\partial^2 z}{\partial y^2}=-\frac{1}{(x^2+y)^2}$.
1893. $\frac{\partial^2 z}{\partial x\,\partial y}=\frac{xy}{(2xy+y^2)^{3/2}}$. **1894.** $\frac{\partial^2 z}{\partial x\,\partial y}=0$. **1895.** $\frac{\partial^2 r}{\partial x^2}=\frac{r^2-x^2}{r^3}$. **1896.** $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial z^2}=0$; $\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial y\,\partial z}=\frac{\partial^2 u}{\partial z\,\partial x}=1$. **1897.** $\frac{\partial^3 u}{\partial x\,\partial y\,\partial z}=\alpha\beta\gamma x^{\alpha-1}y^{\beta-1}z^{\gamma-1}$.
1898. $\frac{\partial^3 z}{\partial x\,\partial y^2}=-x^2y\cos(xy)-2x\sin(xy)$. **1899.** $f''_{x^2}(0, 0)=m(m-1)$; $f''_{xy}(0, 0)=mn$; $f''_{y^2}(0, 0)=n(n-1)$. **1902. Hint.** Using the rules of differentiation and the definition of a partial derivative, verify that $f'_x(x, y)=y\left[\frac{x^2-y^2}{x^2+y^2}+\frac{4x^2y^2}{(x^2+y^2)^2}\right]$ (when $x^2+y^2\neq 0$), $f'_x(0, 0)=0$ and, consequently, that for $x=0$ and for any y, $f'_x(0, y)=-y$. Whence $f''_{xy}(0, y)=-1$; in particular, $f''_{xy}(0, 0)=-1$. Similarly, we find that $f''_{yx}(0, 0)=1$.

1903. $\frac{\partial^2 z}{\partial x^2}=2f'_u(u, v)+4x^2f''_{uu}(u, v)+4xyf''_{uv}(u, v)+y^2f''_{vv}(u, v)$;

$\frac{\partial^2 z}{\partial x\,\partial y}=f'_v(u, v)+4xyf''_{uu}(u, v)+2(x^2+y^2)f''_{uv}(u, v)+xyf''_{vv}(u, v)$;

$\frac{\partial^2 z}{\partial y^2}=2f'_u(u, v)+4y^2f''_{uu}(u, v)+4xyf''_{uv}(u, v)+x^2f''_{vv}(u, v)$.

1904. $\frac{\partial^2 u}{\partial x^2}=f''_{xx}+2f''_{xz}\varphi'_x+f''_{zz}(\varphi'_x)^2+f'_z\varphi''_{xx}$.

1905. $\frac{\partial^2 z}{\partial x^2}=f''_{uu}(\varphi'_x)^2+2f''_{uv}\varphi'_x\psi'_x+f''_{vv}(\psi'_x)^2+f'_u\varphi''_{xx}+f'_v\psi''_{xx}$;

$\frac{\partial^2 z}{\partial x\,\partial y}=f''_{uu}\varphi'_x\varphi'_y+f''_{uv}(\varphi'_x\psi'_y+\psi'_x\varphi'_y)+f''_{vv}\psi'_x\psi'_y+f'_u\varphi''_{xy}+f'_v\psi''_{xy}$;

$\frac{\partial^2 z}{\partial y^2}=f''_{uu}(\varphi'_y)^2+2f''_{uv}\varphi'_y\psi'_y+f''_{vv}(\psi'_y)^2+f'_u\varphi''_{yy}+f'_v\psi''_{yy}$.

1914. $u(x, y)=\varphi(x)+\psi(y)$. **1915.** $u(x, y)=x\varphi(y)+\psi(y)$. **1916.** $d^2z=e^{xy}\times[(y\,dx+x\,dy)^2+2dx\,dy]$. **1917.** $d^2u=2(x\,dy\,dz+y\,dz\,dx+z\,dx\,dy)$. **1918.** $d^2z=4\varphi''(t)(x\,dx+y\,dy)^2+2\varphi'(t)(dx^2+dy^2)$. **1919.** $dz=\left(\frac{x}{y}\right)^{xy}\times\left(y\ln\frac{ex}{y}dx+x\ln\frac{x}{ey}dy\right)$; $d^2z=\left(\frac{x}{y}\right)^{xy}\left[\left(y^2\ln^2\frac{ex}{y}+\frac{y}{x}\right)dx^2+2\left(xy\ln\frac{ex}{y}\ln\frac{x}{ey}+\ln\frac{x}{y}\right)dx\,dy+\left(x^2\ln^2\frac{x}{ey}-\frac{x}{y}\right)dy^2\right]$.

1920. $d^2z=a^2f''_{uu}(u, v)\,dx^2+2abf''_{uv}(u, v)\,dx\,dy+b^2f''_{vv}(u, v)\,dy^2$.

1921. $d^2z=(ye^xf'_v+e^{2y}f''_{uu}+2ye^{x+y}f''_{uv}+y^2e^{2x}f''_{vv})\,dx^2+2(e^yf'_u+e^xf'_v+xe^{2y}f''_{uu}+e^{x+y}(1+xy)f''_{uv}+ye^{2x}f''_{vv})\,dx\,dy+\left(xe^yf'_u+x^2e^{2y}f''_{uu}+2xe^{x+y}f''_{uv}+e^{2x}f''_{vv}\right)dy^2$. **1922.** $d^3z=e^x(\cos y\,dx^3-3\sin y\,dx^2\,dy-3\cos y\,dx\,dy^2+\sin y\,dy^3)$. **1923.** $d^3z=-y\cos x\,dx^3-3\sin x\,dx^2\,dy-3\cos y\,dx\,dy^2+x\sin y\,dy^3$. **1924.** $df(1, 2)=0$; $d^2f(1, 2)=6dx^2+2dx\,dy+4.5\,dy^2$. **1925.** $d^2f(0, 0, 0)=2dx^2+4dy^2+6dz^2-4dx\,dy+8dx\,dz+4dy\,dz$. **1926.** $xy+C$. **1927.** $x^3y-\frac{y^3}{3}+\sin x+C$. **1928.** $\frac{x}{x+y}+\ln(x+y)+C$. **1929.** $\frac{1}{2}\ln(x^2+y^2)+2\arc\tan\frac{x}{y}+C$. **1930.** $\frac{x}{y}+C$. **1931.** $\sqrt{x^2+y^2}+C$. **1932.** $a=-1$, $b=-1$, $z=\frac{x-y}{x^2+y^2}+C$. **1933.** $x^2+y^2+z^2+xy+xz+yz+C$. **1934.** $x^3+2xy^2+3xz+y^2-yz-2z+C$. **1935.** $x^2yz-3xy^2z+4x^2y^2+2x+y+3z+C$. **1936.** $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+C$. **1937.** $\sqrt{x^2+y^2+z^2}+C$

1938. $\lambda=-1$. **Hint.** Write the condition of the total differential for the expression $X\,dx+Y\,dy$. **1939.** $f'_x=f'_y$. **1940.** $u=\int_a^{xy}f(z)\,dz+C$. **1941.** $\frac{dy}{dx}=-\frac{b^2x}{a^2y}$; $\frac{d^2y}{dx^2}=-\frac{b^4}{a^2y^3}$; $\frac{d^3y}{dx^3}=-\frac{3b^6x}{a^4y^5}$. **1942.** The equation defining y is the equation of a pair of straight lines. **1943.** $\frac{dy}{dx}=\frac{y^x\ln y}{1-xy^{x-1}}$. **1944.** $\frac{dy}{dx}=\frac{y}{y-1}$; $\frac{d^2y}{dx^2}=\frac{y}{(1-y)^3}$. **1945.** $\left(\frac{dy}{dx}\right)_{x=1}=3$ or -1; $\left(\frac{d^2y}{dx^2}\right)_{x=1}=8$ or -8.

1946. $\frac{dy}{dx}=\frac{x+ay}{ax-y}$; $\frac{d^2y}{dx^2}=\frac{(a^2+1)(x^2+y^2)}{(ax-y)^3}$. **1947.** $\frac{dy}{dx}=-\frac{y}{x}$; $\frac{d^2y}{dx^2}=\frac{2y}{x^2}$.

1948. $\frac{\partial z}{\partial x}=\frac{x^2-yz}{xy-z^2}$; $\frac{\partial z}{\partial y}=\frac{6y^2-3xz-2}{3(xy-z^2)}$. **1949.** $\frac{\partial z}{\partial x}=\frac{z\sin x-\cos y}{\cos x-y\sin z}$; $\frac{\partial z}{\partial y}=\frac{x\sin y-\cos z}{\cos x-y\sin z}$. **1950.** $\frac{\partial z}{\partial x}=-1$; $\frac{\partial z}{\partial y}=\frac{1}{2}$. **1951.** $\frac{\partial z}{\partial x}=-\frac{c^2x}{a^2z}$; $\frac{\partial z}{\partial y}=-\frac{c^2y}{b^2z}$;

$\frac{\partial^2 z}{\partial x^2}=-\frac{c^4(b^2-y^2)}{a^2b^2z^3}$; $\frac{\partial^2 z}{\partial x\,\partial y}=-\frac{c^4xy}{a^2b^2z^3}$; $\frac{\partial^2 z}{\partial y^2}=-\frac{c^4(a^2-x^2)}{a^2b^2z^3}$. **1953.** $\frac{dz}{dx}=$ $=\frac{\begin{vmatrix}\varphi'_x & \varphi'_y\\ \psi'_x & \psi'_y\end{vmatrix}}{\psi'_y}$. **1954.** $dz=-\frac{x}{z}dx-\frac{y}{z}dy$; $d^2z=\frac{y^2-a^2}{z^3}dx^2-2\frac{xy}{z^3}dx\,dy+$ $+\frac{x^2-a^2}{z^3}dy^2$. **1955.** $dz=0$; $d^2z=\frac{4}{15}(dx^2+dy^2)$. **1956.** $dz=\frac{z}{1-z}(dx+dy)$; $d^2z=\frac{z}{(1-z)^3}(dx^2+2dx\,dy+dy^2)$. **1961.** $\frac{dy}{dx}=\infty$; $\frac{dz}{dx}=\frac{1}{5}$; $\frac{d^2y}{dx^2}=\infty$; $\frac{d^2z}{dx^2}=\frac{4}{25}$. **1962.** $dy=\frac{y(z-x)}{x(y-z)}dx$; $dz=\frac{z(x-y)}{x(y-z)}dx$; $d^2y=-d^2z=-\frac{a}{x^3(y-z)^3}\times$ $\times[(x-y)^2+(y-z)^2+(z-x)^2]\,dx^2$. **1963.** $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=1$; $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial y^2}=0$; $\frac{\partial v}{\partial x}=-1$; $\frac{\partial v}{\partial y}=0$; $\frac{\partial^2 v}{\partial x^2}=2$; $\frac{\partial^2 v}{\partial x\,\partial y}=1$; $\frac{\partial^2 v}{\partial y^2}=0$. **1964.** $du=\frac{y}{1+y}dx+$ $+\frac{v}{1+y}dy$; $dv=\frac{1}{1+y}dx-\frac{v}{1+y}dy$; $d^2u=-d^2v=\frac{2}{(1+y)^2}dx\,dy-$ $-\frac{2v}{(1+y)^2}dy^2$. **1965.** $du=\frac{\psi'_v dx-\varphi'_v dy}{\begin{vmatrix}\varphi'_u & \varphi'_v\\ \psi'_u & \psi'_v\end{vmatrix}}$; $dv=\frac{-\psi'_u dx+\varphi'_u dy}{\begin{vmatrix}\varphi'_u & \varphi'_v\\ \psi'_u & \psi'_v\end{vmatrix}}$. **1966.** a) $\frac{\partial z}{\partial x}=-\frac{c\sin v}{u}$, $\frac{\partial z}{\partial y}=\frac{c\cos v}{u}$; b) $\frac{\partial z}{\partial x}=\frac{1}{2}(v+u)$, $\frac{\partial z}{\partial y}=\frac{1}{2}(v-u)$; c) $dz=\frac{1}{2e^{2u}}[e^{u-v}(v+u)\,dx+e^{u+v}(v-u)\,dy]$. **1967.** $\frac{\partial z}{\partial x}=F'_r(r,\varphi)\cos\varphi-$ $-F'_\varphi(r,\varphi)\frac{\sin\varphi}{r}$; $\frac{\partial z}{\partial y}=F'_r(r,\varphi)\sin\varphi+F'_\varphi(r,\varphi)\frac{\cos\varphi}{r}$. **1968.** $\frac{\partial z}{\partial x}=$ $=-\frac{c}{a}\cos\varphi\cot\psi$; $\frac{\partial z}{\partial y}=-\frac{c}{b}\sin\varphi\cot\psi$. **1969.** $\frac{d^2y}{dt^2}+\frac{dy}{dt}+y=0$. **1970.** $\frac{d^2y}{dt^2}=0$. **1971.** a) $\frac{d^2x}{dy^2}-2y\frac{dx}{dy}=0$; b) $\frac{d^3x}{dy^3}=0$. **1972.** $\tan\mu=\frac{r}{\frac{dr}{d\varphi}}$. **1973.** $K=\frac{r^2+2\left(\frac{dr}{d\varphi}\right)^2-r\frac{d^2r}{d\varphi^2}}{\left[r^2+\left(\frac{dr}{d\varphi}\right)^2\right]^{3/2}}$. **1974.** $\frac{\partial z}{\partial u}=0$. **1975.** $u\frac{\partial z}{\partial u}-z=0$. **1976.** $\frac{\partial^2 u}{\partial r^2}+$ $+\frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{1}{r}\frac{\partial u}{\partial r}=0$. **1977.** $\frac{\partial^2 z}{\partial u\,\partial v}=\frac{1}{2u}\frac{\partial z}{\partial v}$. **1978.** $\frac{\partial w}{\partial v}=0$. **1979.** $\frac{\partial^2 w}{\partial v^2}=0$. **1980.** $\frac{\partial^2 w}{\partial u^2}=\frac{1}{2}$. **1981.** a) $2x-4y-z-5=0$; $\frac{x-1}{2}=\frac{y+2}{-4}=\frac{z-5}{-1}$; b) $3x+4y-$ $-6z=0$; $\frac{x-4}{3}=\frac{y-3}{4}=\frac{z-4}{-6}$; c) $x\cos\alpha+y\sin\alpha-R=0$, $\frac{x-R\cos\alpha}{\cos\alpha}=$ $=\frac{y-R\sin\alpha}{\sin\alpha}=\frac{z-R}{0}$. **1982.** $\pm\frac{a^2}{\sqrt{a^2+b^2+c^2}}$; $\pm\frac{b^2}{\sqrt{a^2+b^2+c^2}}$; $\pm\frac{c^2}{\sqrt{a^2+b^2+c^2}}$.

1983. $3x+4y+12z-169=0$. **1985.** $x+4y+6z=\pm 21$ **1986.** $x+y+z=$ $=\pm\sqrt{a^2+b^2+c^2}$ **1987** At the points $(1, \pm 1, 0)$, the tangent planes are parallel to the xz-plane; at the points (0, 0, 0) and (2, 0, 0), to the yz-plane. There are no points on the surface at which the tangent plane is parallel to the xy-plane. **1991.** $\frac{\pi}{3}$. **1994.** Projection on the xy-plane: $\begin{cases} z=0 \\ x^2+y^2-xy-1=0. \end{cases}$ Projection on the yz-plane: $\begin{cases} x=0 \\ \frac{3y^2}{4}+z^2-1=0 \end{cases}$ Projection on the xz-plane: $\begin{cases} y=0 \\ \frac{3x^2}{4}+z^2-1=0. \end{cases}$ **Hint.** The line of tangency of the surface with the cylinder projecting this surface on some plane is a locus at which the tangent plane to the given surface is perpendicular to the plane of the projection **1996.** $f(x+h, \quad y+k)=ax^2+2bxy+cy^2+2(ax+by)h+2(bx+cy)k+ah^2+$ $+2bhk+ck^2$ **1997.** $f(x, y)=1-(x+2)^2+2(x+2)(y-1)+3(y-1)^2$. **1998.** $\Delta f(x, y)=2h+k+h^2+2hk+h^2k$. **1999.** $f(x, y, z)=(x-1)^2+(y-1)^2+$ $+(z-1)^2+2(x-1)(y-1)-(y-1)(z-1)$. **2000.** $f(x+h, \quad y+k, z+l)=$ $=f(x, y, z)+2[h(x-y-z)+k(y-x-z)+l(z-x-y)]+f(h, k, l)$. **2001.** $y+xy+\frac{3x^2y-y^3}{3!}$. **2002.** $1-\frac{x^2+y^2}{2!}+\frac{x^4+6x^2y^2+y^4}{4!}$. **2003.** $1+(y-1)+$ $+(x-1)(y-1)$. **2004.** $1+[(x-1)+(y+1)]+\frac{[(x-1)+(y+1)]^2}{2!}+$ $+\frac{[(x-1)+(y+1)]^3}{3!}$. **2005.** a) $\arctan\frac{1+\alpha}{1-\beta}\approx\frac{\pi}{4}+\frac{1}{2}(\alpha+\beta)-\frac{1}{4}(\alpha^2-\beta^2)$; b) $\sqrt{\frac{(1+\alpha)^m+(1+\beta)^n}{2}}\approx 1+\frac{1}{4}(m\alpha+n\beta)+\frac{1}{32}[(3m^2-4m)\alpha^2-2mn\alpha\beta+$ $+(3n^2-4n)\beta^2]$. **2006.** a) 1.0081; b) 0.902. **Hint.** Apply Taylor's formula for the functions: a) $f(x, y)=\sqrt{x}\sqrt[3]{y}$ in the neighbourhood of the point (1,1); b) $f(x, y)=y^x$ in the neighbourhood of the point (2,1). **2007.** $z=1+2(x-1)-$ $-(y-1)-8(x-1)^2+10(x-1)(y-1)-3(y-1)^2+\ldots$ **2008.** $z_{min}=0$ when $x=1$, $y=0$ **2009.** No extremum. **2010.** $z_{min}=-1$ when $x=1$, $y=0$. **2011.** $z_{max}=108$ when $x=3$, $y=2$. **2012.** $z_{min}=-8$ when $x=\sqrt{2}$, $y=-\sqrt{2}$ and when $x=$ $=-\sqrt{2}$, $y=\sqrt{2}$. There is no extremum for $x=y=0$. **2013.** $z_{max}=\frac{ab}{3\sqrt{3}}$ at the points $x=\frac{a}{\sqrt{3}}$, $y=\frac{b}{\sqrt{3}}$ and $x=-\frac{a}{\sqrt{3}}$, $y=-\frac{b}{\sqrt{3}}$; $z_{min}=-\frac{ab}{3\sqrt{3}}$ at the points $x=\frac{a}{\sqrt{3}}$, $y=-\frac{b}{\sqrt{3}}$ and $x=-\frac{a}{\sqrt{3}}$, $y=\frac{b}{\sqrt{3}}$. **2014.** $z_{max}=1$ when $x=y=0$. **2015.** $z_{min}=0$ when $x=y=0$; nonrigorous maximum $\left(z=\frac{1}{e}\right)$ at points of the circle $x^2+y^2=1$. **2016.** $z_{max}=\sqrt{3}$ when $x=1$, $y=-1$. **2017.** $u_{min}=-\frac{4}{3}$ when $x=-\frac{2}{3}$, $y=-\frac{1}{3}$, $z=1$. **2018.** $u_{min}=4$ when $x=\frac{1}{2}$, $y=1$, $z=1$. **2019.** The equation defines two functions, of which one has a maximum ($z_{max}=8$) when $x=1$, $y=-2$; the other has a minimum ($z_{min}=-2$) when $x=1$, $y=-2$, at points of the circle $(x-1)^2+(y+2)^2=25$, each of these functions has a boundary extremum ($z=3$). **Hint.** The functions mentioned in the answer are explicitly defined by the equalities

$z=3\pm\sqrt{25-(x-1)^2-(y+2)^2}$ and consequently exist only inside and on the boundary of the circle $(x-1)^2+(y+2)^2=25$, at the points of which both functions assume the value $z=3$. This value is the least for the first function and is the greatest for the second. **2020.** One of the functions defined by the equation has a maximum ($z_{max}=-2$) for $x=-1$, $y=2$, the other has a minimum ($z_{min}=1$) for $x=-1$, $y=2$, both functions have a boundary extremum at the points of the curve $4x^3-4y^2-12x+16y-33=0$. **2021.** $z_{max}=\frac{1}{4}$ for $x=y=\frac{1}{2}$. **2022.** $z_{max}=5$ for $x=1$, $y=2$; $z_{min}=-5$ for $x=-1$, $y=-2$ **2023.** $z_{min}=\frac{36}{13}$ for $x=\frac{18}{13}$, $y=\frac{12}{13}$. **2024.** $z_{max}=\frac{2+\sqrt{2}}{2}$ for $x=\frac{7\pi}{8}+k\pi$, $y=\frac{9\pi}{8}+k\pi$, $z_{min}=\frac{2-\sqrt{2}}{2}$ for $x=\frac{3\pi}{8}+k\pi$, $y=\frac{5\pi}{8}+k\pi$. **2025.** $u_{min}=-9$ for $x=-1$, $y=2$, $z=-2$, $u_{max}=9$ for $x=1$, $y=-2$, $z=2$. **2026.** $u_{max}=a$ for $x=\pm a$, $y=z=0$; $u_{min}=c$ for $x=y=0$ $z=\pm c$. **2027.** $u_{max}=2\cdot4^2\cdot6^3$ for $x=2$, $y=4$, $z=6$. **2028.** $u_{max}=4^4/_{27}$ at the points $\left(\frac{4}{3}, \frac{4}{3}, \frac{7}{3}\right)$; $\left(\frac{4}{3}, \frac{7}{3}, \frac{4}{3}\right)$; $\left(\frac{7}{3}, \frac{4}{3}, \frac{4}{3}\right)$; $u_{min}=4$ at the points (2, 2, 1) (2, 1, 2) (1, 2, 2). **2030.** a) Greatest value $z=3$ for $x=0$, $y=1$; b) smallest value $z=2$ for $x=1$, $y=0$. **2031.** a) Greatest value $z=\frac{2}{3\sqrt{3}}$ for $x=\pm\sqrt{\frac{2}{3}}$, $y=\sqrt{\frac{1}{3}}$; smallest value $z=-\frac{2}{3\sqrt{3}}$ for $x=\pm\sqrt{\frac{2}{3}}$, $y=-\sqrt{\frac{1}{3}}$; b) greatest value $z=1$ for $x=\pm1$, $y=0$; smallest value $z=-1$ for $x=0$, $y=\pm1$. **2032.** Greatest value $z=\frac{3\sqrt{3}}{2}$ for $x=y=\frac{\pi}{3}$ (internal maximum); smallest value $z=0$ for $x=y=0$ (boundary minimum). **2033.** Greatest value $z=13$ for $x=2$, $y=-1$ (boundary maximum); smallest value $z=-2$ for $x=y=1$ (internal minimum) and for $x=0$, $y=-1$ (boundary minimum). **2034.** Cube. **2035.** $\sqrt[3]{2V}$, $\sqrt[3]{2V}$, $\frac{1}{2}\sqrt[3]{2V}$. **2036.** Isosceles triangle. **2037.** Cube. **2038.** $a=\sqrt[4]{a}\cdot\sqrt[4]{a}\cdot\sqrt[4]{a}\cdot\sqrt[4]{a}$. **2039.** $M\left(-\frac{1}{4}, \frac{1}{4}\right)$. **2040.** Sides of the triangle are $\frac{3}{4}p$, $\frac{3}{4}p$, and $\frac{p}{2}$. **2041.** $x=\frac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3}$, $y=\frac{m_1y_1+m_2y_2+m_3y_3}{m_1+m_2+m_3}$. **2042.** $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$. **2043.** The dimensions of the parallelepiped are $\frac{2a}{\sqrt{3}}$, $\frac{2b}{\sqrt{3}}$, $\frac{2c}{\sqrt{3}}$, where a, b, and c are the semi-axes of the ellipsoid. **2044.** $x=y=2\delta+\sqrt[3]{2V}$, $z=\frac{x}{2}$. **2045.** $x=\pm\frac{a}{\sqrt{2}}$, $y=\pm\frac{b}{\sqrt{2}}$ **2046.** Major axis, $2a=6$, minor axis, $2b=2$. **Hint.** The square of the distance of the point (x, y) of the ellipse from its centre (coordinate origin) is equal to x^2+y^2. The problem reduces to finding the extremum of the function x^2+y^2 provided $5x^2+8xy+5y^2=9$. **2047.** The radius of the base of the cylinder is $\frac{R}{2}\sqrt{2+\frac{2}{\sqrt{5}}}$, the altitude $R\sqrt{2-\frac{2}{\sqrt{5}}}$, where R is the radius of the sphere. **2048.** The channel must connect the point of the parabola $\left(\frac{1}{2}, \frac{1}{4}\right)$ with the point of the straight line $\left(\frac{11}{8}, -\frac{5}{8}\right)$; its length is $\frac{7\sqrt{2}}{8}$. **2049.** $\frac{1}{14}\sqrt{2730}$. **2050.** $\frac{\sin\alpha}{\sin\beta}=\frac{v_1}{v_2}$. **Hint.** Obviously, the point M, at which the ray passes from one medium into the other, must lie between A_1 and B_1; $AM=\frac{a}{\cos\alpha}$, $BM=\frac{b}{\cos\beta}$, $A_1M=a\tan\alpha$, $B_1M=b\tan\beta$. The duration of motion of the ray is $\frac{a}{v_1\cos\alpha}+\frac{b}{v_2\cos\beta}$. The problem reduces to finding the minimum of the function $f(\alpha, \beta)=\frac{a}{v_1\cos\alpha}+\frac{b}{v_2\cos\beta}$ provided that $a\tan\alpha+b\tan\beta=c$. **2051.** $\alpha=\beta$. **2052.** $I_1:I_2:I_3=\frac{1}{R_1}:\frac{1}{R_2}:\frac{1}{R_3}$. **Hint.** Find the minimum of the function $f(I_1, I_2, I_3)=I_1^2R_1+I_2^2R_2+I_3^2R_3$ provided that $I_1+I_2+I_3=I$. **2053.** The isolated point (0, 0). **2054.** Cusp of second kind (0, 0). **2055.** Tacnode (0, 0). **2056.** Isolated point (0, 0). **2057.** Node (0, 0). **2058.** Cusp of first kind (0, 0). **2059.** Node (0, 0). **2060.** Node (0, 0). **2061.** Origin is isolated point if $a>b$; it is a cusp of the first kind if $a=b$, and a node if $a<b$. **2062.** If among the quantities a, b, and c, none are equal, then the curve does not have any singular points. If $a=b<c$, then $A(a, 0)$ is an isolated point; if $a<b=c$, then $B(b, 0)$ is a node; if $a=b=c$, then $A(a, 0)$ is a cusp of the first kind. **2063.** $y=\pm x$. **2064.** $y^2=2px$. **2065.** $y=\pm R$. **2066.** $x^{2/3}+y^{2/3}=l^{2/3}$. **2067.** $xy=\frac{1}{2}S$. **2068.** A pair of conjugate equilateral hyperbolas, whose equations, if the axes of symmetry of the ellipses are taken as the coordinate axes, have the form $xy=\pm\frac{S}{2\pi}$. **2069.** a) The discriminant curve $y=0$ is the locus of points of inflection and of the envelope of the given family; b) the discriminant curve $y=0$ is the locus of cusps and of the envelope of the family; c) the discriminant curve $y=0$ is the locus of cusps and is not an envelope; d) the discriminant curve decomposes into the straight lines: $x=0$ (locus of nodes) and $x=a$ (envelope). **2070.** $y=\frac{v_0^2}{2g}-\frac{gx^2}{2v_0^2}$. **2071.** $7\frac{1}{3}$. **2072.** $\sqrt{9+4\pi^2}$. **2073.** $\sqrt{3}\,(e^t-1)$. **2074.** 42. **2075.** 5. **2076.** x_0+z_0. **2077.** $11+\frac{\ln 10}{9}$. **2079.** a) Straight line; b) parabola; c) ellipse; d) hyperbola. **2080.** 1) $\frac{da}{dt}a^0$ 2) $a\frac{da^0}{dt}$; 3) $\frac{da}{dt}a^0+a\frac{da^0}{dt}$. **2081.** $\frac{d}{dt}(abc)=\left(\frac{da}{dt}bc\right)+\left(a\frac{db}{dt}c\right)+\left(ab\frac{dc}{dt}\right)$ **2082.** $4t(t^2+1)$. **2083.** $x=3\cos t$; $y=4\sin t$ (ellipse); for $t=0$, $\boldsymbol{v}=4\boldsymbol{j}$, $\boldsymbol{w}=-3\boldsymbol{i}$; fo $t=\frac{\pi}{4}$, $\boldsymbol{v}=-\frac{3\sqrt{2}}{2}\boldsymbol{i}+2\sqrt{2}\boldsymbol{j}$, $\boldsymbol{w}=-\frac{3\sqrt{2}}{2}\boldsymbol{i}-2\sqrt{2}\boldsymbol{j}$; for $t=\frac{\pi}{2}$, $\boldsymbol{v}=-3\boldsymbol{i}$, $\boldsymbol{w}=-4\boldsymbol{j}$. **2084.** $x=2\cos t$, $y=2\sin t$, $z=3t$ (screw-line); $\boldsymbol{v}=-2\boldsymbol{i}\sin t+2\boldsymbol{j}\cos t+3\boldsymbol{k}$; $v=\sqrt{13}$ for any t; $\boldsymbol{w}=-2\boldsymbol{i}\cos t-2\boldsymbol{j}\sin t$; $w=2$ for any t for $t=0$, $\boldsymbol{v}=2\boldsymbol{j}+3\boldsymbol{k}$, $\boldsymbol{w}=-2\boldsymbol{i}$; for $t=\frac{\pi}{2}$, $\boldsymbol{v}=-2\boldsymbol{i}+3\boldsymbol{k}$, $\boldsymbol{w}=-2\boldsymbol{j}$

2085. $x=\cos\alpha\cos\omega t$; $y=\sin\alpha\cos\omega t$; $z=\sin\omega t$ (circle); $\mathbf{v}=-\omega\mathbf{i}\cos\alpha\sin\omega t-\omega\mathbf{j}\sin\alpha\sin\omega t+\omega\mathbf{k}\cos\omega t$; $v=|\omega|$; $\mathbf{w}=-\omega^2\mathbf{i}\cos\alpha\cos\omega t-\omega^2\mathbf{j}\sin\alpha\cos\omega t-\omega^2\mathbf{k}\sin\omega t$; $w=\omega^2$. **2086.** $v=\sqrt{v_{x_0}^2+v_{y_0}^2+(v_{x_0}-gt)^2}$; $w_x=w_y=0$; $w_z=-g$; $w=g$. **2088.** $\omega\sqrt{a^2+h^2}$, where $\omega=\frac{d\theta}{dt}$ is the angular speed of rotation of the screw. **2089.** $\sqrt{a^2\omega^2+v_0^2-2a\omega v_0\sin\omega t}$. **2090.** $\boldsymbol{\tau}=\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{k})$; $\boldsymbol{\nu}=-\mathbf{j}$; $\boldsymbol{\beta}=\frac{\sqrt{2}}{2}(\mathbf{i}-\mathbf{k})$. **2091.** $\boldsymbol{\tau}=\frac{1}{\sqrt{3}}[(\cos t-\sin t)\mathbf{i}+(\sin t+\cos t)\mathbf{j}+\mathbf{k}]$; $\boldsymbol{\nu}=-\frac{1}{\sqrt{2}}[(\sin t+\cos t)\mathbf{i}+(\sin t-\cos t)\mathbf{j}]$; $\cos(\widehat{\boldsymbol{\tau}, z})=\frac{\sqrt{3}}{3}$; $\cos(\widehat{\boldsymbol{\nu}, z})=0$. **2092.** $\boldsymbol{\tau}=\frac{\mathbf{i}+4\mathbf{j}+2\mathbf{k}}{\sqrt{21}}$; $\boldsymbol{\nu}=\frac{-4\mathbf{i}+5\mathbf{j}-8\mathbf{k}}{\sqrt{105}}$; $\boldsymbol{\beta}=\frac{-2\mathbf{i}+\mathbf{k}}{\sqrt{5}}$. **2093.** $\frac{x-a\cos t}{-a\sin t}=\frac{y-a\sin t}{a\cos t}=\frac{z-bt}{b}$ (tangent); $\frac{x-a\cos t}{b\sin t}=\frac{y-a\sin t}{-b\cos t}=\frac{z-bt}{a}$ (binormal); $\frac{x-a\cos t}{\cos t}=\frac{y-a\sin t}{\sin t}=\frac{z-bt}{0}$ (principal normal). The direction cosines of the tangent are $\cos\alpha=-\frac{a\sin t}{\sqrt{a^2+b^2}}$; $\cos\beta=\frac{a\cos t}{\sqrt{a^2+b^2}}$; $\cos\gamma=\frac{b}{\sqrt{a^2+b^2}}$. The direction cosines of the principal normal are $\cos\alpha_1=\cos t$; $\cos\beta_1=\sin t$; $\cos\gamma_1=0$. **2094.** $2x-z=0$ (normal plane); $y-1=0$ (osculating plane); $x+2z-5=0$ (rectifying plane). **2095.** $\frac{x-2}{1}=\frac{y-4}{4}=\frac{z-8}{12}$ (tangent); $x+4y+12z-114=0$ (normal plane); $12x-6y+z-8=0$ (osculating plane). **2096.** $\frac{x-\frac{t^2}{4}}{t^2}=\frac{y-\frac{t^3}{3}}{t}=\frac{z-\frac{t^2}{2}}{1}$ (tangent); $\frac{x-\frac{t^4}{4}}{t^3+2t}=\frac{y-\frac{t^3}{3}}{1-t^4}=\frac{z-\frac{t^2}{2}}{-2t^3-t}$ (principal normal); $\frac{x-\frac{t^4}{4}}{1}=\frac{y-\frac{t^3}{3}}{-2t}=\frac{z-\frac{t^2}{2}}{t^2}$ (binormal); $M_1\left(\frac{1}{4}, -\frac{1}{3}, \frac{1}{2}\right)$; $M_2\left(4, -\frac{8}{3}, 2\right)$. **2097.** $\frac{x-2}{1}=\frac{y+2}{-1}=\frac{z-2}{2}$ (tangent); $x+y=0$ (osculating plane); $\frac{x-2}{1}=\frac{y+2}{-1}=\frac{z-2}{-1}$ (principal normal); $\frac{x-2}{+1}=\frac{y+2}{1}=\frac{z-2}{0}$ (binormal); $\cos\alpha_2=\frac{1}{\sqrt{2}}$; $\cos\beta_2=\frac{1}{\sqrt{2}}$, $\cos\gamma_2=0$. **2098.** a) $\frac{x-\frac{R}{2}}{2}=\frac{y-\frac{R}{2}}{0}=\frac{z-\frac{\sqrt{2}}{2}R}{-\sqrt{2}}$ (tangent); $x\sqrt{2}-z=0$ (normal plane); b) $\frac{x-1}{1}=\frac{y-1}{1}=\frac{z-2}{4}$ (tangent); $x+y+4z-10=0$ (normal plane); c) $\frac{x-2}{2\sqrt{3}}=\frac{y-2\sqrt{3}}{1}=\frac{z-3}{-2\sqrt{3}}$ (tangent); $2\sqrt{3}\,x+y-2\sqrt{3}z=0$ (normal plane); **2099.** $x+y=0$. **2100.** $x-y-z\sqrt{2}=0$. **2101.** a) $4x-y-z-9=0$; b) $9x-6y+2z-18=0$; c) $b^2x_0^3x-a^2y_0^3y+(a^2-b^2)z_0^3z=a^2b^2(a^2-b^2)$. **2102.** $6x-8y-z+3=0$ (osculating plane); $\frac{x-1}{31}=\frac{y-1}{26}=\frac{z-1}{-22}$ (principal normal); $\frac{x-1}{-6}=\frac{y-1}{8}=\frac{z-1}{1}$

(binormal). **2103.** $bx-z=0$ (osculating plane); $\left.\begin{matrix}x=0,\\ z=0\end{matrix}\right\}$ (principal normal); $\left.\begin{matrix}x+bz=0,\\ y=0\end{matrix}\right\}$ (binormal); $\tau=\frac{i+bk}{\sqrt{1+b^2}}$; $\beta=\frac{-bi+k}{\sqrt{1+b^2}}$; $\nu=j$. **2106.** $2x+3y+19z-27=0$. **2107.** a) $\sqrt{2}$; b) $\frac{\sqrt{6}}{4}$. **2108.** a) $K=\frac{e^{-t}\sqrt{2}}{3}$; $T=\frac{e^{-t}}{3}$; b) $K=T=\frac{1}{2a\cosh^2 t}$. **2109.** a) $R=\varrho=\frac{(y+a)^2}{a}$; b) $R=\varrho=\frac{(p^4+2x^4)^2}{8p^4x^3}$. **2111.** $\frac{av^2}{a^2+b^2}$. **2112.** When $t=0$, $K=2$, $w_\tau=0$, $w_n=2$; when $t=1$, $K=\frac{1}{7}\sqrt{\frac{19}{14}}$, $w_\tau=\frac{22}{\sqrt{14}}$, $w_n=2\sqrt{\frac{19}{14}}$.

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2113. $4\frac{2}{3}$. **2114.** $\ln\frac{25}{24}$ **2115.** $\frac{\pi}{12}$. **2116.** $\frac{9}{4}$. **2117.** 50.4. **2118.** $\frac{\pi a^2}{2}$. **2119.** 2.4. **2120.** $\frac{\pi}{6}$. **2121.** $x=\frac{y^2}{4}-1$; $x=2-y$; $y=-6$; $y=2$. **2122.** $y=x^2$; $y=x+9$; $x=1$; $x=3$. **2123.** $y=x$; $y=10-x$; $y=0$; $y=4$. **2124.** $y=\frac{x}{3}$; $y=2x$; $x=1$; $x=3$. **2125.** $y=0$; $y=\sqrt{25-x^2}$; $x=0$; $x=3$. **2126.** $y=x^2$; $y=x+2$; $x=-1$; $x=2$. **2127.** $\int_0^1 dy\int_0^2 f(x, y)\,dx=\int_0^2 dx\int_0^1 f(x, y)\,dy$.

2128. $\int_0^1 dy\int_y^1 f(x, y)\,dx=\int_0^1 dx\int_0^x f(x, y)\,dy$. **2129.** $\int_0^1 dy\int_0^{2-y} f(x, y)\,dx=$ $=\int_0^1 dx\int_0^1 f(x, y)\,dy+\int_1^2 dx\int_0^{2-x} f(x, y)\,dy$. **2130.** $\int_1^2 dx\int_{2x}^{2x+3} f(x, y)\,dy=$ $=\int_2^4 dy\int_1^{\frac{y}{2}} f(x, y)\,dx+\int_4^5 dy\int_1^2 f(x, y)\,dx+\int_5^7 dy\int_{\frac{y-3}{2}}^2 f(x, y)\,dx$.

2131. $\int_0^1 dy\int_{-y}^{y} f(x, y)\,dx+\int_1^{\sqrt{2}} dy\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} f(x, y)\,dx=\int_{-1}^0 dx\int_{-x}^{\sqrt{2-x^2}} f(x, y)\,dy+$ $+\int_0^1 dx\int_x^{\sqrt{2-x^2}} f(x, y)\,dy$. **2132.** $\int_{-1}^1 dx\int_{2x^2}^2 f(x, y)\,dy=\int_0^2 dy\int_{-\sqrt{\frac{y}{2}}}^{\sqrt{\frac{y}{2}}} f(x, y)\,dx$.

2133. $\int\limits_{-2}^{-1} dx \int\limits_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y)\,dy + \int\limits_{-1}^{1} dx \int\limits_{-\sqrt{4-x^2}}^{-\sqrt{1-x^2}} f(x, y)\,dy + \int\limits_{-1}^{1} dx \int\limits_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} f(x, y)\,dy +$

$+\int\limits_{1}^{2} dx \int\limits_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y)\,dy = \int\limits_{-2}^{-1} dy \int\limits_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x, y)\,dx + \int\limits_{-1}^{1} dy \int\limits_{-\sqrt{4-y^2}}^{-\sqrt{1-y^2}} f(x, y)\,dx +$

$+\int\limits_{-1}^{1} dy \int\limits_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} f(x, y)\,dx + \int\limits_{1}^{2} dy \int\limits_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x, y)\,dx.$

2134. $\int\limits_{-3}^{-2} dx \int\limits_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x, y)\,dy + \int\limits_{-2}^{2} dx \int\limits_{-\sqrt{1+x^2}}^{\sqrt{1+x^2}} f(x, y)\,dy +$

$+\int\limits_{2}^{3} dx \int\limits_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x, y)\,dy = \int\limits_{-\sqrt{5}}^{-1} dy \int\limits_{-\sqrt{9-y^2}}^{-\sqrt{y^2-1}} f(x, y)\,dx +$

$+\int\limits_{-\sqrt{5}}^{-1} dy \int\limits_{\sqrt{y^2-1}}^{\sqrt{9-y^2}} f(x, y)\,dx + \int\limits_{-1}^{1} dy \int\limits_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x, y)\,dx + \int\limits_{1}^{\sqrt{5}} dy \int\limits_{-\sqrt{9-y^2}}^{-\sqrt{y^2-1}} f(x, y)\,dy +$

$+\int\limits_{1}^{\sqrt{5}} dy \int\limits_{\sqrt{y^2-1}}^{\sqrt{9-y^2}} f(x, y)\,dx.$ 2135. a) $\int\limits_{0}^{1} dx \int\limits_{0}^{1-x} f(x, y)\,dy = \int\limits_{0}^{1} dy \int\limits_{0}^{1-y} f(x, y)\,dx;$

b) $\int\limits_{-a}^{a} dx \int\limits_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x, y)\,dy = \int\limits_{-a}^{a} dy \int\limits_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} f(x, y)\,dx;$ c) $\int\limits_{0}^{1} dx \int\limits_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} f(x, y)\,dy =$

$= \int\limits_{-1/2}^{1/2} dy \int\limits_{\frac{1-\sqrt{1-4y^2}}{2}}^{\frac{1+\sqrt{1-4y^2}}{2}} f(x, y)\,dx;$ d) $\int\limits_{-1}^{1} dx \int\limits_{x}^{1} f(x, y)\,dy = \int\limits_{-1}^{1} dy \int\limits_{-1}^{y} f(x, y)\,dx;$

e) $\int\limits_{0}^{a} dy \int\limits_{y}^{y+2a} f(x, y)\,dx = \int\limits_{0}^{a} dx \int\limits_{0}^{x} f(x, y)\,dy + \int\limits_{a}^{2a} dx \int\limits_{0}^{a} f(x, y)\,dy + \int\limits_{2a}^{3a} dx \int\limits_{x-2a}^{a} f(x, y)\,dy.$

2136. $\int\limits_{0}^{48} dy \int\limits_{\frac{y}{12}}^{\sqrt{\frac{y}{3}}} f(x, y)\,dx.$ 2137. $\int\limits_{0}^{2} dy \int\limits_{\frac{y}{3}}^{\frac{y}{2}} f(x, y)\,dx + \int\limits_{2}^{3} dy \int\limits_{\frac{y}{3}}^{1} f(x, y)\,dx.$

2138. $\int\limits_{0}^{\frac{a}{2}} dy \int\limits_{\sqrt{a^2-2ay}}^{\sqrt{a^2-y^2}} f(x, y)\,dx + \int\limits_{\frac{a}{2}}^{a} dy \int\limits_{0}^{\sqrt{a^2-y^2}} f(x, y)\,dx.$

(binormal). **2103.** $bx - z = 0$ (osculating plane); $\left.\begin{matrix} x=0, \\ z=0 \end{matrix}\right\}$ (principal normal); $\left.\begin{matrix} x+bz=0, \\ y=0 \end{matrix}\right\}$ (binormal); $\boldsymbol{\tau} = \frac{\mathbf{i}+b\mathbf{k}}{\sqrt{1+b^2}}$; $\boldsymbol{\beta} = \frac{-b\mathbf{i}+\mathbf{k}}{\sqrt{1+b^2}}$; $\boldsymbol{\nu} = \mathbf{j}$. **2106.** $2x + 3y + 19z - 27 = 0$. **2107.** a) $\sqrt{2}$; b) $\frac{\sqrt{6}}{4}$. **2108.** a) $K = \frac{e^{-t}\sqrt{2}}{3}$; $T = \frac{e^{-t}}{3}$; *b*) $K = T = \frac{1}{2a\cosh^2 t}$. **2109.** a) $R = \varrho = \frac{(y+a)^2}{a}$; b) $R = \varrho = \frac{(p^4+2x^4)^2}{8p^4x^3}$. **2111.** $\frac{av^2}{a^2+b^2}$. **2112.** When $t=0$, $K=2$, $w_\tau = 0$, $w_n = 2$; when $t=1$, $K = \frac{1}{7}\sqrt{\frac{19}{14}}$, $w_\tau = \frac{22}{\sqrt{14}}$, $w_n = 2\sqrt{\frac{19}{14}}$.

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2113. $4\frac{2}{3}$. **2114.** $\ln\frac{25}{24}$. **2115.** $\frac{\pi}{12}$ **2116.** $\frac{9}{4}$. **2117.** 50.4. **2118.** $\frac{\pi a^2}{2}$. **2119.** 2.4. **2120.** $\frac{\pi}{6}$. **2121.** $x = \frac{y^2}{4} - 1$; $x = 2 - y$; $y = -6$; $y = 2$. **2122.** $y = x^2$; $y = x + 9$; $x = 1$; $x = 3$. **2123.** $y = x$; $y = 10 - x$; $y = 0$; $y = 4$. **2124.** $y = \frac{x}{3}$; $y = 2x$; $x = 1$; $x = 3$. **2125.** $y = 0$; $y = \sqrt{25 - x^2}$; $x = 0$; $x = 3$. **2126.** $y = x^2$; $y = x + 2$; $x = -1$; $x = 2$. **2127.** $\int_0^1 dy \int_0^2 f(x, y)\,dx = \int_0^2 dx \int_0^1 f(x, y)\,dy$.

2128. $\int_0^1 dy \int_y^1 f(x, y)\,dx = \int_0^1 dx \int_0^x f(x, y)\,dy$. **2129.** $\int_0^1 dy \int_0^{2-y} f(x, y)\,dx = \int_0^1 dx \int_0^1 f(x, y)\,dy + \int_1^2 dx \int_0^{2-x} f(x, y)\,dy$. **2130.** $\int_1^2 dx \int_{2x}^{2x+3} f(x, y)\,dy = \int_2^4 dy \int_1^{\frac{y}{2}} f(x, y)\,dx + \int_4^5 dy \int_1^2 f(x, y)\,dx + \int_5^7 dy \int_{\frac{y-3}{2}}^2 f(x, y)\,dx$.

2131. $\int_0^1 dy \int_{-y}^{y} f(x, y)\,dx + \int_1^{\sqrt{2}} dy \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} f(x, y)\,dx = \int_{-1}^0 dx \int_{-x}^{\sqrt{2-x^2}} f(x, y)\,dy + \int_0^1 dx \int_x^{\sqrt{2-x^2}} f(x, y)\,dy$. **2132.** $\int_{-1}^1 dx \int_{2x^2}^2 f(x, y)\,dy = \int_0^2 dy \int_{-\sqrt{\frac{y}{2}}}^{\sqrt{\frac{y}{2}}} f(x, y)\,dx$.

2133. $$\int_{-2}^{-1} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y)\,dy + \int_{-1}^{1} dx \int_{-\sqrt{4-x^2}}^{-\sqrt{1-x^2}} f(x, y)\,dy + \int_{-1}^{1} dx \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} f(x, y)\,dy + \int_{1}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y)\,dy = \int_{-2}^{-1} dy \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x, y)\,dx + \int_{-1}^{1} dy \int_{-\sqrt{4-y^2}}^{-\sqrt{1-y^2}} f(x, y)\,dx + \int_{-1}^{1} dy \int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} f(x, y)\,dx + \int_{1}^{2} dy \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x, y)\,dx.$$

2134. $$\int_{-3}^{-2} dx \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x, y)\,dy + \int_{-2}^{2} dx \int_{-\sqrt{1+x^2}}^{\sqrt{1+x^2}} f(x, y)\,dy + \int_{2}^{3} dx \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x, y)\,dy = \int_{-\sqrt{5}}^{-1} dy \int_{-\sqrt{9-y^2}}^{-\sqrt{y^2-1}} f(x, y)\,dx + \int_{-\sqrt{5}}^{-1} dy \int_{\sqrt{y^2-1}}^{\sqrt{9-y^2}} f(x, y)\,dx + \int_{-1}^{1} dy \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x, y)\,dx + \int_{1}^{\sqrt{5}} dy \int_{-\sqrt{9-y^2}}^{-\sqrt{y^2-1}} f(x, y)\,dy + \int_{1}^{\sqrt{5}} dy \int_{\sqrt{y^2-1}}^{\sqrt{9-y^2}} f(x, y)\,dx.$$

2135. a) $\int_0^1 dx \int_0^{1-x} f(x, y)\,dy = \int_0^1 dy \int_0^{1-y} f(x, y)\,dx$;

b) $\int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x, y)\,dy = \int_{-a}^{a} dy \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} f(x, y)\,dx$; c) $\int_0^1 dx \int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} f(x, y)\,dy = \int_{-1/2}^{1/2} dy \int_{\frac{1-\sqrt{1-4y^2}}{2}}^{\frac{1+\sqrt{1-4y^2}}{2}} f(x, y)\,dx$; d) $\int_{-1}^{1} dx \int_{x}^{1} f(x, y)\,dy = \int_{-1}^{1} dy \int_{-1}^{y} f(x, y)\,dx$;

e) $\int_0^a dy \int_y^{y+2a} f(x, y)\,dx = \int_0^a dx \int_0^x f(x, y)\,dy + \int_a^{2a} dx \int_0^a f(x, y)\,dy + \int_{2a}^{3a} dx \int_{x-2a}^{a} f(x, y)\,dy$.

2136. $\int_0^{48} dy \int_{\frac{y}{12}}^{\sqrt{\frac{y}{3}}} f(x, y)\,dx$. **2137.** $\int_0^2 dy \int_{\frac{y}{3}}^{\frac{y}{2}} f(x, y)\,dx + \int_2^3 dy \int_{\frac{y}{3}}^{1} f(x, y)\,dx$.

2138. $\int_0^{\frac{a}{2}} dy \int_{\sqrt{a^2-2ay}}^{\sqrt{a^2-y^2}} f(x, y)\,dx + \int_{\frac{a}{2}}^{a} dy \int_0^{\sqrt{a^2-y^2}} f(x, y)\,dx$.

2139. $\int\limits_0^{\frac{a\sqrt{3}}{2}} dy \int\limits_{\frac{a}{2}}^{a} f(x, y)\,dx + \int\limits_{\frac{a\sqrt{3}}{2}}^{a} dy \int\limits_{a-\sqrt{a^2-y^2}}^{a} f(x, y)\,dx.$

2140. $\int\limits_0^{a} dy \int\limits_{\frac{y^2}{4a}}^{a-\sqrt{a^2-y^2}} f(x, y)\,dx + \int\limits_0^{a} dy \int\limits_{a+\sqrt{a^2-y^2}}^{2a} f(x, y)\,dx + \int\limits_0^{2\sqrt{2}a} dy \int\limits_{\frac{y^2}{4a}}^{2a} f(x, y)\,dx.$

2141. $\int\limits_{-1}^{0} dx \int\limits_0^{\sqrt{1-x^2}} f(x, y)\,dy + \int\limits_0^{1} dx \int\limits_0^{1-x} f(x, y)\,dy.$ **2142.** $\int\limits_0^{\frac{1}{2}} dx \int\limits_0^{\sqrt{2x}} f(x, y)\,dy +$ $+ \int\limits_{\frac{1}{2}}^{\sqrt{2}} dx \int\limits_0^{1} f(x, y)\,dy + \int\limits_{\sqrt{2}}^{\sqrt{3}} dx \int\limits_0^{\sqrt{3-x^2}} f(x, y)\,dy.$ **2143.** $\int\limits_0^{\frac{R\sqrt{2}}{2}} dy \int\limits_y^{\sqrt{R^2-y^2}} f(x, y)\,dx.$

2144. $\int\limits_0^{1} dy \int\limits_{\arcsin y}^{\pi-\arcsin y} f(x, y)\,dx.$ **2145.** $\frac{1}{6}$. **2146.** $\frac{1}{6}$. **2147.** $\frac{\pi}{2}a$. **2148.** $\frac{\pi}{6}$.

2149. 6. **2150.** $\frac{1}{2}$. **2151.** $\ln 2$ **2152.** a) $\frac{4}{3}$; b) $\frac{15\pi-16}{150}$; c) $2\frac{2}{5}$.

2153. $\frac{8\sqrt{2}}{21}p^5$. **2154.** $\int\limits_1^{3} dx \int\limits_0^{\sqrt{1-(x-2)^2}} xy\,dy = \frac{4}{3}$. **2155.** $\frac{8}{3}a\sqrt{2a}.$

2156. $\frac{5}{2}\pi R^3$. Hint. $\iint\limits_{(S)} y\,dx\,dy = \int\limits_0^{2\pi R} dx \int\limits_0^{y=f(x)} y\,dy =$ $= \int\limits_0^{2\pi} R(1-\cos t)\,dt \int\limits_0^{R(1-\cos t)} y\,dy$, where the last integral is obtained from the preceding one by the substitution $x = R(t-\sin t)$. **2157.** $\frac{R^4}{80}$. **2158.** $\frac{1}{6}$.

2159. $a^2 + \frac{R^2}{2}$. **2160.** $\int\limits_0^{\frac{\pi}{4}} d\varphi \int\limits_0^{\frac{1}{\cos\varphi}} rf(r\cos\varphi, r\sin\varphi)\,dr +$ $+ \int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\varphi \int\limits_0^{\frac{1}{\sin\varphi}} rf(r\cos\varphi, r\sin\varphi)\,dr.$ **2161.** $\int\limits_0^{\frac{\pi}{4}} d\varphi \int\limits_0^{\frac{2}{\cos\varphi}} rf(r^2)\,dr.$

2162. $\int\limits_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi \int\limits_0^{\frac{1}{\sin\varphi}} rf(r\cos\varphi,\ r\sin\varphi)\,dr.$ **2163.** $\int\limits_0^{\frac{\pi}{4}} f(\tan\varphi)\,d\varphi \int\limits_0^{\frac{\sin\varphi}{\cos^2\varphi}} r\,dr +$

$$+\int\limits_{\frac{\pi}{4}}^{\frac{3\pi}{4}} f(\tan\varphi)\,d\varphi \int\limits_0^{\frac{1}{\sin\varphi}} r\,dr + \int\limits_{\frac{3\pi}{4}}^{\pi} f(\tan\varphi)\,d\varphi \int\limits_0^{\frac{\sin\varphi}{\cos^2\varphi}} r\,dr.$$

2164. $\int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \int\limits_0^{a\sqrt{\cos 2\varphi}} rf(r\cos\varphi,\ r\sin\varphi)\,dr + \int\limits_{\frac{3\pi}{4}}^{5\pi} d\varphi \int\limits_0^{a\sqrt{\cos 2\varphi}} rf(r\cos\varphi,\ r\sin\varphi)\,dr.$

2165. $\int\limits_0^{\frac{\pi}{2}} d\varphi \int\limits_0^{a\cos\varphi} r^2\sin\varphi\,dr = \frac{a^3}{12}.$ **2166.** $\frac{3}{2}\pi a^4.$ **2167.** $\frac{\pi a^3}{3}.$

2168. $\left(\frac{22}{9}+\frac{\pi}{2}\right)a^3.$ **2169.** $\frac{\pi a^3}{6}.$ **2170.** $\left(\frac{\pi}{3}-\frac{16\sqrt{2}-20}{9}\right)\frac{a^3}{2}.$

2171. $\frac{2}{3}\pi ab.$ **Hint.** The Jacobian is $I=abr$. The limits of integration are $0\leqslant\varphi\leqslant 2\pi$, $0\leqslant r\leqslant 1$. **2172.** $\int\limits_{\frac{\alpha}{1+\alpha}}^{\frac{\beta}{1+\beta}} dv \int\limits_0^{\frac{c}{1-v}} f(u-uv,\ uv)\,u\,du.$ **Solution.** We have $x=u(1-v)$ and $y=uv$; the Jacobian is $I=u$. We define the limits u as functions of v: when $x=0$, $u(1-v)=0$, whence $u=0$ (since $1-v\neq 0$); when $x=c$, $u=\frac{c}{1-v}$. Limits of variation of v: since $y=\alpha x$, it follows that $uv=\alpha u(1-v)$, whence $v=\frac{\alpha}{1+\alpha}$; for $y=\beta x$ we find $v=\frac{\beta}{1+\beta}$. **2173.** $I=\frac{1}{2}\left[\int\limits_0^1 du\int\limits_{-u}^{u} f\left(\frac{u+v}{2},\ \frac{u-v}{2}\right)dv+\right.$

$$\left.+\int\limits_1^2 du\int\limits_{u-2}^{2-u} f\left(\frac{u+v}{2},\ \frac{u-v}{2}\right)dv\right]=\frac{1}{2}\left[\int\limits_{-1}^{0} dv\int\limits_{-v}^{2+v} f\left(\frac{u+v}{2},\ \frac{u-v}{2}\right)du+\right.$$

$\left.+\int\limits_0^1 dv\int\limits_v^{2-v} f\left(\frac{u+v}{2},\ \frac{u-v}{2}\right)du\right].$ **Hint.** After change of variables, the equations of the sides of the square will be $u=v$; $u+v=2$; $u-v=2$; $u=-v$.

2174. $ab\left[\left(\frac{a^2}{h^2}-\frac{b^2}{k^2}\right)\arctan\frac{ak}{bh}+\frac{ab}{hk}\right].$ **Solution.** The equation of the curve

$r^4=r^2\left(\frac{a^2}{h^2}\cos^2\varphi-\frac{b^2}{k^2}\sin^2\varphi\right)$, whence the lower limit for r will be 0 and the upper limit, $r=\sqrt{\frac{a^2}{h^2}\cos^2\varphi-\frac{b^2}{k^2}\sin^2\varphi}$. Since r must be real, it follows that $\frac{a^2}{h^2}\cos^2\varphi-\frac{b^2}{k^2}\sin^2\varphi\geqslant 0$; whence for the first quadrantal angle we have $\tan\varphi\leqslant\frac{ak}{bh}$. Due to symmetry of the region of integration relative to the axes, we can compute $\frac{1}{4}$ of the entire integral, confining ourselves to the first quadrant: $\iint\limits_{(S)} dx\,dy=4\int\limits_0^{\arctan\frac{ak}{bh}} d\varphi \int\limits_0^{\sqrt{\frac{a^4}{h^2}\cos^2\varphi-\frac{b^2}{k^2}\sin^2\varphi}} abr\,dr$.

2175. a) $4\frac{1}{2}$; $\int\limits_0^1 dy\int\limits_{-\sqrt{y}}^{\sqrt{y}} dx+\int\limits_1^2 dy\int\limits_{y-2}^{\sqrt{y}} dx$; b) $\frac{\pi a^2}{4}-\frac{a^2}{2}$; $\int\limits_0^a dx\int\limits_{a-x}^{\sqrt{a^2-x^2}} dy$.

2176. a) $\frac{9}{2}$; b) $\left(2+\frac{\pi}{4}\right)a^2$. **2177.** $\frac{7a^2}{120}$. **2178.** $\frac{10}{3}a^2$. **2179.** π **Hint.** $-1\leqslant x\leqslant 1$. **2180.** $\frac{16}{3}\sqrt{15}$. **2181.** $3\left(\frac{\pi}{4}+\frac{1}{2}\right)$. **2182.** $\frac{4\pi}{3}-\sqrt{3}$. **2183.** $\frac{5}{4}\pi a^2$. **2184.** 6. **2185.** 10π. **Hint.** Change the variables $x-2y=u$, $3x+4y=v$. **2186.** $\frac{1}{3}(b-a)(\beta-\alpha)$. **2187.** $\frac{1}{3}(\beta-\alpha)\ln\frac{b}{a}$. **2188.** $v=\int\limits_0^1 dy\int\limits_y^1(1-x)\,dx=\int\limits_0^1 dx\int\limits_0^x(1-x)\,dy$. **2193.** $\frac{\pi a^3}{6}$. **2194.** $\frac{3}{4}$. **2195.** $\frac{1}{6}$. **2196.** $\frac{a^3}{3}$. **2197.** $\frac{\pi r^4}{4a}$. **2198.** $\frac{48\sqrt{6}}{5}$. **2199.** $\frac{88}{105}$. **2200.** $\frac{a^3}{18}$. **2201.** $\frac{abc}{3}$. **2202.** $\pi a^3(\alpha-\beta)$. **2203.** $\frac{4}{3}\pi a^3(2\sqrt{2}-1)$. **2204.** $\frac{4}{3}\pi a^3(\sqrt{2}-1)$. **2205.** $\frac{\pi a^3}{3}$. **2206.** $\frac{4}{3}\pi abc$. **2207.** $\frac{\pi a^3}{3}(6\sqrt{3}-5)$. **2208.** $\frac{32}{9}a^3$. **2209.** $\pi a(1-e^{-R^2})$. **2210.** $\frac{3\pi ab}{2}$. **2211.** $\frac{3\sqrt{3}-2}{2}$. **2212.** $\frac{\sqrt{2}}{2}(2\sqrt{2}-1)$. **Hint.** Change the variables $xy=u$, $\frac{y}{x}=v$. **2213.** $\frac{1}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2}$ **2214.** $4(m-n)R^2$. **2215.** $\frac{\sqrt{2}}{2}a^2$. **Hint.** Integrate in the yz-plane. **2216.** $4a^2$. **2217.** $8a^2\arcsin\frac{b}{a}$. **2218.** $\frac{1}{3}\pi a^2(3\sqrt{3}-1)$. **2219.** $8a^2$. **2220.** $3\pi a^2$. Hint. Pass to polar coordinates. **2221.** $\sigma=\frac{2}{3}\pi a^2\left[\left(1+\frac{R^2}{a^2}\right)^{\frac{3}{2}}-1\right]$. **Hint.** Pass to polar coordinates. 2222. $\frac{16}{9}a^2$ and $8a^2$. **Hint.** Pass to polar coordinates.

2223. $8a^2 \arctan \frac{\sqrt{2}}{5}$ **Hint.** $\sigma = \int_0^{\frac{a}{2}} dx \int_0^{\frac{a}{2}} \frac{a\,dy}{\sqrt{a^2-x^2-y^2}} = 8a \int_0^{\frac{a}{2}} \arcsin \frac{a}{2\sqrt{a^2-x^2}} dx$.

Integrate by parts, and then change the variable $x = \frac{a\sqrt{3}}{2} \sin t$; transform the answer. 2224 $\frac{\pi}{4}\left(b\sqrt{b^2+c^2} - a\sqrt{a^2+c^2} + c^2 \ln \frac{b+\sqrt{b^2+c^2}}{a+\sqrt{a^2+c^2}}\right)$. **Hint.** Pass to polar coordinates 2225. $\frac{2\pi\delta R^2}{3}$. 2226. $\frac{a^3 b}{12}$; $\frac{a^2b^2}{24}$. 2227. $\bar{x} = \frac{12-\pi^2}{3(4-\pi)}$; $\bar{y} = \frac{\pi}{6(4-\pi)}$. 2228. $\bar{x} = \frac{5}{6}a$; $\bar{y} = 0$. 2229. $\bar{x} = \frac{2a \sin \alpha}{3\alpha}$; $\bar{y} = 0$. 2230. $\bar{x} = \frac{2}{5}$; $\bar{y} = 0$. 2231. $I_X = 4$ 2232. a) $I_0 = \frac{\pi}{32}(D^4 - d^4)$; b) $I_X = \frac{\pi}{64}(D^4 - d^4)$.

2233. $I = \frac{2}{3}a^4$. 2234. $\frac{8}{5}a^4$. **Hint.** $I = \int_0^a dx \int_{-\sqrt{ax}}^{\sqrt{ax}} (y+a)^2\,dy$.

2235. $16 \ln 2 - 9\frac{3}{8}$. **Hint.** The distance of the point (x, y) from the straight line $x = y$ is equal to $d = \frac{x-y}{\sqrt{2}}$ and is found by means of the normal equation of the straight line. 2236. $I = \frac{1}{40}ka^5[7\sqrt{2} + 3\ln(\sqrt{2}+1)]$, where k is the proportionality factor. **Hint.** Placing the coordinate origin at the vertex, the distance from which is proportional to the density of the lamina, we direct the coordinate axes along the sides of the square. The moment of inertia is determined relative to the x-axis Passing to polar coordinates, we have

$$I_x = \int_0^{\frac{\pi}{4}} d\varphi \int_0^{a \sec \varphi} kr(r \sin \varphi)^2 r\,dr + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\varphi \int_0^{a \operatorname{cosec} \varphi} kr(r \sin \varphi)^2 r\,dr$$

2237. $I_0 = \frac{35}{16}\pi a^4$.

2238. $I_0 = \frac{\pi a^4}{2}$. 2239. $\frac{35}{12}\pi a^4$. **Hint.** For the variables of integration take t and y (see Problem 2156). 2240. $\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} f(x, y, z)\,dz$

2241. $\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_0^H f(x, y, z)\,dz$.

2242. $\int_{-a}^{a} dx \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} dy \int_{c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}}^{c} f(x, y, z)\,dz$.

2243. $\int\limits_{-1}^{1} dx \int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int\limits_{0}^{\sqrt{1-x^2-y^2}} f(x, y, z)\,dz.$

2244. $\frac{8}{15}(31+12\sqrt{2}-27\sqrt{3})$. **2245.** $\frac{4\pi\sqrt{2}}{3}$. **2246.** $\frac{\pi^2 a^3}{8}$. **2247.** $\frac{1}{720}$. **2248.** $\frac{1}{2}\ln 2-\frac{5}{16}$. **2249.** $\frac{\pi a^5}{5}\left(18\sqrt{3}-\frac{97}{6}\right)$. **2250.** $\frac{59}{480}\pi R^5$. **2251.** $\frac{\pi abc^3}{4}$. **2252.** $\frac{4}{5}\pi abc$. **2253.** $\frac{\pi h^2 R^2}{4}$. **2254.** πR^3. **2255.** $\frac{8}{9}a^2$. **2256.** $\frac{8}{3}r^3\left(\pi-\frac{4}{3}\right)$. **2257.** $\frac{4}{15}\pi R^5$. **2258.** $\frac{\pi}{10}$. **2259.** $\frac{32}{9}a^2 h$. **2260.** $\frac{3}{4}\pi a^3$. Solution. $v=$

$$=2\int\limits_0^{2a} dx \int\limits_0^{\sqrt{2ax-x^2}} dy \int\limits_0^{\frac{x^2+y^2}{2a}} dz = 2\int\limits_0^{\frac{\pi}{2}} d\varphi \int\limits_0^{2a\cos\varphi} r\,dr \int\limits_0^{\frac{r^2}{2a}} dh =$$

$$=2\int\limits_0^{\frac{\pi}{2}} d\varphi \int\limits_0^{2a\cos\varphi} \frac{r^3 dr}{2a} = \frac{1}{a}\int\limits_0^{\frac{\pi}{2}} \frac{(2a\cos\varphi)^4}{4}\,d\varphi = \frac{3}{4}\pi a^3.$$

2261. $\frac{2\pi a^3\sqrt{2}}{3}$. **Hint.** Pass to spherical coordinates. **2262.** $\frac{19}{6}\pi$. **Hint.** Pass to cylindrical coordinates. **2263.** $\frac{a^3}{9}(3\pi-4)$. **2264.** πabc. **2265.** $\frac{abc}{2}(a+b+c)$. **2266.** $\frac{ab}{24}(6c^2-a^2-b^2)$. **2267.** $\bar{x}=0$; $\bar{y}=0$; $\bar{z}=\frac{2}{5}a$. **Hint.** Introduce spherical coordinates. **2268.** $\bar{x}=\frac{4}{3}$, $\bar{y}=0$, $\bar{z}=0$. **2269.** $\frac{\pi a^2 h}{12}(3a^2+4h^2)$. **Hint.** For the axis of the cylinder we take the z-axis, for the plane of the base of the cylinder, the xy-plane The moment of inertia is computed about the x-axis. After passing to cylindrical coordinates, the square of the distance of an element $r\,d\varphi\,dr\,dz$ from the x-axis is equal to $r^2\sin^2\varphi+z^2$. **2270.** $\frac{\pi\varrho h a^2}{60}(2h^2+3a^2)$. **Hint.** The base of the cone is taken for the xy-plane, the axis of the cone, for the z-axis. The moment of inertia is computed about the x-axis. Passing to cylindrical coordinates, we have for points of the surface of the cone: $r=\frac{a}{h}(h-z)$; and the square of the distance of the element $r\,d\varphi\,dr\,dz$ from the x-axis is equal to $r^2\sin^2\varphi+z^2$. **2271.** $2\pi k\varrho h(1-\cos\alpha)$, where k is the proportionality factor and ϱ is the density. **Solution.** The vertex of the cone is taken for the coordinate origin and its axis is the z-axis. If we introduce spherical coordinates, the equation of the lateral surface of the cone will be $\psi=\frac{\pi}{2}-\alpha$, and the equation of the plane of the base will be $r=\frac{h}{\sin\psi}$. From the symmetry it follows that the resulting stress is directed along the z-axis. The mass of an element of volume $dm=\varrho r^2\cos\psi\,d\varphi\,d\psi\,dr$, where ϱ is the density. The component of attraction, along the z-axis, by this element of unit mass lying at the point 0 is equal to $\frac{k\,dm}{r^2}\sin\psi=k\varrho\sin\psi\cos\psi\,d\psi\,d\varphi\,dr$.

The resulting attraction is equal to $\int\limits_0^{2\pi} d\varphi \int\limits_0^{\frac{\pi}{2}-\alpha} d\psi \int\limits_0^{h\,\mathrm{cosec}\,\psi} k\varrho \sin\psi \cos\psi\, dr$.

2272. Solution. We introduce cylindrical coordinates (ϱ, φ, z) with origin at the centre of the sphere and with the z-axis passing through a material point whose mass we assume equal to m. We denote by ξ the distance of this point from the centre of the sphere. Let $r=\sqrt{\varrho^2+(\xi-z)^2}$ be the distance from the element of volume dv to the mass m. The attractive force of the element of volume dv of the sphere and the material point m is directed along r and is numerically equal to $-k\gamma m\frac{dv}{r^2}$, where $\gamma=\frac{M}{\frac{4}{3}\pi R^3}$ is the density of the sphere and $dv=\varrho\, d\varphi\, d\varrho\, dz$ is the element of volume. The projection of this force on the z-axis is

$$dF=-\frac{km\gamma\, dv}{r^2}\cos\widehat{(rz)}=-km\gamma\frac{\xi-z}{r^3}\varrho\, d\varphi\, d\varrho\, dz.$$

Whence

$$F=-km\gamma\int\limits_0^{2\pi} d\varphi \int\limits_{-R}^{R} (\xi-z)\, dz \int\limits_0^{\sqrt{R^2-z^2}} \frac{\varrho\, d\varrho}{r^3}=km\gamma\frac{4}{3}\pi R^3\frac{1}{\xi^2}.$$

But since $\frac{4}{3}\gamma\pi R^3=M$, it follows that $F=\frac{kMm}{\xi^2}$. **2273.** $-\int\limits_x^{\infty} y^2e^{-xy^2}\, dy-e^{-x^3}$.

2275. a) $\frac{1}{p}$ $(p>0)$; b) $\frac{1}{p-\alpha}$ for $p>\alpha$; c) $\frac{\beta}{p^2+\beta^2}$ $(p>0)$; d) $\frac{p}{p^2+\beta^2}$ $(p>0)$

2276. $-\frac{1}{n^2}$. **2277.** $\frac{2}{p^3}$. **Hint.** Differentiate $\int\limits_0^{\infty} e^{-pt}\, dt=\frac{1}{p}$ twice. **2278.** $\ln\frac{\beta}{\alpha}$.

2279. $\arctan\frac{\beta}{m}-\arctan\frac{\alpha}{m}$. **2280.** $\frac{\pi}{2}\ln(1+\alpha)$. **2281.** $\pi(\sqrt{1-\alpha^2}-1)$.

2282. $\mathrm{arc\,cot}\frac{\alpha}{\beta}$. **2283.** 1. **2284.** $\frac{1}{2}$. **2285.** $\frac{\pi}{4}$. **2286.** $\frac{\pi}{4a^2}$. **Hint.** Pass to polar coordinates. **2287.** $\frac{\sqrt{\pi}}{2}$. **2288.** $\frac{\pi^2}{8}$. **2289.** Converges. **Solution.** Eliminate from S the coordinate origin together with its ε-neighbourhood, that is, consider $I_\varepsilon=\iint\limits_{(S_\varepsilon)} \ln\sqrt{x^2+y^2}\, dx\, dy$, where the eliminated region is a circle of radius ε with centre at the origin. Passing to polar coordinates, we have

$$I_\varepsilon=\int\limits_0^{2\pi} d\varphi \int\limits_\varepsilon^1 r\ln r\, dr=\int\limits_0^{2\pi}\left[\frac{r^2}{2}\ln r\,\Big|_\varepsilon^1-\frac{1}{2}\int\limits_\varepsilon^1 r\, dr\right] d\varphi=2\pi\left(\frac{\varepsilon^2}{4}-\frac{\varepsilon^2}{2}\ln\varepsilon-\frac{1}{4}\right).$$

Whence $\lim\limits_{\varepsilon\to 0} I_\varepsilon=-\frac{\pi}{2}$. **2290.** Converges for $\alpha>1$. **2291.** Converges. **Hint.** Surround the straight line $y=x$ with a narrow strip and put $\iint\limits_{(S)} \frac{dx\, dy}{\sqrt[3]{(x-y)^2}}=$

$=\lim_{\varepsilon\to 0}\int_0^1 dx\int_0^{x-\varepsilon}\frac{dy}{\sqrt[3]{(x-y)^2}}+\lim_{\delta\to 0}\int_0^1 dx\int_{x+\delta}^1\frac{dy}{\sqrt[3]{(x-y)^2}}$. **2292.** Converges for $\alpha>\frac{3}{2}$. **2293.** 0. **2294.** $\ln\frac{\sqrt{5}+3}{2}$. **2295.** $\frac{ab(a^2+ab+b^2)}{3(a+b)}$. **2296.** $\frac{256}{15}a^3$. **2297.** $\frac{a^2}{3}\left[(1+4\pi^2)^{\frac{3}{2}}-1\right]$. **2298.** $\frac{a^5\sqrt{1+m^2}}{5m}$. **2299.** $a^2\sqrt{2}$. **2300.** $\frac{1}{54}(56\sqrt{7}-1)$. **2301.** $\frac{\sqrt{a^2+b^2}}{ab}\arctan\frac{2\pi b}{a}$. **2302.** $2\pi a^2$. **2303.** $\frac{16}{27}(10\sqrt{10}-1)$. **Hint.** $\int_C f(x, y)\,ds$ may be interpreted geometrically as the area of a cylindrical surface with generatrix parallel to the z-axis, with base, the contour of integration, and with altitudes equal to the values of the integrand. Therefore, $S=\int_C x\,ds$, where C is the arc OA of the parabola $y=\frac{3}{8}x^2$ that connects the points (0, 0) and (4,6). **2304.** $a\sqrt{3}$. **2305.** $2\left(b^2+\frac{a^2b}{\sqrt{a^2-b^2}}\arcsin\frac{\sqrt{a^2-b^2}}{a}\right)$. **2306.** $\sqrt{a^2+b^2}\left(\pi\sqrt{a^2+4\pi^2b^2}+\frac{a^2}{2b}\ln\frac{2\pi b+\sqrt{a^2+4\pi^2b^2}}{a}\right)$. **2307.** $\left(\frac{4}{3}a, \frac{4}{3}a\right)$. **2308.** $2\pi a^2\sqrt{a^2+b^2}$. **2309.** $\frac{kMmb}{\sqrt{(a^2+b^2)^3}}$. **2310.** $40\frac{19}{30}$. **2311.** $-2\pi a^2$. **2312.** a) $\frac{4}{3}$; b) 0; c) $\frac{12}{5}$; d) -4; e) 4. **2313.** In all cases 4. **2314.** -2π. **Hint.** Use the parametric equations of a circle. **2315.** $\frac{4}{3}ab^2$. **2316.** $-2\sin 2$. **2317.** 0. **2318.** a) 8; b) 12; c) 2; d) $\frac{3}{2}$; e) $\ln(x+y)$; f) $\int_{x_1}^{x_2}\varphi(x)\,dx+\int_{y_1}^{y_2}\psi(y)\,dy$. **2319.** a) 62; b) 1; c) $\frac{1}{4}+\ln 2$; d) $1+\sqrt{2}$. **2320.** $\sqrt{1+a^2}-\sqrt{1+b^2}$. **2322.** a) $x^2+3xy-2y^2+C$; b) $x^3-x^2y+xy^2-y^3+C$; c) $e^{x-y}(x+y)+C$; d) $\ln|x+y|+C$. **2323.** $-2\pi\alpha(a+b)$. **2324.** $-\pi R^2\cos^2\alpha$ **2325.** $\left(\frac{1}{6}+\frac{\pi\sqrt{2}}{16}\right)R^3$. **2326.** a) -20; b) $abc-1$; c) $5\sqrt{2}$; d) 0. **2327.** $I=\iint_{(S)} y^2\,dx\,dy$. **2328.** $-\frac{4}{3}$. **2329.** $\frac{\pi R^4}{2}$. **2330.** $-\frac{1}{3}$. **2331.** 0. **2332.** a) 0; b) $2n\pi$. **Hint** In Case (b), Green's formula is used in the region between the contour C and a circle of sufficiently small radius with centre at the coordinate origin **2333. Solution.** If we consider that the direction of the tangent coincides with that of positive circulation of the contour, then $\cos(X, n)=\cos(Y, t)=\frac{dy}{ds}$, hence, $\oint_C\cos(X, n)\,ds=\oint_C\frac{dy}{ds}\,ds=\oint_C dy=0$ **2334.** $2S$, where S is the area bounded by the contour C. **2335.** -4. **Hint.** Green's formula is not applicable. **2336.** πab. **2337.** $\frac{3}{8}\pi a^2$. **2338.** $6\pi a^2$. **2339.** $\frac{3}{2}a^2$. **Hint.** Put

$y=tx$, where t is a parameter. **2340.** $\frac{a^2}{60}$. **2341.** $\pi(R+r)(R+2r)$; $6\pi R^2$ for $R=r$ **Hint.** The equation of an epicycloid is of the form $x=(R+r)\cos t - r\cos\frac{R+r}{r}t$, $y=(R+r)\sin t - r\sin\frac{R+r}{r}t$, where t is the angle of turn of the radius of a stationary circle drawn to the point of tangency. **2342.** $\pi(R-r)(R-2r)$, $\frac{3}{8}\pi R^2$ for $r=\frac{R}{4}$ **Hint.** The equation of the hypocycloid is obtained from the equation of the corresponding epicycloid (see Problem 2341) by replacing r by $-r$ **2343.** FR. **2344.** $mg(z_1-z_2)$. **2345.** $\frac{k}{2}(a^2-b^2)$, where k is a proportionality factor. **2346.** a) Potential, $U=mgz$, work, $mg(z_1-z_2)$; b) potential, $U=\frac{\mu}{r}$, work, $\frac{\mu}{\sqrt{a^2+b^2+c^2}}$; c) potential, $U=-\frac{k^2}{2}(x^2+y^2+z^2)$, work, $\frac{k^2}{2}(R^2-r^2)$. **2347.** $\frac{8}{3}\pi a^4$. **2348.** $\frac{2\pi a^2\sqrt{a^2+b^2}}{3}$. **2349.** 0. **2350.** $\frac{4}{3}\pi abc$. **2351.** $\frac{\pi a^4}{2}$. **2352.** $\frac{3}{4}$. **2353.** $\frac{25\sqrt{5}+1}{10(5\sqrt{5}-1)}a$. **2354.** $\frac{\pi\sqrt{2}}{2}h^4$. **2355.** a) 0; b) $-\iint\limits_{(S)}(\cos\alpha+\cos\beta+\cos\gamma)\,dS$. **2356.** 0. **2357.** 4π. **2358.** $-\pi a^2$. **2359.** $-a^3$. **2360.** $\frac{\partial R}{\partial y}=\frac{\partial Q}{\partial z}$, $\frac{\partial P}{\partial z}=\frac{\partial R}{\partial x}$, $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$. **2361.** 0. **2362.** $2\iiint\limits_{(V)}(x+y+z)\,dx\,dy\,dz$.

2363. $2\iiint\limits_{(V)}\frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}}$. **2364.** $\iiint\limits_{(V)}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)dx\,dy\,dz$.

2365. $3a^4$ **2366.** $\frac{a^3}{2}$. **2367.** $\frac{12}{5}\pi a^5$. **2368.** $\frac{\pi a^2b^2}{2}$ **2371.** Spheres; cylinders. **2372.** Cones. **2373.** Circles, $x^2+y^2=c_1^2$, $z=c_2$. **2376.** $\operatorname{grad} U(A)=9\boldsymbol{i}-3\boldsymbol{j}-3\boldsymbol{k}$; $|\operatorname{grad} U(A)|=\sqrt{99}=3\sqrt{11}$; $z^2=xy$; $x=y=z$. **2377.** a) $\frac{\boldsymbol{r}}{r}$; b) $2\boldsymbol{r}$. c) $-\frac{\boldsymbol{r}}{r^3}$; d) $f'(r)\frac{\boldsymbol{r}}{r}$ **2378.** $\operatorname{grad}(\boldsymbol{cr})=\boldsymbol{c}$; the level surfaces are planes perpendicular to the vector $\boldsymbol{c}$. **2379.** $\frac{\partial U}{\partial r}=\frac{2U}{r}$, $\frac{\partial U}{\partial r}=|\operatorname{grad} U|$ when $a=b=c$. **2380.** $\frac{\partial U}{\partial l}=-\frac{\cos(\boldsymbol{l},\boldsymbol{r})}{r^2}$; $\frac{\partial U}{\partial l}=0$ for $\boldsymbol{l}\perp\boldsymbol{r}$. **2382.** $\frac{2}{r}$. **2383.** $\operatorname{div}\boldsymbol{a}=\frac{2}{r}f(r)+f'(r)$. **2385.** a) $\operatorname{div}\boldsymbol{r}=3$, $\operatorname{rot}\boldsymbol{r}=0$; b) $\operatorname{div}(\boldsymbol{rc})=\frac{\boldsymbol{rc}}{r}$, $\operatorname{rot}(\boldsymbol{rc})=\frac{\boldsymbol{r}\times\boldsymbol{c}}{r}$; c) $\operatorname{div}(f(r)\boldsymbol{c})=\frac{f'(r)}{r}(\boldsymbol{c},\boldsymbol{r})$, $\operatorname{rot}(f(r)\boldsymbol{c})=\frac{f'(r)}{r}\boldsymbol{c}\times\boldsymbol{r}$. **2386.** $\operatorname{div}\boldsymbol{v}=0$; $\operatorname{rot}\boldsymbol{v}=2\boldsymbol{\omega}$, where $\boldsymbol{\omega}=\omega\boldsymbol{k}$ **2387.** $2\omega\boldsymbol{n}^\circ$, where $\boldsymbol{n}^\circ$ is a unit vector parallel to the axis of rotation. **2388.** $\operatorname{div}\operatorname{grad} U=\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}$; $\operatorname{rot}\operatorname{grad} U=0$. **2391.** $3\pi R^2H$. **2392.** a) $\frac{1}{10}\pi R^2H(3R^2+2H^2)$; b) $\frac{3}{10}\pi R^2H(R^2+2H^2)$. **2393.** $\operatorname{div}\boldsymbol{F}=0$ at all points except the origin. The flux is equal to $-4\pi m$. **Hint.** When calculating the flux, use the Ostrogradsky-Gauss theorem. **2394.** $2\pi^2h^2$. **2395.** $\frac{-\pi R^6}{8}$.

2396. $U=\int\limits_{r_0}^{r} rf(r)\,dr$. **2397.** $\frac{m}{r}$. **2398.** a) No potential; b) $U=xyz+C$; c) $U=xy+xz+yz+C$. **2400.** Yes.

Chapter VIII

2401. $\frac{1}{2n-1}$. **2402.** $\frac{1}{2n}$. **2403.** $\frac{n}{2^{n-1}}$. **2404.** $\frac{1}{n^2}$. **2405.** $\frac{n+2}{(n+1)^2}$. **2406.** $\frac{2n}{3n+2}$. **2407.** $\frac{1}{n(n+1)}$. **2408.** $\frac{1\cdot 3\cdot 5\ldots(2n-1)}{1\cdot 4\cdot 7\ldots(3n-2)}$. **2409.** $(-1)^{n+1}$. **2410.** $n^{(-1)^{n+1}}$.

2416. Diverges. **2417.** Converges. **2418.** Diverges. **2419.** Diverges. **2420.** Diverges. **2421.** Diverges. **2422.** Diverges. **2423.** Diverges. **2424.** Diverges. **2425.** Converges. **2426.** Converges. **2427.** Converges. **2428.** Converges. **2429.** Converges. **2430.** Converges. **2431.** Converges. **2432.** Converges. **2433.** Converges. **2434.** Diverges. **2435.** Diverges. **2436.** Converges. **2437.** Diverges. **2438.** Converges. **2439.** Converges. **2440.** Converges. **2441.** Diverges. **2442.** Converges. **2443.** Converges. **2444.** Converges. **2445.** Converges. **2446.** Converges. **2447.** Converges. **2448.** Converges. **2449.** Converges. **2450.** Diverges. **2451.** Converges. **2452.** Diverges. **2453.** Converges. **2454.** Diverges. **2455.** Diverges. **2456.** Converges. **2457.** Diverges. **2458.** Converges. **2459.** Diverges. **2460.** Converges. **2461.** Diverges. **2462.** Converges. **2463.** Diverges. **2464.** Converges. **2465.** Converges. **2466.** Converges. **2467.** Diverges. **2468.** Diverges. **Hint.** $\frac{a_{n+1}}{a_n}>1$ **2470.** Converges conditionally. **2471.** Converges conditionally. **2472.** Converges absolutely **2473.** Diverges. **2474.** Converges conditionally. **2475.** Converges absolutely. **2476.** Converges conditionally. **2477.** Converges absolutely. **2478.** Converges absolutely. **2479.** Diverges. **2480.** Converges absolutely. **2481.** Converges conditionally. **2482.** Converges absolutely. **2484.** a) Diverges; b) converges absolutely; c) diverges; d) converges conditionally. **Hint.** In examples (a) and (d) consider the series $\sum\limits_{k=1}^{\infty}(a_{2k-1}+a_{2k})$ and in examples (b) and (c) investigate separately the series $\sum\limits_{k=1}^{\infty}a_{2k-1}$ and $\sum\limits_{k=1}^{\infty}a_{2k}$. **2485.** Diverges. **2486.** Converges absolutely. **2487.** Converges absolutely. **2488.** Converges conditionally. **2489.** Diverges. **2490.** Converges absolutely. **2491.** Converges absolutely. **2492.** Converges absolutely. **2493.** Yes. **2494.** No. **2495.** $\sum\limits_{n=1}^{\infty}\frac{1+(-1)^n}{3^n}$; converges. **2496.** $\sum\limits_{n=1}^{\infty}\frac{1}{2n(2n-1)}$; converges. **2497.** Diverges. **2499.** Converges. **2500.** Converges.

2501. $|R_4|<\frac{1}{120}$, $|R_5|<\frac{1}{720}$; $R_4<0$, $R_5>0$. **2502.** $R_n<\frac{a_n}{2n+1}=\frac{1}{2^n(2n+1)\,n!}$ **Hint.** The remainder of the series may be evaluated by means of the sum of a geometric progression exceeding this remainder: $R_n=a_n\left[\frac{1}{2}\frac{1}{n+1}+\left(\frac{1}{2}\right)^2\frac{1}{(n+1)(n+2)}+\ldots\right]<a_n\left[\frac{1}{2}\cdot\frac{1}{n+1}+\left(\frac{1}{2}\right)^2\cdot\frac{1}{(n+1)^2}+\ldots\right]$.

2503. $R_n < \frac{n+2}{(n+1)(n+1)!}$; $R_{10} < 3\cdot 10^{-8}$. **2504.** $\frac{1}{n+1} < R_n < \frac{1}{n}$. **Solution.**
$R_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots > \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} + \ldots =$
$= \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \ldots = \frac{1}{n+1}$, $R_n < \frac{1}{n(n+1)} +$
$+ \frac{1}{(n+1)(n+2)} + \ldots = \frac{1}{n}$. **2505.** For the given series it is easy to find the exact value of the remainder:

$$R_n = \frac{1}{15}\left(n + \frac{16}{15}\right)\left(\frac{1}{4}\right)^{2n-2}.$$

Solution. $R_n = (n+1)\left(\frac{1}{4}\right)^{2n} + (n+2)\left(\frac{1}{4}\right)^{2n+2} + \ldots$

We multiply by $\left(\frac{1}{4}\right)^2$:

$$\frac{1}{16}R_n = (n+1)\left(\frac{1}{4}\right)^{2n+2} + (n+2)\left(\frac{1}{4}\right)^{2n+4} + \ldots$$

Whence we obtain

$$\frac{15}{16}R_n = n\left(\frac{1}{4}\right)^{2n} + \left(\frac{1}{4}\right)^{2n} + \left(\frac{1}{4}\right)^{2n+2} + \left(\frac{1}{4}\right)^{2n+4} + \ldots =$$

$$= n\left(\frac{1}{4}\right)^{2n} + \frac{\left(\frac{1}{4}\right)^{2n}}{1 - \frac{1}{16}} = \left(n + \frac{16}{15}\right)\left(\frac{1}{4}\right)^{2n}.$$

From this we find the above value of R_n. Putting $n = 0$, we find the sum of the series $S = \left(\frac{16}{15}\right)^2$. **2506.** 99; 999. **2507.** 2; 3; 5. **2508.** $S = 1$. **Hint.** $a_n = \frac{1}{n} - \frac{1}{n+1}$ **2509.** $S = 1$ when $x > 0$, $S = -1$ when $x < 0$; $S = 0$ when $x = 0$. **2510.** Converges absolutely for $x > 1$, diverges for $x \leqslant 1$. **2511.** Converges absolutely for $x > 1$, converges conditionally for $0 < x \leqslant 1$, diverges for $x \leqslant 0$. **2512.** Converges absolutely for $x > e$, converges conditionally for $1 < x \leqslant e$, diverges for $x \leqslant 1$. **2513.** $-\infty < x < \infty$. **2514.** $-\infty < x < \infty$. **2515.** Converges absolutely for $x > 0$, diverges for $x \leqslant 0$. **Solution.** 1) $|a_n| \leqslant \frac{1}{e^{nx}}$; and when $x > 0$ the series with general term $\frac{1}{e^{nx}}$ converges; 2) $\frac{1}{e^{nx}} \geqslant 1$ for $x \leqslant 0$, and $\cos nx$ does not tend to zero as $n \to \infty$, since from $\cos nx \to 0$ it would follow that $\cos 2nx \to -1$; thus, the necessary condition for convergence is violated when $x \leqslant 0$. **2516.** Converges absolutely when $2k\pi < x < (2k+1)\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$; at the remaining points it diverges. **2517.** Diverges everywhere. **2518.** Converges absolutely for $x \neq 0$. **2519.** $x > 1$, $x \leqslant -1$. **2520.** $x > 3$, $x < 1$. **2521.** $x \geqslant 1$, $x \leqslant -1$. **2522.** $x \geqslant 5\frac{1}{3}$, $x < 4\frac{2}{3}$. **2523.** $x > 1$, $x < -1$. **2524.** $-1 < x < -\frac{1}{2}$, $\frac{1}{2} < x < 1$. **Hint.** For these values of x, both the series $\sum_{k=1}^{\infty} x^k$ and the series $\sum_{k=1}^{\infty} \frac{1}{2^k x^k}$ converge. When $|x| \geqslant 1$

and when $|x| \leqslant \frac{1}{2}$, the general term of the series does not tend to zero
2525. $-1 < x < 0$, $0 < x < 1$. **2526.** $-1 < x < 1$. **2527.** $-2 \leqslant x < 2$.
2528. $-1 < x < 1$ **2529.** $-\frac{1}{\sqrt{2}} \leqslant x \leqslant \frac{1}{\sqrt{2}}$. **2530.** $-1 < x \leqslant 1$. **2531.** $-1 < x < 1$
2532. $-1 < x < 1$. **2533.** $-\infty < x < \infty$. **2534.** $x = 0$. **2535.** $-\infty < x < \infty$.
2536. $-4 < x < 4$. **2537.** $-\frac{1}{3} < x < \frac{1}{3}$. **2538.** $-2 < x < 2$. **2539.** $-e < x < e$.
2540. $-3 \leqslant x < 3$. **2541.** $-1 < x < 1$ **2542.** $-1 < x < 1$ **Solution.** The divergence of the series for $|x| \geqslant 1$ is obvious (it is interesting, however, to note that the divergence of the series at the end-points of the interval of convergence $x = \pm 1$ is detected not only with the aid of the necessary condition of convergence, but also by means of the d'Alembert test). When $|x| < 1$ we have

$$\lim_{n \to \infty} \left| \frac{(n+1)!\, x^{(n+1)!}}{n!\, x^{n!}} \right| = \lim_{n \to \infty} |(n+1)\, x^{n!\, n}| \leqslant \lim_{n \to \infty} (n+1)\, |x|^n = \lim_{n \to \infty} \frac{n+1}{\left|\frac{1}{x}\right|^n} = 0$$

(this equality is readily obtained by means of l'Hospital's rule).
2543. $-1 \leqslant x \leqslant 1$ **Hint.** Using the d'Alembert test, it is possible not only to find the interval of convergence, but also to investigate the convergence of the given series at the extremities of the interval of convergence. **2544.** $-1 \leqslant x \leqslant 1$. **Hint.** Using the Cauchy test, it is possible not only to find the interval of convergence, but also to investigate the convergence of the given series at the extremities of the interval of convergence. **2545.** $2 < x \leqslant 8$.
2546. $-2 \leqslant x < 8$. **2547.** $-2 < x < 4$. **2548.** $1 \leqslant x \leqslant 3$ **2549.** $-4 \leqslant x \leqslant -2$.
2550. $x = -3$ **2551.** $-7 < x < -3$ **2552.** $0 \leqslant x < 4$. **2553.** $-\frac{5}{4} < x < \frac{13}{4}$.
2554. $-e-3 < x < e-3$. **2555.** $-2 \leqslant x \leqslant 0$. **2556.** $2 < x < 4$ **2557.** $1 < x \leqslant 3$.
2558. $-3 \leqslant x \leqslant -1$ **2559.** $1 - \frac{1}{e} < x < 1 + \frac{1}{e}$ **Hint.** For $x = 1 \pm \frac{1}{e}$ the series diverges, since $\lim_{n \to \infty} \frac{\left(1 + \frac{1}{n}\right)^{n^2}}{e^n} = \frac{1}{\sqrt{e}} \neq 0$ **2560.** $-2 < x < 0$
2561. $1 < x \leqslant 3$ **2562.** $1 \leqslant x < 5$. **2563.** $2 \leqslant x \leqslant 4$. **2564.** $|z| < 1$ **2565.** $|z| < 1$
2566. $|z - 2i| < 3$ **2567.** $|z| < \sqrt{2}$ **2568.** $z = 0$ **2569.** $|z| < \infty$. **2570.** $|z| < \frac{1}{2}$
2576. $-\ln(1-x)$ $(-1 \leqslant x < 1)$ **2577.** $\ln(1+x)$ $(-1 < x \leqslant 1)$.
2578. $\frac{1}{2} \ln \frac{1+x}{1-x}$ $(|x| < 1)$ **2579.** $\arctan x$ $(|x| \leqslant 1)$. **2580.** $\frac{1}{(x-1)^2}$ $(|x| < 1)$.
2581. $\frac{1-x^2}{(1+x^2)^2}$ $(|x| < 1)$ **2582.** $\frac{2}{(1-x)^3}$ $(|x| < 1)$. **2583** $\frac{x}{(x-1)^2}$ $(|x| > 1)$.
2584. $\frac{1}{2}\left(\arctan x - \frac{1}{2} \ln \frac{1-x}{1+x}\right)$ $(|x| < 1)$. **2585.** $\frac{\pi\sqrt{3}}{6}$. **Hint.** Consider the sum of the series $x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$ (see Problem 2579) for $x = \frac{1}{\sqrt{3}}$.
2586. 3. **2587.** $a^x = 1 + \sum_{n=1}^{\infty} \frac{x^n \ln^n a}{n!}$, $-\infty < x < \infty$. **2588.** $\sin\left(x + \frac{\pi}{4}\right) =$
$= \frac{\sqrt{2}}{2}\left[1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \dots + (-1)^{\frac{n^2-n}{2}} \frac{x^n}{n!} + \dots\right]$.

2589. $\cos(x+a)=\cos a - x\sin a - \frac{x^2}{2!}\cos a + \frac{x^3}{3!}\sin a + \frac{x^4}{4!}\cos a + \ldots$ $\ldots + \frac{x^n}{n!}\sin\left[a + \frac{(n+1)\pi}{2}\right] + \ldots, -\infty < x < \infty$. **2590.** $\sin^2 x = \frac{2x^2}{2!} - \frac{2^3x^4}{4!} + \frac{2^5x^6}{6!} - \ldots$ $\ldots + (-1)^{n-1}\frac{2^{2n-1}x^{2n}}{(2n)!} + \ldots, -\infty < x < \infty$. **2591** $\ln(2+x) = \ln 2 + \frac{x}{2} - \frac{x^2}{2\cdot 2^2} + \frac{x^3}{3\cdot 2^3} - \ldots + (-1)^{n-1}\frac{x^n}{n\cdot 2^n} + \ldots, -2 < x \leqslant 2$. **Hint.** When investigating the remainder, use the theorem on integrating a power series **2592.** $\frac{2x-3}{(x-1)^2} = -\sum_{n=0}^{\infty}(n+3)x^n, |x| < 1$. **2593.** $\frac{3x-5}{x^2-4x+3} = -\sum_{n=0}^{\infty}\left(1+\frac{2}{3^{n+1}}\right)x^n, |x| < 1$. **2594.** $xe^{-2x} = x + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}2^{n-1}x^n}{(n-1)!}$, $-\infty < x < \infty$. **2595.** $e^{x^2} = 1 + \sum_{n=1}^{\infty}\frac{x^{2n}}{n!}, -\infty < x < \infty$ **2596.** $\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$ $(-\infty < x < \infty)$ **2597.** $1 + \sum_{n=1}^{\infty}(-1)^n\frac{2^n x^{2n}}{(2n)!}$. **2598.** $1 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!}$ $-\infty < x < \infty$. **2599.** $2\sum_{n=0}^{\infty}(-1)^n\frac{(n+2)3^{2n}\cdot x^{2n+1}}{(2n+1)!}$ $(-\infty < x < \infty)$.

2600. $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{9^{n+1}}$ $(-3 < x < 3)$. **2601.** $\frac{1}{2} + \frac{1}{2}\cdot\frac{x^2}{2^3} + \frac{1\cdot 3}{2\cdot 4}\,\frac{x^4}{2^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{x^6}{2^7} + \ldots + \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}\,\frac{x^{2n}}{2^{2n+1}} + \ldots$ $(-2 < x < 2)$

2602. $2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}$ $(|x| < 1)$ **2603.** $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^n - 1}{n}x^n\left(-\frac{1}{2} < x \leqslant \frac{1}{2}\right)$.

2604 $x + \sum_{n=2}^{\infty}(-1)^n\frac{x^n}{(n-1)n}$ $(|x| \leqslant 1)$. **2605.** $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ $(|x| \leqslant 1)$.

2606. $x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\,\frac{x^5}{5} + \ldots + \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}\,\frac{x^{2n+1}}{2n+1} + \ldots$ $(|x| \leqslant 1)$.

2607. $x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\,\frac{x^5}{5} - \ldots + (-1)^n\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}\,\frac{x^{2n+1}}{2n+1} + \ldots$ $(|x| \leqslant 1)$.

2608. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{4n-3}x^{2n}}{(2n)!}$ $(-\infty < x < \infty)$. **2609.** $1 + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{n-1}{n!}x^n$ $(-\infty < x < \infty)$. **2610.** $8 + 3\sum_{n=1}^{\infty}\frac{1+2^n+3^{n-1}}{n!}x^n$ $(-\infty < x < \infty)$.

2611. $2 + \frac{x}{2^2\cdot 3\cdot 1!} - \frac{2\cdot x^2}{2^5\cdot 3^2\cdot 2!} + \frac{2\cdot 5x^3}{2^8\cdot 3^3\cdot 3!} + \ldots + (-1)^{n-1}\frac{2\cdot 5\cdot 8\ldots(3n-4)x^n}{2^{3n-1}\cdot 3^n\cdot n!} + \ldots$

$(-\infty < x < \infty)$. **2612.** $\frac{1}{6} - \sum_{n=1}^{\infty} \left(\frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} \right) x^n$ $(-2 < x < 2)$.

2613. $1 + \frac{3}{4} \sum_{n=1}^{\infty} \frac{(1+3^{2n-1}) x^{2n}}{(2n)!}$ $(|x| < \infty)$. **2614.** $\sum_{n=0}^{\infty} \frac{x^{4n}}{4^{n+1}}$ $(|x| < \sqrt{2})$.

2615. $\ln 2 + \sum_{n=1}^{\infty} (-1)^{n-1} (1+2^{-n}) \frac{x^n}{n}$ $(-1 < x \leqslant 1)$. **2616.** $\sum_{n=0}^{\infty} (-1)^n \times$ $\times \frac{x^{2n+1}}{(2n+1)(2n+1)!}$ $(-\infty < x < \infty)$. **2617.** $x + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)\, n!}$ $(|x| < \infty)$.

2618. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n^2}$ $(|x| \leqslant 1)$. **2619.** $x + \frac{1}{2 \cdot 5} x^5 + \frac{1 \cdot 3}{2^2 \cdot 9 \cdot 2!} x^9 + \ldots +$ $+ \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^n (4n+1)\, n!} x^{4n+1} + \ldots$ $(|x| < 1)$. **2620.** $x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$

2621. $x - \frac{x^3}{3} + \frac{2x^5}{15} - \ldots$ **2622.** $e \left(1 - \frac{x^2}{2} + \frac{x^4}{6} - \ldots \right)$. **2623.** $1 + \frac{x^2}{2} + \frac{5x^4}{24} + \ldots$

2624. $-\left(\frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \ldots \right)$. **2625.** $x + x^2 + \frac{1}{3} x^3 + \ldots$ **2626.** **Hint.** Proceeding from the parametric equations of the ellipse $x = a \cos \varphi$, $y = b \sin \varphi$, compute the length of the ellipse and expand the expression obtained in a series of powers of ε. **2628.** $x^3 - 2x^2 - 5x - 2 = -78 + 59(x+4) - 14(x+4)^2 + (x+4)^3$ $(-\infty < x < \infty)$. **2629.** $f(x+h) = 5x^3 - 4x^2 - 3x + 2 + (15x^2 - 8x - 3)h + (15x - 4)h^2 + 5h^3$ $(-\infty < x < \infty;\ -\infty < h < \infty)$.

2630. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}$ $(0 < x \leqslant 2)$. **2631.** $\sum_{n=0}^{\infty} (-1)^n (x-1)^n$ $(0 < x < 2)$.

2632. $\sum_{n=0}^{\infty} (n+1)(x+1)^n$ $(-2 < x < 0)$. **2633.** $\sum_{n=0}^{\infty} (2^{-n-1} - 3^{-n-1})(x+4)^n$ $(-6 < x < -2)$. **2634.** $\sum_{n=0}^{\infty} (-1)^n \frac{(x+2)^{2n}}{3^{n+1}}$ $(-2 - \sqrt{3} < x < -2 + \sqrt{3})$.

2635. $e^{-2} \left[1 + \sum_{n=1}^{\infty} \frac{(x+2)^n}{n!} \right]$ $(|x| < \infty)$. **2636.** $2 + \frac{x-4}{2^2} - \frac{1}{4} \frac{(x-4)^2}{2^4} + \frac{1 \cdot 3}{4 \cdot 6} \frac{(x-4)^3}{2^6} - \frac{1 \cdot 3 \cdot 5}{4 \cdot 6 \cdot 8} \frac{(x-4)^4}{2^8} + \ldots + (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \ldots (2n-3)}{4 \cdot 6 \cdot 8 \ldots 2n} \frac{(x-4)^n}{2^{2n}} + \ldots$ $(0 \leqslant x \leqslant 8)$. **2637.** $\sum_{n=1}^{\infty} (-1)^n \frac{\left(x - \frac{\pi}{2} \right)^{2n-1}}{(2n-1)!}$ $(|x| < \infty)$. **2638.** $\frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{4^{n-1} \left(x - \frac{\pi}{4} \right)^{2n-1}}{(2n-1)!}$ $(|x| < \infty)$. **2639.** $-2 \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{1-x}{1+x} \right)^{2n+1}$ $(0 < x < \infty)$.

Hint. Make the substitution $\frac{1-x}{1+x}=t$ and expand $\ln x$ in powers of t. **2640.** $\frac{x}{1+x}+\frac{1}{2}\left(\frac{x}{1+x}\right)^2+\frac{1\cdot 3}{2\cdot 4}\left(\frac{x}{1+x}\right)^3+\ldots+\frac{1\cdot 3\cdot 5\ldots(2n-3)}{2\cdot 4\cdot 6\ldots(2n-2)}\left(\frac{x}{1+x}\right)^n+\ldots$ $\ldots\left(-\frac{1}{2}\leqslant x<\infty\right)$. **2641.** $|R|<\frac{e}{5!}<\frac{1}{40}$. **2642.** $|R|<\frac{1}{11}$. **2643.** $\frac{\pi}{6}\approx$ $\approx\frac{1}{2}+\frac{1}{2}\frac{\left(\frac{1}{2}\right)^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{\left(\frac{1}{2}\right)^5}{5}\approx 0.523$. **Hint.** To prove that the error does not exceed 0.001, it is necessary to evaluate the remainder by means of a geometric progression that exceeds this remainder. **2644.** Two terms, that is, $1-\frac{x^2}{2}$. **2645.** Two terms, i.e., $x-\frac{x^3}{6}$. **2646.** Eight terms, i.e., $1+\sum_{n=1}^{7}\frac{1}{n!}$. **2647.** 99; 999. **2648.** 1.92 **2649.** 4.8 $|R|<0.005$. **2650.** 2.087. **2651.** $|x|<0.69$; $|x|<0.39$; $|x|<0.22$. **2652.** $|x|<0.39$; $|x|<0.18$ **2653.** $\frac{1}{2}-\frac{1}{2^3\cdot 3\cdot 3!}\approx 0.4931$. **2654.** 0.7468. **2655.** 0.608 **2656.** 0.621 **2657.** 0.2505 **2658.** 0.026.

2659. $1+\sum_{n=1}^{\infty}(-1)^n\frac{(x-y)^{2n}}{(2n)!}$ $(-\infty<x<\infty;\ -\infty<y<\infty)$.

2660. $\sum_{n=1}^{\infty}(-1)^n\frac{(x-y)^{2n}-(x+y)^{2n}}{2\cdot(2n)!}$ $(-\infty<x<\infty;\ -\infty<y<\infty)$.

2661. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x^2+y^2)^{2n-1}}{(2n-1)!}$ $(-\infty<x<\infty;\ -\infty<y<\infty)$.

2662. $1+2\sum_{n=1}^{\infty}(y-x)^n$; $|x-y|<1$ **Hint.** $\frac{1-x+y}{1+x-y}=-1+\frac{2}{1-(y-x)}$. Use a geometric progression **2663.** $-\sum_{n=1}^{\infty}\frac{x^n+y^n}{n}$ $(-1\leqslant x<1;\quad -1\leqslant y<1)$. **Hint.** $1-x-y+xy=(1-x)(1-y)$. **2664.** $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}+y^{2n+1}}{2n+1}$ $(-1\leqslant x\leqslant 1;$ $-1\leqslant y\leqslant 1)$. **Hint.** $\arctan\frac{x+y}{1-xy}=\arctan x+\arctan y$ (for $|x|\leqslant 1,\ |y|\leqslant 1$). **2665.** $f(x+h,\ y+k)=ax^2+2bxy+cy^2+2(ax+by)h+2(bx+cy)k+ah^2+2bhk+ck^2$. **2666.** $f(1+h,\ 2+k)-f(1,\ 2)=9h-21k+3h^2+3hk-12k^2+h^3-2k^3$. **2667.** $1+\sum_{n=1}^{\infty}\frac{[(x-2)+(y+2)]^n}{n!}$. **2668.** $1+\sum_{n=1}^{\infty}(-1)^n\frac{\left[x+\left(y-\frac{\pi}{2}\right)\right]^{2n}}{(2n)!}$.

2669. $1+x+\frac{x^2-y^2}{2!}+\frac{x^3-3xy^2}{3!}+\ldots$ **2670.** $1+x+xy+\frac{1}{2}x^2y+\ldots$

2671. $\frac{c_1+c_2}{2}-\frac{2(c_1-c_2)}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}$; $S(0)=\frac{c_1+c_2}{2}$; $S(\pm\pi)=\frac{c_1+c_2}{2}$.

2672. $\frac{b-a}{4}\pi-\frac{2(b-a)}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}+(a+b)\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin nx}{n}$; $S(\pm\pi)=\frac{b-a}{2}\pi$. 2673. $\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$; $S(\pm\pi)=\pi^2$. 2674. $\frac{2}{\pi}\sinh a\pi\times\times\left[\frac{1}{2a}+\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}(a\cos nx-n\sin nx)\right]$; $S(\pm\pi)=\cosh a\pi$. 2675. $\frac{2\sin a\pi}{\pi}\times\times\sum_{n=1}^{\infty}(-1)^n\frac{n\sin nx}{a^2-n^2}$ if a is nonintegral; $\sin ax$ if a is an integer; $S(\pm\pi)=0$. 2676. $\frac{2\sin a\pi}{\pi}\left[\frac{1}{2a}+\sum_{n=1}^{\infty}(-1)^n\frac{a\cos nx}{a^2-n^2}\right]$ if a is nonintegral; $\cos ax$ if a is an integer; $S(\pm\pi)=\cos a\pi$. 2677. $\frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n\sin nx}{a^2+n^2}$; $S(\pm\pi)=0$. 2678. $\frac{2\sinh a\pi}{\pi}\left[\frac{1}{2a}+\sum_{n=1}^{\infty}(-1)^n\frac{a\cos nx}{a^2+n^2}\right]$; $S(\pm\pi)=\cosh a\pi$. 2679. $\sum_{n=1}^{\infty}\frac{\sin nx}{n}$. 2680. $\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}$; a) $\frac{\pi}{4}$; b) $\frac{\pi}{3}$; c) $\frac{\pi}{2\sqrt{3}}$. 2681. a) $2\sum_{n=1}^{\infty}(-1)^{n-1}\times\times\frac{\sin nx}{n}$; b) $\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$; $\frac{\pi^2}{8}$. 2682. a) $\sum_{n=1}^{\infty}b_n\sin nx$, where $b_{2k-1}=\frac{2\pi}{2k-1}-\frac{8}{\pi(2k-1)^3}$ and $b_{2k}=-\frac{\pi}{k}$; b) $\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$; 1) $\frac{\pi^2}{6}$. 2) $\frac{\pi^2}{12}$. 2683. a) $\frac{2}{\pi}\sum_{n=1}^{\infty}[1-(-1)^n e^{a\pi}]\frac{n\sin nx}{a^2+n^2}$; b) $\frac{e^{a\pi}-1}{a\pi}+\frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n e^{a\pi}-1]\cos nx}{a^2+n^2}$. 2684. a) $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos\frac{n\pi}{2}}{n}\sin nx$; b) $\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n}\cos nx$. 2685. a) $\frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin(2n-1)x}{(2n-1)^2}$; b) $\frac{\pi}{4}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2(2n-1)x}{(2n-1)^2}$ 2686. $\sum_{n=1}^{\infty}b_n\sin nx$, where $b_{2k}=(-1)^{k-1}\frac{1}{2k}$, $b_{2k+1}==(-1)^k\frac{2}{\pi(2k+1)^2}$ 2687. $\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^3}$. 2688. $\frac{8}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n\sin nx}{4n^2-1}$.

2689. $\frac{2h}{\pi}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\frac{\sin nh}{nh}\cos nx\right)$. **2690.** $\frac{2h}{\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{\sin nh}{nh}\right)^2\cos nx\right]$.

2691. $1-\frac{\cos x}{2}+2\sum_{n=2}^{\infty}(-1)^{n-1}\frac{\cos nx}{n^2-1}$. **2692.** $\frac{4}{\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos 2nx}{4n^2-1}\right]$.

2694. **Solution.** 1) $a_{2n}=\frac{2}{\pi}\int_0^{\pi} f(x)\cos 2nx\,dx=\frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x)\cos 2nx\,dx+$ $+\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} f(x)\cos 2nx\,dx$. If we make the substitution $t=\frac{\pi}{2}-x$ in the first integral and $t=x-\frac{\pi}{2}$ in the second, then, taking advantage of the assumed identity $f\left(\frac{\pi}{2}+t\right)=-f\left(\frac{\pi}{2}-t\right)$, it will readily be seen that $a_{2n}=0$ $(n=0,\ 1,\ 2,\ \ldots)$;

2) $b_{2n}=\frac{2}{\pi}\int_0^{\pi} f(x)\sin 2nx\,dx=\frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x)\sin 2nx\,dx+\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} f(x)\sin 2nx\,dx.$

The same substitution as in Case (1), with account taken of the assumed identity $f\left(\frac{\pi}{2}+t\right)=f\left(\frac{\pi}{2}-t\right)$ leads to the equalities $b_{2n}=0$ $(n=1,\ 2,\ \ldots)$.

2695. $\frac{1}{2}-\frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi x}{(2n+1)^2}$. **2696.** $1-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2n\pi x}{n}$.

2697. $\sinh l\left[\frac{1}{l}+2\sum_{n=1}^{\infty}(-1)^n\frac{l\cos\frac{n\pi x}{l}-\pi n\sin\frac{n\pi x}{l}}{l^2+n^2\pi^2}\right]$.

2698. $\frac{10}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{\sin\frac{n\pi x}{5}}{n}$ **2699.** a) $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2(n-1)\pi x}{2n-1}$; b) 1 **2700** a) $\frac{2l}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin\frac{n\pi x}{l}}{n}$; b) $\frac{l}{2}-\frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\frac{(2n-1)\pi x}{l}}{(2n-1)^2}$. **2701.** a) $\sum_{n=1}^{\infty}b_n\sin\frac{nx}{2}$, where $b_{2k+1}=\frac{8}{\pi}\left[\frac{\pi^2}{2k+1}-\frac{4}{(2k+1)^3}\right]$, $b_{2k}=-\frac{4\pi}{k}$; b) $\frac{4\pi^2}{3}-$ $-16\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos\frac{nx}{2}}{n^2}$. **2702.** a) $\frac{8}{\pi^2}\sum_{n=0}^{\infty}(-1)^n\frac{\sin\frac{(2n+1)\pi x}{2}}{(2n+1)^2}$, b) $\frac{1}{2}-$ $-\frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi x}{(2n+1)^2}$. **2703.** $\frac{2}{3}-\frac{9}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{3}+\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{\cos 2n\pi x}{n^2}$.

Chapter IX

2704. Yes. **2705.** No. **2706.** Yes. **2707.** Yes. **2708.** Yes. **2709.** a) Yes; b) no. **2710.** Yes. **2714.** $y-xy'=0$. **2715.** $xy'-2y=0$. **2716.** $y-2xy'=0$. **2717.** $x\,dx+y\,dy=0$. **2718.** $y'=y$. **2719.** $3y^2-x^2=2xyy'$. **2720.** $xyy'(xy^2+1)=1$. **2721.** $y=xy'\ln\frac{x}{y}$. **2722.** $2xy''+y'=0$. **2723.** $y''-y'-2y=0$. **2724.** $y''+4y=0$. **2725.** $y''-2y'+y=0$. **2726.** $y''=0$. **2727.** $y'''=0$. **2728.** $(1+y'^2)y'''-3y'y''^2=0$. **2729.** $y^2-x^2=25$. **2730.** $y=xe^{2x}$. **2731.** $y=-\cos x$. **2732.** $y=\frac{1}{6}(-5e^{-x}+9e^{x}-4e^{2x})$. **2738.** 2.593 (exact value $y=e$). **2739.** 4.780 [exact value $y=3(e-1)$]. **2740.** 0.946 (exact value $y=1$). **2741.** 1.826 (exact value $y=\sqrt{3}$) **2742.** $\cot^2 y=\tan^2 x+C$. **2743.** $x=\frac{Cy}{\sqrt{1+y^2}}$; $y=0$. **2744.** $x^2+y^2=\ln Cx^2$. **2745.** $y=a+\frac{Cx}{1+ax}$. **2746.** $\tan y=C(1-e^{x})^3$; $x=0$. **2747.** $y=C\sin x$. **2748.** $2e^{\frac{y^2}{2}}=\sqrt{e}\,(1+e^{x})$. **2749.** $1+y^2=\frac{2}{1-x^2}$. **2750.** $y=1$. **2751.** $\arctan(x+y)=x+C$. **2752.** $8x+2y+1=2\tan(4x+C)$. **2753.** $x+2y+3\ln|2x+3y-7|=C$. **2754.** $5x+10y+C=3\ln|10x-5y+6|$. **2755.** $\varrho=\frac{C}{1-\cos\varphi}$ or $y^2=2Cx+C^2$. **2756.** $\ln\varrho=\frac{1}{2\cos^2\varphi}-\ln|\cos\varphi|+C$ or $\ln|x|-\frac{y^2}{2x^2}=C$. **2757.** Straight line $y=Cx$ or hyperbola $y=\frac{C}{x}$. **Hint.** The segment of the tangent is equal to $\sqrt{y^2+\left(\frac{y}{y'}\right)^2}$. **2758.** $y^2-x^2=C$. **2759.** $y=Ce^{\frac{x}{a}}$. **2760.** $y^2=2px$. **2761.** $y=ax^2$. **Hint.** By hypothesis $\frac{\int_0^x xy\,dx}{\int_0^x y\,dx}=\frac{3}{4}x$. Differentiating twice with respect to x, we get a differential equation. **2762.** $y^2=\frac{1}{3}x$.

2763. $y=\sqrt{4-x^2}+2\ln\frac{2-\sqrt{4-x^2}}{x}$. **2764.** Pencil of lines $y=kx$. **2765.** Family of similar ellipses $2x^2+y^2=C^2$. **2766.** Family of hyperbolas $x^2-y^2=C$. **2767.** Family of circles $x^2+(y-b)^2=b^2$. **2768.** $y=x\ln\frac{C}{x}$. **2769.** $y=\frac{C}{x}-\frac{x}{2}$. **2770.** $x=Ce^{\frac{x}{y}}$. **2771.** $(x-C)^2-y^2=C^2$; $(x-2)^2-y^2=4$; $y=\pm x$. **2772.** $\sqrt{\frac{x}{y}}+\ln|y|=C$. **2773.** $y=\frac{C}{2}x^2-\frac{1}{2C}$; $x=0$. **2774.** $(x^2+y^2)^3(x+y)^2C$. **2775.** $y=x\sqrt{1-\frac{3}{8}x}$. **2776.** $(x+y-1)^3=C(x-y+3)$. **2777.** $3x+y+2\times\ln|x+y-1|=C$. **2778.** $\ln|4x+8y+5|+8y-4x=C$. **2779.** $x^2=1-2y$.

2780. Paraboloid of revolution. **Solution.** By virtue of symmetry the sought-for mirror is a surface of revolution. The coordinate origin is located in the source of light; the x-axis is the direction of the pencil of rays. If a tangent at any point $M(x, y)$ of the curve, generated by the desired surface being cut by the xy-plane, forms with the x-axis an angle φ, and the segment connecting the origin with the point $M(x, y)$ forms an angle α, then $\tan\alpha=\tan 2\varphi=\frac{2\tan\varphi}{1-\tan^2\varphi}$. But $\tan\alpha=\frac{y}{x}$; $\tan\varphi=y'$. The desired differential equation is $y-yy'^2=2xy'$ and its solution is $y^2=2Cx+C^2$. The plane section is a parabola. The desired surface is a paraboloid of revolution. **2781.** $(x-y)^2-Cy=0$. **2782.** $x^2=C(2y+C)$. **2783.** $(2y^2-x^2)^3=Cx^2$. **Hint.** Use the fact that the area is equal to $\int\limits_a^x y\,dx$. **2784.** $y=Cx-x\ln|x|$. **2785.** $y=Cx+x^2$. **2786.** $y=\frac{1}{6}x^4+\frac{C}{x^2}$. **2787.** $x\sqrt{1+y^2}+\cos y=C$. **Hint.** The equation is linear with respect to x and $\frac{dx}{dy}$. **2788.** $x=Cy^2-\frac{1}{y}$. **2789.** $y=\frac{e^x}{x}+\frac{ab-e^a}{x}$. **2790.** $y=\frac{1}{2}(x\sqrt{1-x^2}+\arcsin x)\sqrt{\frac{1+x}{1-x}}$. **2791.** $y=\frac{x}{\cos x}$. **2792.** $y(x^2+Cx)=1$. **2793.** $y^2=x\ln\frac{C}{x}$. **2794.** $x^2=\frac{1}{y+Cy^2}$. **2795.** $y^3(3+Ce^{\cos x})=x$ **2797.** $xy=Cy^2+a^2$. **2798.** $y^2+x+ay=0$. **2799.** $x=y\ln\frac{y}{a}$. **2800.** $\frac{a}{x}+\frac{b}{y}=1$. **2801.** $x^2+y^2-Cy+a^2=0$. **2802.** $\frac{x^2}{2}+xy+y^2=C$. **2803.** $\frac{x^3}{3}+xy^2+x^2=C$. **2804.** $\frac{x^4}{4}-\frac{3}{2}x^2y^2+2x+\frac{y^3}{3}=C$. **2805.** $x^2+y^2-2\arctan\frac{y}{x}=C$. **2806.** $x^2-y^2=Cy^3$. **2807.** $\frac{x^2}{2}+ye^{\frac{x}{y}}=2$ **2808.** $\ln|x|-\frac{y^2}{x}=C$. **2809.** $\frac{x}{y}+\frac{x^2}{2}=C$. **2810.** $\frac{1}{y}\ln x+\frac{1}{2}y^2=C$. **2811.** $(x\sin y+y\cos y-\sin y)e^x=C$. **2812.** $(x^2C^2+1-2Cy)\times(x^2+C^2-2Cy)=0$; singular integral $x^2-y^2=0$. **2813.** General integral $(y+C)^2=x^3$; there is no singular integral. **2814.** General integral $\left(\frac{x^2}{2}-y+C\right)\times\left(x-\frac{y^2}{2}+C\right)=0$; there is no singular integral. **2815.** General integral $y^2+C^2=2Cx$; singular integral $x^2-y^2=0$. **2816.** $y=\frac{1}{2}\cos x\pm\frac{\sqrt{3}}{2}\sin x$. **2817.** $\begin{cases}x=\sin p+\ln p,\\ y=p\sin p+\cos p+p+C.\end{cases}$ **2818.** $\begin{cases}x=e^p+pe^p+C,\\ y=p^2e^p.\end{cases}$ **2819.** $\begin{cases}x=2p-\frac{2}{p}+C,\\ y=p^2+2\ln p.\end{cases}$ Singular solution: $y=0$. **2820.** $4y=x^2+p^2$, $\ln|p-x|=C+\frac{x}{p-x}$. **2821.** $\ln\sqrt{p^2+y^2}+\arctan\frac{p}{y}=C$, $x=\ln\frac{y^2+p^2}{2p}$. Singular solution: $y=e^x$.

2822. $y=\frac{1}{2}Cx^2+\frac{2}{C}$; $y=\pm 2x$. 2824. $\begin{cases} x=Ce^{-p}-2p+2, \\ y=C(1+p)e^{-p}-p^2+2. \end{cases}$

2823. $\begin{cases} x=\ln|p|-\arcsin p+C, \\ y=p+\sqrt{1-p^2}. \end{cases}$ 2825. $\begin{cases} x=\frac{1}{3}(Cp^{-\frac{1}{2}}-p), \\ y=\frac{1}{6}(2Cp^{\frac{1}{2}}+p^2). \end{cases}$ **Hint.** The differential equation from which x is defined as a function of p is homogeneous. 2826. $y=Cx+C^2$; $y=-\frac{x^2}{4}$. 2827. $y=Cx+C$; no singular solution 2828. $y=Cx+\sqrt{1+C^2}$; $x^2+y^2=1$. 2829. $y=Cx+\frac{1}{C}$; $y^2=4x$. 2830. $xy=C$ 2831. A circle and the family of its tangents. 2832. The astroid $x^{2/3}+y^{2/3}=a^{2/3}$. 2833. a) Homogeneous, $y=xu$; b) linear in x; $x=uv$; c) linear in y; $y=uv$; d) Bernoulli's equation; $y=uv$; e) with variables separable; f) Clairaut's equation; reduce to $y=xy'\pm\sqrt{y'^2}$; g) Lagrange's equation; differentiate with respect to x; h) Bernoulli's equation; $y=uv$; i) leads to equation with variables separable; $u=x+y$; j) Lagrange's equation; differentiate with respect to x; k) Bernoulli's equation in x; $x=uv$; l) exact differential equation; m) linear; $y=uv$; n) Bernoulli's equation; $y=uv$. 2834. a) $\sin\frac{y}{x}=-\ln|x|+C$; b) $x=y\cdot e^{Cy+1}$. 2835. $x^2+y^4=Cy^2$. 2836. $y=\frac{x}{x^2+C}$. 2837. $xy\left(C-\frac{1}{2}\ln^2 x\right)=1$. 2838. $y=Cx+C\ln C$; singular solution, $y=e^{-(x+1)}$. 2839. $y=Cx+\sqrt{-aC}$; singular solution, $y=\frac{a}{4x}$. 2840. $3y+\ln\frac{|x^2-1|}{(y+1)^6}=C$. 2841. $\frac{1}{2}e^{2x}-e^y-\arctan y-\frac{1}{2}\ln(1+y^2)=C$. 2842. $y=x^2\left(1+Ce^{\frac{1}{x}}\right)$. 2843. $x=y^2(C-e^{-y})$. 2844. $y=Ce^{-\sin x}+\sin x-1$. 2845. $y=ax+C\sqrt{1-x^2}$. 2846. $y=\frac{x}{x+1}(x+\ln|x|+C)$. 2847. $x=Ce^{\sin y}-2a(1+\sin y)$. 2848. $\frac{x^2}{2}+3x+y+\ln[(x-3)^{10}|y-1|^3]=C$. 2849. $2\arctan\frac{y-1}{2x}=\ln Cx$. 2850. $x^2=1-\frac{2}{y}+Ce^{-\frac{2}{y}}$. 2851. $x^3=Ce^y-y-2$ 2852. $\sqrt{\frac{y}{x}}+\ln|x|=C$. 2853. $y=x\arcsin(Cx)$. 2854. $y^2=Ce^{-2x}+\frac{2}{5}\sin x+\frac{4}{5}\cos x$. 2855. $xy=C(y-1)$. 2856. $x=Ce^y-\frac{1}{2}(\sin y+\cos y)$. 2857. $py=C(p-1)$. 2858. $x^4=Ce^{4y}-y^3-\frac{3}{4}y^2-\frac{3}{8}y-\frac{3}{32}$ 2859. $(xy+C)(x^2y+C)=0$. 2860. $\sqrt{x^2+y^2}-\frac{x}{y}=C$. 2861. $xe^y-y^2=C$. 2862. $\begin{cases} x=\frac{C}{p^2}-\frac{\sqrt{1+p^2}}{2p}+\frac{1}{2p^2}\ln(p+\sqrt{1+p^2}), \\ y=2px+\sqrt{1+p^2}. \end{cases}$

2863. $y=xe^{Cx}$. 2864. $2e^x-y^4=Cy^2$. 2865. $\ln|y+2|+2\arctan\frac{y+2}{x-3}=C$. 2866.

$y^2+Ce^{-\frac{y^2}{2}}+\frac{1}{x}-2=0$. **2867.** $x^2\cdot y=Ce^{\frac{y}{a}}$. **2868.** $x+\frac{x}{y}=C$. **2869.** $y=\frac{C-x^4}{4(x^2-1)^{3/2}}$. **2870.** $y=C\sin x-a$. **2871.** $y=\frac{a^2\ln(x+\sqrt{a^2+x^2})+C}{x+\sqrt{a^2+x^2}}$. **2872.** $(y-Cx)\cdot(y^2-x^2+C)=0$. **2873.** $y=Cx+\frac{1}{C^2}$, $y=\frac{3}{2}\sqrt[3]{2x^2}$. **2874.** $x^3+x^2y-y^2x-y^3=C$. **2875.** $p^2+4y^2=Cy^3$. **2876.** $y=x-1$. **2877.** $y=x$. **2878.** $y=2$. **2879.** $y=0$. **2880.** $y=\frac{1}{2}(\sin x+\cos x)$. **2881.** $y=\frac{1}{4}(2x^2+2x+1)$. **2882.** $y=e^{-x}+2x-2$. **2883.** a) $y=x$; b) $y=Cx$, where C is arbitrary; the point (0,0) is a singular point of the differential equation. **2884.** a) $y^2=x$; b) $y^2=2px$; (0,0) is a singular point. **2885.** a) $(x-C)^2+y^2=C^2$; b) no solution; c) $x^2+y^2=x$; (0,0) is a singular point. **2886.** $y=e^{\frac{x}{y}}$. **2887.** $y=(\sqrt{2a}\pm\sqrt{x})^2$. **2888.** $y^2=1-e^{-x}$. **2889.** $r=Ce^{a\varphi}$. **Hint.** Pass to polar coordinates. **2890.** $3y^2-2x=0$ **2891.** $r=k\varphi$ **2892.** $x^2+(y-b)^2=b^2$. **2893.** $y^2+16x=0$. **2894.** Hyperbola $y^2-x^2=C$ or circle $x^2+y^2=C^2$. **2895.** $y=\frac{1}{2}(e^x+e^{-x})$. **Hint.** Use the fact that the area is equal to $\int_0^x y\,dx$ and the arc length, to $\int_0^x\sqrt{1+y'^2}\,dx$. **2896.** $x=\frac{a^2}{y}+Cy$. **2897.** $y^2=4C(C+a-x)$. **2898.** **Hint.** Use the fact that the resultant of the force of gravity and the centrifugal force is normal to the surface. Taking the y-axis as the axis of rotation and denoting by ω the angular velocity of rotation, we get for the plane axial cross-section of the desired surface the differential equation $g\frac{dy}{dx}=\omega^2x$. **2899.** $p=e^{-0.000167h}$. **Hint.** The pressure at each level of a vertical column of air may be considered as due solely to the pressure of the upper-lying layers Use the law of Boyle-Mariotte, according to which the density is proportional to the pressure. The sought-for differential equation is $dp=-kp\,dh$. **2900.** $s=\frac{1}{2}kl\omega$. **Hint.** Equation $ds=kw\frac{l-x}{l}dx$. **2901.** $s=\left(p+\frac{1}{2}w\right)kl$. **2902.** $T=a+(T_0-a)e^{-kt}$. **2903.** In one hour. **2904.** $\omega=100\left(\frac{3}{5}\right)^t$ rpm. **2905.** 4.2% of the initial quantity Q_0 will decay in 100 years. **Hint.** Equation $\frac{dQ}{dt}=kQ$. $Q=Q_0\left(\frac{1}{2}\right)^{\frac{t}{1600}}$. **2906.** $t\approx 35.2$ sec. **Hint.** Equation $\pi(h^2-2h)\,dh=\pi\left(\frac{1}{10}\right)^2v\,dt$. **2907.** $\frac{1}{1024}$. **Hint.** $dQ=-kQ\,dh$. $Q=Q_0\left(\frac{1}{2}\right)^{\frac{h}{3}}$. **2908.** $v\to\sqrt{\frac{gm}{k}}$ as $t\to\infty$ (k is a proportionality factor). **Hint.** Equation $m\frac{dv}{dt}=mg-kv^2$; $v=\sqrt{\frac{gm}{k}}\tanh\left(t\sqrt{\frac{gk}{m}}\right)$. **2909.** 18.1 kg. **Hint.** Equation $\frac{dx}{dt}=k\left(\frac{1}{3}-\frac{x}{300}\right)$. **2910.** $i=\frac{E}{R^2+L^2\omega^2}[(R\sin\omega t-$

$-L\omega\cos\omega t)+L\omega e^{-\frac{R}{L}t}]$. **Hint.** Equation $Ri+L\frac{di}{dt}=E\sin\omega t$. **2911.** $y=x\ln|x|+C_1x+C_2$. **2912.** $1+C_1y^2=\left(C_2+\frac{C_1x}{\sqrt{2}}\right)^2$. **2913.** $y=\ln|e^{2x}+C_1|-x+C_2$. **2914.** $y=C_1+C_2\ln|x|$. **2915.** $y=C_1e^{C_2x}$ **2916.** $y=\pm\sqrt{C_1x+C_2}$. **2917.** $y=(1+C_1^2)\ln|x+C_1|-C_1x+C_2$. **2918.** $(x-C_1)=a\ln\left|\sin\frac{y-C_2}{a}\right|$. **2919.** $y=\frac{1}{2}(\ln|x|)^2+C_1\ln|x|+C_2$. **2920.** $x=\frac{1}{C_1}\ln\left|\frac{y}{y+C_1}\right|+C_2$; $y=C$. **2921.** $y=C_1e^{C_2x}+\frac{1}{C_2}$. **2922.** $y=\pm\frac{1}{2}\left[x\sqrt{C_1^2-x^2}+C_1^2\arcsin\frac{x}{C_1}\right]+C_2$ **2923.** $y=(C_1e^x+1)x+C_2$. **2924.** $y=(C_1x-C_1^2)e^{\frac{x}{C_1}+1}+C_2$; $y=\frac{e}{2}x^2+C$ (singular solution). **2925.** $y=C_1x(x-C_1)+C_2$; $y=\frac{x^3}{3}+C$ (singular solution). **2926.** $y=\frac{x^3}{12}+\frac{x^2}{2}+C_1x\ln|x|+C_2x+C_3$. **2927.** $y=\pm\sin(C_1\pm x)+C_2x+C_3$. **2928.** $y=x^3+3x$. **2929.** $y=\frac{1}{2}(x^2+1)$. **2930.** $y=x+1$. **2931.** $y=Cx^2$. **2932.** $y=C_1\times\frac{1+C_2e^x}{1-C_2e^x}$; $y=C$. **2933.** $x=C_1+\ln\left|\frac{y-C_2}{y+C_2}\right|$. **2934.** $x=C_1-\frac{1}{C_2}\ln\left|\frac{y}{y+C_2}\right|$. **2935.** $x=C_1y^2+y\ln y+C_2$. **2936.** $2y^2-4x^2=1$. **2937.** $y=x+1$. **2938.** $y=\frac{x^2-1}{2(e^2-1)}-\frac{e^2-1}{4}\ln|x|$ or $y=\frac{1-x^2}{2(e^2+1)}+\frac{e^2+1}{4}\ln|x|$. **2939.** $y=\frac{1}{2}x^2$. **2940.** $y=\frac{1}{2}x^2$. **2941.** $y=2e^x$. **2942.** $x=-\frac{3}{2}(y+2)^{\frac{2}{3}}$. **2943.** $y=e^x$. **2944.** $y^2=\frac{e}{e-1}+\frac{e^{-x}}{1-e}$. **2945.** $y=\frac{2\sqrt{2}}{3}x^{\frac{3}{2}}-\frac{8}{3}$. **2946.** $y=\frac{3e^{3x}}{2+e^{3x}}$. **2947.** $y=\sec^2 x$. **2948.** $y=\sin x+1$ **2949.** $y=\frac{x^2}{4}-\frac{1}{2}$. **2950.** $x=-\frac{1}{2}e^{-y^2}$. **2951.** No solution. **2952.** $y=e^x$. **2953.** $y=2\ln|x|-\frac{2}{x}$. **2954.** $y=\frac{(x+C_1^2+1)^2}{2}+\frac{4}{3}C_1(x+1)^{\frac{3}{2}}+C_2$. Singular solution, $y=C$. **2955.** $y=C_1\frac{x^2}{2}+(C_1-C_1^2)x+C_2$. Singular solution, $y=\frac{(x+1)^3}{12}+C$. **2956.** $y=\frac{1}{12}(C_1+x)^4+C_2x+C_3$. **2957.** $y=C_1+C_2e^{C_1x}$; $y=1-e^x$; $y=-1+e^{-x}$; singular solution, $y=\frac{4}{C-x}$. **2958.** Circles. **2959.** $(x-C_1)^2-C_2y^2+kC_2^2=0$. **2960.** Catenary, $y=a\cosh\frac{x-x_0}{2}$. Circle, $(x-x_0)^2+y^2=a^2$. **2961.** Parabola, $(x-x_0)^2=2ay-a^2$. Cycloid, $x-x_0=a(t-\sin t)$, $y=a(1-\cos t)$. **2962.** $e^{ay+C_2}=\sec(ax+C_1)$. **2963.** Parabola. **2964.** $y=\frac{C_1}{2}\frac{H}{q}e^{\frac{q}{H}x}+\frac{1}{2C_1}\frac{H}{q}e^{-\frac{q}{H}x}+C_2=a\times$

$\times\cosh\frac{x+C}{a}+C_2$, where H is a constant horizontal tension, and $\frac{H}{q}=a$. **Hint.** The differential equation $\frac{d^2y}{dx^2}=\frac{q}{H}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$. **2965.** Equation of motion, $\frac{d^2s}{dt^2}=g(\sin\alpha-\mu\cos\alpha)$. Law of motion, $s=\frac{gt^2}{2}(\sin\alpha-\mu\cos\alpha)$ **2966.** $s=\frac{m}{k}\times$ $\times\ln\cosh\left(t\sqrt{g\frac{k}{m}}\right)$. **Hint.** Equation of motion, $m\frac{d^2s}{dt^2}=mg-k\left(\frac{ds}{dt}\right)^2$. **2967.** In 6.45 seconds. **Hint.** Equation of motion, $\frac{300}{g}\frac{d^2x}{dt^2}=-10\,v$. **2968.** a) No, b) yes, c) yes, d) yes, e) no, f) no, g) no, h) yes **2969.** a) $y''+y=0$; b) $y''-2y'+y=0$; c) $x^2y''-2xy'+2y=0$, d) $y'''-3y''+4y'-2y=0$ **2970.** $y=3x-5x^2+2x^3$. **2971.** $y=$ $=\frac{1}{x}(C_1\sin x+C_2\cos x)$. **Hint.** Use the substitution $y=y_1u$. **2972.** $y=C_1x+$ $+C_2\ln x$. **2973.** $y=A+Bx^2+x^3$. **2974.** $y=\frac{x^2}{3}+Ax+\frac{B}{x}$. **Hint.** Particular solutions of the homogeneous equation $y_1=x$, $y_2=\frac{1}{x}$. By the method of the variation of parameters we find: $C_1=\frac{x}{2}+A$, $C_2=-\frac{x^3}{6}+B$ **2975.** $y=A+$ $+B\sin x+C\cos x+\ln|\sec x+\tan x|+\sin x\ln|\cos x|-x\cos x$. **2976.** $y=C_1e^{2x}+$ $+C_2e^{3x}$ **2977.** $y=C_1e^{-3x}+C_2e^{3x}$. **2978.** $y=C_1+C_2e^{x}$ **2979.** $y=C_1\cos x+C_2\sin x$. **2980.** $y=e^{x}(C_1\cos x+C_2\sin x)$ **2981.** $y=e^{-2x}(C_1\cos 3x+C_2\sin 3x)$ **2982.** $y=$ $=(C_1+C_2x)e^{-x}$. **2983.** $y=e^{2x}(C_1e^{x\sqrt{2}}+C_2e^{-x\sqrt{2}})$. **2984.** If $k>0$, $y=$ $=C_1e^{x\sqrt{k}}+C_2e^{-x\sqrt{k}}$; if $k<0$, $y=C_1\cos\sqrt{-k}x+C_2\sin\sqrt{-k}x$. **2985.** $y=e^{-\frac{x}{2}}\left(C_1e^{\frac{\sqrt{5}}{2}x}+C_2e^{-\frac{\sqrt{5}}{2}x}\right)$ **2986.** $y=e^{\frac{x}{6}}\left(C_1\cos\frac{\sqrt{11}}{6}x+C_2\sin\frac{\sqrt{11}}{6}x\right)$. **2987.** $y=4e^{x}+e^{3x}$. **2988.** $y=e^{-x}$. **2989.** $y=\sin 2x$. **2990.** $y=1$. **2991.** $y=a\cosh\frac{x}{a}$. **2992.** $y=0$ **2993.** $y=C\sin\pi x$ **2994.** a) $xe^{2x}(Ax^2+Bx+C)$; b) $A\cos 2x+$ $+B\sin 2x$; c) $A\cos 2x+B\sin 2x+Cx^2e^{2x}$; d) $e^{x}(A\cos x+B\sin x)$, e) $e^{x}\times$ $\times(Ax^2+Bx+C)+xe^{2x}(Dx+E)$; f) $xe^{x}[(Ax^2+Bx+C)\cos 2x+(Dx^2+Ex+F)\times$ $\times\sin 2x]$ **2995.** $y=(C_1+C_2x)e^{2x}+\frac{1}{8}(2x^2+4x+3)$. **2996.** $y=e^{\frac{x}{2}}\left(C_1\cos\frac{x\sqrt{3}}{2}+\right.$ $\left.+C_2\sin\frac{x\sqrt{3}}{2}\right)+x^3+3x^2$. **2997.** $y=(C_1+C_2x)e^{-x}+\frac{1}{9}e^{2x}$. **2998.** $y=C_1e^{x}+C_2e^{7x}+2$ **2999.** $y=C_1e^{x}+C_2e^{-x}+\frac{1}{2}xe^{x}$. **3000.** $y=C_1\cos x+$ $+C_2\sin x+\frac{1}{2}x\sin x$. **3001.** $y=C_1e^{x}+C_2e^{-2x}-\frac{2}{5}(3\sin 2x+\cos 2x)$. **3002.** $y=$ $=C_1e^{2x}+C_2e^{-3x}+x\left(\frac{x}{10}-\frac{1}{25}\right)e^{2x}$. **3003.** $y=(C_1+C_2x)e^{x}+\frac{1}{2}\cos x+\frac{x^2}{4}e^{x}-$ $-\frac{1}{8}e^{-x}$ **3004.** $y=C_1+C_2e^{-x}+\frac{1}{2}x+\frac{1}{20}(2\cos 2x-\sin 2x)$. **3005.** $y=e^{x}\times$ $\times(C_1\cos 2x+C_2\sin 2x)+\frac{x}{4}e^{x}\sin 2x$. **3006.** $y=\cos 2x+\frac{1}{3}(\sin x+\sin 2x)$.

3007. 1) $x=C_1\cos\omega t+C_2\sin\omega t+\frac{A}{\omega^2-p^2}\sin pt$; 2) $x=C_1\cos\omega t+C_2\sin\omega t-\frac{A}{2\omega}t\cdot\cos\omega t$. **3008.** $y=C_1e^{3x}+C_2e^{4x}-xe^{4x}$. **3009.** $y=C_1+C_2e^{2x}+\frac{x}{4}-\frac{x^2}{4}-\frac{x^3}{6}$. **3010.** $y=e^x(C_1+C_2x+x^2)$. **3011.** $y=C_1+C_2e^{2x}+\frac{1}{2}xe^{2x}-\frac{5}{2}x$. **3012.** $y=C_1e^{-2x}+C_2e^{4x}-\frac{1}{9}e^x+\frac{1}{5}(3\cos 2x+\sin 2x)$. **3013.** $y=C_1+C_2e^{-x}+e^x+\frac{5}{2}x^2-5x$. **3014.** $y=C_1+C_2e^x-3xe^x-x-x^2$. **3015.** $y=\left(C_1+C_2x+\frac{1}{2}x^2\right)\times e^{-x}+\frac{1}{4}e^x$. **3016.** $y=(C_1\cos 3x+C_2\sin 3x)e^x+\frac{1}{37}(\sin 3x+6\cos 3x)+\frac{e^x}{9}$. **3017.** $y=(C_1+C_2x+x^2)e^{2x}+\frac{x+1}{8}$. **3018.** $y=C_1+C_2e^{3x}-\frac{1}{10}(\cos x+3\sin x)-\frac{x^2}{6}-\frac{x}{9}$. **3019.** $y=\frac{1}{8}e^{2x}(4x+1)-\frac{x^3}{6}-\frac{x^2}{4}+\frac{x}{4}$. **3020.** $y=C_1e^x+C_2e^{-x}-x\sin x-\cos x$. **3021.** $y=C_1e^{-2x}+C_2e^{2x}-\frac{e^{2x}}{20}(\sin 2x+2\cos 2x)$. **3022.** $y=C_1\cos 2x+C_2\sin 2x-\frac{x}{4}(3\sin 2x+2\cos 2x)+\frac{1}{4}$. **3023.** $y=e^x(C_1\cos x+C_2\sin x-2x\cos x)$. **3024.** $y=C_1e^x+C_2e^{-x}+\frac{1}{4}(x^2-x)e^x$. **3025.** $y=C_1\cos 3x+C_2\sin 3x+\frac{1}{4}x\sin x-\frac{1}{16}\cos x+\frac{1}{54}(3x-1)e^{3x}$. **3026.** $y=C_1e^{3x}+C_2e^{-x}+\frac{1}{9}\times(2-3x)+\frac{1}{16}(2x^2-x)e^{3x}$. **3027.** $y=C_1+C_2e^{2x}-2xe^x-\frac{3}{4}x-\frac{3}{4}x^2$ **3028.** $y=\left(C_1+C_2x+\frac{x^3}{6}\right)e^{2x}$. **3029.** $y=C_1e^{-3x}+C_2e^x-\frac{1}{8}(2x^2+x)e^{-3x}+\frac{1}{16}\times(2x^2+3x)e^x$. **3030.** $y=C_1\cos x+C_2\sin x+\frac{x}{4}\cos x+\frac{x^2}{4}\sin x-\frac{x}{8}\cos 3x+\frac{3}{32}\sin 3x$. **Hint.** Transform the product of cosines to the sum of cosines. **3031.** $y=C_1e^{-x\sqrt{2}}+C_2e^{x\sqrt{2}}+xe^x\sin x+e^x\cos x$. **3032.** $y=C_1\cos x+C_2\sin x+\cos x\ln\left|\cot\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$. **3033.** $y=C_1\cos x+C_2\sin x+\sin x\cdot\ln\left|\tan\frac{x}{2}\right|$. **3034.** $y=(C_1+C_2x)e^x+xe^x\ln|x|$. **3035.** $y=(C_1+C_2x)e^{-x}+xe^{-x}\ln|x|$. **3036.** $y=C_1\cos x+C_2\sin x+x\sin x+\cos x\ln|\cos x|$ **3037.** $y=C_1\cos x+C_2\sin x-x\cos x+\sin x\ln|\sin x|$. **3038.** a) $y=C_1e^x+C_2e^{-x}+(e^x+e^{-x})\times\text{arc tan}\,e^x$; b) $y=C_1e^{x\sqrt{2}}+C_2e^{-x\sqrt{2}}+e^{x^2}$. **3040.** Equation of motion, $\frac{2}{g}\left(\frac{d^2x}{dt^2}\right)=2-k(x+2)$; $(k=1)$; $T=2\pi\sqrt{\frac{2}{g}}$ sec. **3041.** $x=\frac{2g\sin 30t-60\sqrt{g}\sin\sqrt{g}t}{g-900}$ cm. **Hint.** If x is reckoned from the position of rest of the load, then $\frac{4}{g}x''=4-k(x_0+x-y-l)$, where x_0 is the distance of the point of rest of the load from the initial point of suspension of the spring, l is the length of the spring at rest; therefore, $k(x_0-l)=4$, hence, $\frac{4}{g}\frac{d^2x}{dt^2}=-k(x-y)$, where $k=4$, $g=981$ cm/sec². **3042.** $m\frac{d^2x}{dt^2}=k(b-x)-k(b+x)$

and $x=c\cos\left(t\sqrt{\frac{2k}{m}}\right)$. **3043.** $6\frac{d^2s}{dt^2}=gs$; $t=\sqrt{\frac{6}{g}}\ln(6+\sqrt{35})$. **3044.** a) $r=\frac{a}{2}(e^{\omega t}+e^{-\omega t})$; b) $r=\frac{v_0}{2\omega}(e^{\omega t}-e^{-\omega t})$ **Hint.** The differential equation of motion is $\frac{d^2r}{dt^2}=\omega^2 r$. **3045.** $y=C_1+C_2e^x+C_3e^{12x}$. **3046.** $y=C_1+C_2e^{-x}+C_3e^x$.

3047. $y=C_1e^{-x}+e^{\frac{x}{2}}\left(C_2\cos\frac{\sqrt{3}}{2}x+C_3\sin\frac{\sqrt{3}}{2}x\right)$

3048. $y=C_1+C_2x+C_3e^{x\sqrt{2}}+C_4e^{-x\sqrt{2}}$ **3049.** $y=e^x(C_1+C_2x+C_3x^2)$.

3050. $y=e^x(C_1\cos x+C_2\sin x)+e^{-x}(C_3\cos x+C_4\sin x)$

3051. $y=(C_1+C_2x)\cos 2x+(C_3+C_4x)\sin 2x$

3052. $y=C_1+C_2e^{-x}+e^{\frac{x}{2}}\left(C_3\cos\frac{\sqrt{3}}{2}x+C_4\sin\frac{\sqrt{3}}{2}x\right)$.

3053. $y=(C_1+C_2x)e^{-x}+(C_3+C_4x)e^x$.

3054. $y=C_1e^{ax}+C_2e^{-ax}+C_3\cos ax+C_4\sin ax$

3055. $y=(C_1+C_2x)e^{\sqrt{3}x}+(C_3+C_4x)e^{-\sqrt{3}x}$. **3056.** $y=C_1+C_2x+C_3\cos ax+C_4\sin ax$. **3057.** $y=C_1+C_2x+(C_3+C_4x)e^{-x}$. **3058.** $y=(C_1+C_2x)\cos x+(C_3+C_4x)\sin x$. **3059.** $y=e^{-x}(C_1+C_2x+\ldots+C_nx^{n-1})$.

3060. $y=C_1+C_2x+\left(C_3+C_4x+\frac{x^2}{2}\right)e^x$.

3061. $y=C_1+C_2x+12x^2+3x^3+\frac{1}{2}x^4+\frac{1}{20}x^5+(C_3+C_4x)e^x$.

3062. $y=C_1e^x+e^{-\frac{x}{2}}\left(C_2\cos\frac{\sqrt{3}}{2}x+C_3\sin\frac{\sqrt{3}}{2}x\right)-x^3-5$.

3063. $y=C_1+C_2x+C_3x^2+C_4e^{-x}+\frac{1}{1088}(4\cos 4x-\sin 4x)$

3064. $y=C_1e^{-x}+C_2+C_3x+\frac{3}{2}x^2-\frac{1}{3}x^3+\frac{1}{12}x^4+e^x\left(\frac{3}{2}x-\frac{15}{4}\right)$.

3065. $y=C_1e^{-x}+C_2\cos x+C_3\sin x+e^x\left(\frac{x}{4}-\frac{3}{8}\right)$.

3066. $y=C_1+C_2\cos x+C_3\sin x+\sec x+\cos x\ln|\cos x|-\tan x\sin x+x\sin x$.

3067. $y=e^{-x}+e^{-\frac{x}{2}}\left(\cos\frac{\sqrt{3}}{2}x+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}x\right)+x-2$.

3068. $y=(C_1+C_2\ln x)\cdot\frac{1}{x}$. **3069.** $y=C_1x^3+\frac{C_2}{x}$.

3070. $y=C_1\cos(2\ln x)+C_2\sin(2\ln x)$.

3071. $y=C_1x+C_2x^2+C_3x^3$. **3072.** $y=C_1+C_2(3x+2)^{-4/3}$.

3073. $y=C_1x^2+\frac{C_2}{x}$. **3074.** $y=C_1\cos(\ln x)+C_2\sin(\ln x)$.

3075. $y=C_1x^3+C_2x^2+\frac{1}{2}x$. **3076.** $y=(x+1)^2[C_1+C_2\ln(x+1)]+(x+1)^3$.

3077. $y=x(\ln x+\ln^2 x)$. **3078.** $y=C_1\cos x+C_2\sin x$, $z=C_2\cos x-C_1\sin x$.

3079. $y=e^{-x}(C_1\cos x+C_2\sin x)$, $z=\frac{1}{5}e^{-x}[(C_2-2C_1)\cos x-(C_1+2C_2)\sin x]$.

3080. $y=(C_1-C_2-C_1x)e^{-2x}$, $z=(C_1x+C_2)e^{-2x}$.

3081. $x=C_1e^t+e^{-\frac{t}{2}}\left(C_2\cos\frac{\sqrt{3}}{2}t+C_3\sin\frac{\sqrt{3}}{2}t\right)$,

$y=C_1e^t+e^{-\frac{t}{2}}\left(\frac{C_3\sqrt{3}-C_2}{2}\cos\frac{\sqrt{3}}{2}t-\frac{C_2\sqrt{3}+C_3}{2}\sin\frac{\sqrt{3}}{2}t\right)$,

$z=C_1e^t+e^{-\frac{t}{2}}\left(\frac{-C_3\sqrt{3}-C_2}{2}\cos\frac{\sqrt{3}}{2}t+\frac{C_2\sqrt{3}-C_3}{2}\sin\frac{\sqrt{3}}{2}t\right)$.

3082. $x=C_1e^{-t}+C_2e^{2t}$, $y=C_3e^{-t}+C_2e^{2t}$, $z=-(C_1+C_3)e^{-t}+C_2e^{2t}$.

3083. $y=C_1+C_2e^{2x}-\frac{1}{4}(x^2+x)$, $z=C_2e^{2x}-C_1+\frac{1}{4}(x^2-x-1)$.

3084. $y=C_1+C_2x+2\sin x$, $z=-2C_1-C_2(2x+1)-3\sin x-2\cos x$.

3085. $y=(C_2-2C_1-2C_2x)e^{-x}-6x+14$, $z=(C_1+C_2x)e^{-x}+5x-9$;

$C_1=9$, $C_2=4$,

$y=14(1-e^{-x})-2x(3+4e^{-x})$, $z=-9(1-e^{-x})+x(5+4e^{-x})$.

3086. $x=10e^{2t}-8e^{3t}-e^t+6t-1$; $y=-20e^{2t}+8e^{3t}+3e^t+12t+10$.

3087. $y=\frac{2C_1}{(C_2-x)^2}$, $z=\frac{C_1}{C_2-x}$. **3088*.** a) $\frac{(x^2+y^2)y}{x}=C_1$, $\frac{z}{y}=C_2$; b) $\ln\sqrt{x^2+y^2}=\arctan\frac{y}{x}+C_1$, $\frac{z}{\sqrt{x^2+y^2}}=C_2$. **Hint.** Integrating the homogeneous equation $\frac{dx}{x-y}=\frac{dx}{x+y}$, we find the *first integral* $\ln\sqrt{x^2+p^2}=$ $=\arctan\frac{y}{x}+C_1$. Then, using the properties of derivative proportions, we have $\frac{dz}{z}=\frac{x\,dx}{x(x-y)}=\frac{y\,dy}{y(x+y)}=\frac{x\,dx+y\,dy}{x^2+y^2}$ Whence $\ln z=\frac{1}{2}\ln(x^2+y^2)+\ln C_2$ and, hence, $\frac{z}{\sqrt{x^2+y^2}}=C_2$; c) $x+y+z=0$, $x^2+y^2+z^2=6$. **Hint.** Applying the properties of derivative proportions, we have $\frac{dx}{y-z}=\frac{dy}{z-x}=\frac{dz}{x-y}=\frac{dx+dy+dz}{0}$; whence $dx+dy+dz=0$ and, consequently, $x+y+z=C_1$. Similarly, $\frac{x\,dx}{x(y-z)}=$ $=\frac{y\,dy}{y(z-x)}=\frac{z\,dz}{z(x-y)}=\frac{x\,dx+y\,dy+z\,dz}{0}$; $x\,dx+y\,dy+z\,dz=0$ and x^2+y^2+ $+z^2=C_2$. Thus, the integral curves are the circles $x+y+z=C_1$, $x^2+y^2+z^2=C_2$. From the initial conditions, $x=1$, $y=1$, $z=-2$, we will have $C_1=0$, $C_2=6$.

3089. $y=C_1x^2+\frac{C_2}{x}-\frac{x^2}{18}(3\ln^2x-2\ln x)$,

$z=1-2C_1x+\frac{C_2}{x^2}+\frac{x}{9}(3\ln^2x+\ln x-1)$.

3090. $y=C_1e^{x\sqrt{2}}+C_2e^{-x\sqrt{2}}+C_3\cos x+C_4\sin x+e^x-2x$,

$z=-C_1e^{x\sqrt{2}}-C_2e^{-x\sqrt{2}}-\frac{C_3}{4}\cos x-\frac{C_4}{4}\sin x-\frac{1}{2}e^x+x$.

3091. $x=\frac{v_0m\cos\alpha}{k}\left(1-e^{-\frac{k}{m}t}\right)$, $y=\frac{m}{k^2}(kv_0\sin\alpha+mg)\left(1-e^{-\frac{k}{m}t}\right)-\frac{mgt}{k}$.

Solution. $m\frac{dv_x}{dt}=-kv_x$; $m\frac{dv_y}{dt}=-kv_y-mg$ for the initial conditions: when

$t=0$, $x_0=y_0=0$, $v_{x_0}=v_0\cos\alpha$, $v_{y_0}=v_0\sin\alpha$. Integrating, we obtain $v_x=$ $=v_0\cos\alpha e^{-\frac{k}{m}t}$, $kv_y+mg=(kv_0\sin\alpha+mg)e^{-\frac{k}{m}t}$. **3092.** $x=a\cos\frac{k}{\sqrt{m}}t$, $y=$ $=\frac{v_0\sqrt{m}}{k}\sin\frac{k}{\sqrt{m}}t$, $\frac{x^2}{a^2}+\frac{k^2y^2}{mv_0^2}=1$. **Hint.** The differential equations of motion: $m\frac{d^2x}{dt^2}=-k^2x$, $m\frac{d^2y}{dt^2}=-k^2y$.

3093. $y=-2-2x-x^2$. **3094.** $y=\left(y_0+\frac{1}{4}\right)e^{2(x-1)}-\frac{1}{2}x+\frac{1}{4}$.

3095. $y=\frac{1}{2}+\frac{1}{4}x+\frac{1}{8}x^2+\frac{1}{16}x^3+\frac{9}{32}x^4+\frac{21}{320}x^5+\ldots$

3096. $y=\frac{1}{3}x^3-\frac{1}{7\cdot 9}x^7+\frac{2}{7\cdot 11\cdot 27}x^{11}-\ldots$

3097. $y=x+\frac{x^2}{1\cdot 2}+\frac{x^3}{2\cdot 3}+\frac{x^4}{3\cdot 4}+\ldots$; the series converges for $-1\leqslant x\leqslant 1$.

3098. $y=x-\frac{x^2}{(1!)^2\cdot 2}+\frac{x^3}{(2!)^2\cdot 3}-\frac{x^4}{(3!)^2\cdot 4}+\ldots$; the series converges for $-\infty<$ $<x<+\infty$. **Hint.** Use the method of undetermined coefficients.

3099. $y=1-\frac{1}{3!}x^3+\frac{1\cdot 4}{6!}x^6-\frac{1\cdot 4\cdot 7}{9!}x^9\ldots$; the series converges for $-\infty<x<+\infty$.

3100. $y=\frac{\sin x}{x}$. **Hint.** Use the method of undetermined coefficients.

3101. $y=1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\frac{x^6}{2^2\cdot 4^2\cdot 6^2}+\ldots$; the series converges for $|x|<\infty$. **Hint.** Use the method of undetermined coefficients. **3102.** $x=a\left(1-\frac{1}{2!}t^2+\right.$ $\left.+\frac{2}{4!}t^4-\frac{9}{6!}t^6+\frac{55}{8!}t^8-\ldots\right)$. **3103.** $u=A\cos\frac{a\pi t}{l}\sin\frac{\pi x}{l}$. **Hint.** Use the conditions: $u(0,t)=0$, $u(l,t)=0$, $u(x,0)=A\sin\frac{\pi x}{l}$, $\frac{\partial u(x,0)}{\partial t}=0$.

3104. $u=\frac{2l}{\pi^2 a}\sum_{n=1}^{\infty}\frac{1}{n^2}(1-\cos n\pi)\sin\frac{n\pi at}{l}\sin\frac{n\pi x}{l}$. **Hint.** Use the conditions: $u(0,t)=0$, $u(l,t)=0$, $u(x,0)=0$, $\frac{\partial u(x,0)}{\partial t}=1$.

3105. $u=\frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\cos\frac{n\pi at}{l}\sin\frac{n\pi x}{l}$. **Hint.** Use the conditions:

$$\frac{\partial u(x,0)}{\partial t}=0,\quad u(0,t)=0,\quad u(l,t)=0,\quad u(x,0)=\begin{cases}\frac{2hx}{l} & \text{for } 0<x\leqslant\frac{l}{2},\\ 2h\left(1-\frac{x}{l}\right) & \text{for } \frac{l}{2}<x<l.\end{cases}$$

3106. $u=\sum_{n=0}^{\infty}A_n\cos\frac{(2n+1)a\pi t}{2l}\sin\frac{(2n+1)\pi x}{2l}$, where the coefficients $A_n=$

$=\frac{2}{l}\int_0^l \frac{x}{l}\sin\frac{(2n+1)\pi x}{2l}\,dx$. **Hint.** Use the conditions

$$u(0,\ t)=0,\quad \frac{\partial u(l,\ t)}{\partial x}=0,\quad u(x,\ 0)\ \frac{x}{l},\quad \frac{\partial u(x,\ 0)}{\partial t}=0.$$

3107. $u=\frac{400}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}(1-\cos n\pi)\sin\frac{n\pi x}{100}\cdot e^{-\frac{a^2n^2\pi^2t}{100^2}}$.

Hint. Use the conditions: $u(0,\ t)=0$, $u(100,\ t)=0$, $u(x,\ 0)=0.01\,x(100-x)$.

Chapter X

3108. a) $\leqslant 1''$; $\leqslant 0.0023\%$; b) $\leqslant 1$ mm; $\leqslant 0.26\%$; c) $\leqslant 1$ gm; $\leqslant 0.0016\%$. **3109.** a) $\leqslant 0.05$; $\leqslant 0.021\%$; b) $\leqslant 0.0005$; $\leqslant 1.45\%$; c) $\leqslant 0.005$; $\leqslant 0.16\%$. **3110.** a) two decimals; $48\cdot 10^3$ or $49\cdot 10^3$, since the number lies between 47,877 and 48,845; b) two decimals; 15; c) one decimal; $6\cdot 10^2$. For practical purposes there is sense in writing the result in the form $(5.9\pm 0.1)\cdot 10^2$. **3111.** a) 29.5; b) $1.6\cdot 10^2$; c) 43.2. **3112.** a) 84.2; b) 18.5 or 18.47 ± 0.01; c) the result of subtraction does not have any correct decimals, since the difference is equal to one hundredth with a possible absolute error of one hundredth. **3113*.** 1.8 ± 0.3 cm^2. **Hint.** Use the formula for increase in area of a square. **3114.** a) 30.0 ± 0.2; b) 43.7 ± 0.1; c) 0.3 ± 0.1. **3115.** 19.9 ± 0.1 m^2. **3116.** a) 1.1295 ± 0.0002; b) 0.120 ± 0.006; c) the quotient may vary between 48 and 62. Hence, not a single decimal place in the quotient may be considered certain. **3117.** 0.480. The last digit may vary by unity. **3118.** a) 0.1729; b) $277\cdot 10^3$; c) 2. **3119.** $(2.05\pm 0.01)\cdot 10^3$ cm^2. **3120.** a) 1.648; b) 4.025 ± 0.001; c) 9.006 ± 0.003. **3121.** $4.01\cdot 10^3$ cm^2. Absolute error, 6 5 cm^2. Relative error, 0.16%. **3122.** The side is equal to 13.8 ± 0.2 cm; $\sin\alpha=0.44\pm 0.01$, $\alpha=26°15'\pm\pm 35'$. **3123.** $2\ 7\pm 0.1$. **3124.** 0.27 ampere **3125.** The length of the pendulum should be measured to within 0.3 cm; take the numbers π and g to three decimals (on the principle of equal effects). **3126.** Measure the radii and the generatrix with relative error 1/300. Take the number π to three decimal places (on the principle of equal effects). **3127.** Measure the quantity l to within 0.2%, and s to within 0.7% (on the principle of equal effects).
3128.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	3	7	−2	−6	14	−23
2	10	5	−8	8	−9	
3	15	−3	0	−1		
4	12	−3	−1			
5	9	−4				
6	5					

3129.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	−4	−12	32	48
3	−16	20	80	48
5	4	100	128	48
7	104	228	176	
9	332	404		
11	736			

3130.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	0	−4	−42	−24	24
1	−4	−46	−66	0	24
2	−50	−112	−66	24	24
3	−162	−178	−42	48	24
4	−340	−220	6	72	24
5	−560	−214	78	96	24
6	−774	−136	174	120	24
7	−910	38	294	144	
8	−872	332	438		
9	−540	770			
10	230				

Hint. Compute the first five values of y and, after obtaining $\Delta^4 y_0 = 24$, repeat the number 24 throughout the column of fourth differences. After this the remaining part of the table is filled in by the operation of addition (moving from right to left).

3131. a) 0.211; 0.389; 0.490; 0.660; b) 0.229; 0.399; 0.491; 0.664. **3132.** 0 1822; 0.1993; 0.2165; 0.2334; 0.2503. **3133.** $1+x+x^2+x^3$. **3134.** $y=\frac{1}{96}x^4-\frac{11}{48}x^3+\frac{65}{24}x^2-\frac{85}{12}x+8$; $y\approx 22$ for $x=5.5$; $y=20$ for $x\approx 5.2$. **Hint.** When computing x for $y=20$ take $y_0=11$. **3135.** The interpolating polynomial is $y=x^2-10x+1$; $y=1$ when $x=0$. **3136.** 158 kgf (approximately). **3137.** a) $y(0.5)=-1$, $y(2)=11$; b) $y(0.5)=-\frac{15}{16}$, $y(2)=-3$. **3138.** —1.325 **3139.** 1.01. **3140.** —1.86; —0.25; 2.11. **3141.** 2.09. **3142.** 2 45 and 0 019. **3143.** 0.31 and 4 **3144.** 2.506. **3145.** 0.02. **3146.** 0 24. **3147.** 1 27 **3148.** —1.88; 0 35; 1 53 **3149.** 1.84. **3150.** 1.31 and —0.67. **3151.** 7.13. **3152.** 0.165. **3153.** 1.73 and 0. **3154.** 1.72. **3155.** 1 38 **3156.** $x=0.83$; $y=0\ 56$; $x=-0.83$; $y=-0.56$ **3157.** $x=1.67$; $y=1\ 22$. **3158.** 4 493. **3159.** ± 1 1997 **3160.** By the trapezoidal formula, 11.625; by Simpson's formula, 11 417. **3161.** —0 995; —1; 0.005; 0.5%; $\Delta=0.005$. **3162.** 0.3068; $\Delta=1.3\cdot 10^{-5}$. **3163.** 0 69 **3164.** 0.79. **3165.** 0.84. **3166.** 0.28. **3167.** 0.10. **3168.** 1 61. **3169.** 1.85 **3170.** 0.09. **3171.** 0.67. **3172.** 0.75. **3173.** 0.79. **3174.** 4.93. **3175.** 1 29. **Hint.** Make use of the parametric equation of the ellipse $x=\cos t$, $y=0.6222\sin t$ and transform the formula of the arc length to the form $\int\limits_0^{\frac{\pi}{2}}\sqrt{1-\varepsilon^2\cos^2 t}\cdot dt$, where ε is the eccentricity of the ellipse. **3176.** $y_1(x)=\frac{x^3}{3}$, $y_2(x)=\frac{x^3}{3}+\frac{x^7}{63}$, $y_3(x)=\frac{x^3}{3}+\frac{x^7}{63}+\frac{2x^{11}}{2079}+\frac{x^{15}}{59535}$. **3177.** $y_1(x)=\frac{x^2}{2}-x+1$, $y_2(x)=\frac{x^3}{6}+\frac{3x^2}{2}-x+1$, $y_3(x)=\frac{x^4}{12}-\frac{x^3}{6}+\frac{3x^2}{2}-x+1$; $z_1(x)=3x-2$, $z_2(x)=\frac{x^3}{6}-2x^2+3x-2$, $z_3(x)=\frac{7x^3}{6}-2x^2+3x-2$. **3178.** $y_1(x)=x$, $y_2(x)=x-\frac{x^3}{6}$, $y_3(x)=x-\frac{x^3}{6}+\frac{x^5}{120}$. **3179.** $y(1)=3.36$. **3180.** $y(2)=0.80$. **3181.** $y(1)=3.72$; $z(1)=2.72$ **3182.** $y=1.80$. **3183.** 3.15. **3184.** 0.14. **3185.** $y(0.5)=3\ 15$; $z(0\ 5)=-3\ 15$. **3186.** $y(0.5)=0.55$; $z(0\ 5)=-0.18$. **3187.** 1.16. **3188.** 0 87. **3189.** $x(\pi)=3.58$; $x'(\pi)=0.79$. **3190.** $429+1739\cos x-1037\sin x-6321\cos 2x+1263\sin 2x-1242\cos 3x-33\sin 3x$. **3191.** $6\ 49-1\ 96\cos x+2.14\sin x-1.68\cos 2x+0.53\sin 2x-1.13\cos 3x+0.04\sin 3x$. **3192.** $0.960+0.851\cos x+0.915\sin x+0.542\cos 2x+0.620\sin 2x+0.271\cos 3x+0.100\sin 3x$. **3193.** a) $0\ 608\sin x+0.076\sin 2x+0.022\sin 3x$; b) $0.338+0.414\cos x+0.111\cos 2x+0.056\cos 3x$.

APPENDIX

I. Greek Alphabet

Alpha—Αα	Iota—Ιι	Rho—Ρρ
Beta—Ββ	Kappa—Κκ	Sigma—Σσ
Gamma—Γγ	Lambda—Λλ	Tau—Ττ
Delta—Δδ	Mu—Μμ	Upsilon—Υυ
Epsilon—Εε	Nu—Νν	Phi—Φφ
Zeta—Ζζ	Xi—Ξξ	Chi—Χχ
Eta—Ηη	Omicron—Οο	Psi—Ψψ
Theta—Θθ	Pi—Ππ	Omega—Ωω

II. Some Constants

Quantity	x	$\log x$	Quantity	x	$\log x$
π	3.14159	0.49715	$\frac{1}{e}$	0.36788	$\bar{1}.56571$
2π	6.28318	0.79818	e^2	7.38906	0.86859
$\frac{\pi}{2}$	1.57080	0.19612	$\sqrt{e}$	1.64872	0.21715
$\frac{\pi}{4}$	0.78540	$\bar{1}.89509$	$\sqrt[3]{e}$	1.39561	0.14476
$\frac{1}{\pi}$	0.31831	$\bar{1}.50285$	$M = \log e$	0.43429	$\bar{1}.63778$
π^2	9.86960	0.99430	$\frac{1}{M} = \ln 10$	2.30258	0.36222
$\sqrt{\pi}$	1.77245	0.24857	1 radian	57°17′45″	
$\sqrt[3]{\pi}$	1.46459	0.16572	arc 1°	0.01745	$\bar{2}.24188$
e	2.71828	0.43429	g	9.81	0.99167

III. Inverse Quantities, Powers, Roots, Logarithms

x	$\frac{1}{x}$	x^2	x^3	$\sqrt{x}$	$\sqrt{10x}$	$\sqrt[3]{x}$	$\sqrt[3]{10x}$	$\sqrt[3]{100x}$	log x (mantissas)	ln x
1.0	1.000	1.000	1.000	1.000	3.162	1.000	2.154	4.642	0000	0.0000
1.1	0.909	1.210	1.331	1.049	3.317	1.032	2.224	4.791	0414	0.0953
1.2	0.833	1.440	1.728	1.095	3.464	1.063	2.289	4.932	0792	0.1823
1.3	0.769	1.690	2.197	1.140	3.606	1.091	2.351	5.066	1139	0.2624
1.4	0.714	1.960	2.744	1.183	3.742	1.119	2.410	5.192	1461	0.3365
1.5	0.667	2.250	3.375	1.225	3.873	1.145	2.466	5.313	1761	0.4055
1.6	0.625	2.560	4.096	1.265	4.000	1.170	2.520	5.429	2041	0.4700
1.7	0.588	2.890	4.913	1.304	4.123	1.193	2.571	5.540	2304	0.5306
1.8	0.556	3.240	5.832	1.342	4.243	1.216	2.621	5.646	2553	0.5878
1.9	0.526	3.610	6.859	1.378	4.359	1.239	2.668	5.749	2788	0.6419
2.0	0.500	4.000	8.000	1.414	4.472	1.260	2.714	5.848	3010	0.6931
2.1	0.476	4.410	9.261	1.449	4.583	1.281	2.759	5.944	3222	0.7419
2.2	0.454	4.840	10.65	1.483	4.690	1.301	2.802	6.037	3424	0.7885
2.3	0.435	5.290	12.17	1.517	4.796	1.320	2.844	6.127	3617	0.8329
2.4	0.417	5.760	13.82	1.549	4.899	1.339	2.884	6.214	3802	0.8755
2.5	0.400	6.250	15.62	1.581	5.000	1.357	2.924	6.300	3979	0.9163
2.6	0.385	6.760	17.58	1.612	5.099	1.375	2.962	6.383	4150	0.9555
2.7	0.370	7.290	19.68	1.643	5.196	1.392	3.000	6.463	4314	0.9933
2.8	0.357	7.840	21.95	1.673	5.292	1.409	3.037	6.542	4472	1.0296
2.9	0.345	8.410	24.39	1.703	5.385	1.426	3.072	6.619	4624	1.0647
3.0	0.333	9.000	27.00	1.732	5.477	1.442	3.107	6.694	4771	1.0986
3.1	0.323	9.610	29.79	1.761	5.568	1.458	3.141	6.768	4914	1.1314
3.2	0.312	10.24	32.77	1.789	5.657	1.474	3.175	6.840	5051	1.1632
3.3	0.303	10.89	35.94	1.817	5.745	1.489	3.208	6.910	5185	1.1939
3.4	0.294	11.56	39.30	1.844	5.831	1.504	3.240	6.980	5315	1.2238
3.5	0.286	12.25	42.88	1.871	5.916	1.518	3.271	7.047	5441	1.2528
3.6	0.278	12.96	46.66	1.897	6.000	1.533	3.302	7.114	5563	1.2809
3.7	0.270	13.69	50.65	1.924	6.083	1.547	3.332	7.179	5682	1.3083
3.8	0.263	14.44	54.87	1.949	6.164	1.560	3.362	7.243	5798	1.3350
3.9	0.256	15.21	59.32	1.975	6.245	1.574	3.391	7.306	5911	1.3610
4.0	0.250	16.00	64.00	2.000	6.325	1.587	3.420	7.368	6021	1.3863
4.1	0.244	16.81	68.92	2.025	6.403	1.601	3.448	7.429	6128	1.4110
4.2	0.238	17.64	74.09	2.049	6.481	1.613	3.476	7.489	6232	1.4351
4.3	0.233	18.49	79.51	2.074	6.557	1.626	3.503	7.548	6335	1.4586
4.4	0.227	19.36	85.18	2.098	6.633	1.639	3.530	7.606	6435	1.4816
4.5	0.222	20.25	91.12	2.121	6.708	1.651	3.557	7.663	6532	1.5041
4.6	0.217	21.16	97.34	2.145	6.782	1.663	3.583	7.719	6628	1.5261
4.7	0.213	22.09	103.8	2.168	6.856	1.675	3.609	7.775	6721	1.5476
4.8	0.208	23.04	110.6	2.191	6.928	1.687	3.634	7.830	6812	1.5686
4.9	0.204	24.01	117.6	2.214	7.000	1.698	3.659	7.884	6902	1.5892
5.0	0.200	25.00	125.0	2.236	7.071	1.710	3.684	7.937	6990	1.6094
5.1	0.196	26.01	132.7	2.258	7.141	1.721	3.708	7.990	7076	1.6292
5.2	0.192	27.04	140.6	2.280	7.211	1.732	3.733	8.041	7160	1.6487
5.3	0.189	28.09	148.9	2.302	7.280	1.744	3.756	8.093	7243	1.6677
5.4	0.185	29.16	157.5	2.324	7.348	1.754	3.780	8.143	7324	1.6864

Continued

x	$\frac{1}{x}$	x^2	x^3	$\sqrt{x}$	$\sqrt{10x}$	$\sqrt[3]{x}$	$\sqrt[3]{10x}$	$\sqrt[3]{100x}$	log x (mantissas)	ln x
5.5	0 182	30 25	166 4	2.345	7.416	1.765	3.803	8.193	7404	1.7047
5 6	0.179	31 36	175.6	2 366	7 483	1 776	3.826	8.243	7482	1 7228
5.7	0.175	32 49	185 2	2 387	7 550	1.786	3.849	8.291	7559	1.7405
5.8	0.172	33.64	195.1	2 408	7 616	1.797	3.871	8.340	7634	1.7579
5.9	0 169	34 81	205.4	2 429	7.681	1.807	3 893	8 387	7709	1.7750
6.0	0.167	36.00	216.0	2.449	7 746	1.817	3.915	8.434	7782	1.7918
6.1	0.164	37 21	227.0	2.470	7.810	1 827	3.936	8 481	7853	1.8083
6.2	0 161	38.44	238 3	2 490	7 874	1.837	3.958	8 527	7924	1.8245
6.3	0 159	39.69	250 0	2.510	7.937	1.847	3.979	8 573	7993	1.8405
6 4	0.156	40 96	262.1	2.530	8.000	1.857	4.000	8 618	8062	1.8563
6.5	0.154	42.25	274 6	2.550	8.062	1.866	4 021	8.662	8129	1.8718
6 6	0.151	43.56	287 5	2 569	8.124	1 876	4.041	8.707	8195	1.8871
6 7	0.149	44 89	300 8	2 588	8.185	1 885	4 062	8 750	8261	1 9021
6 8	0.147	46 24	314 4	2 608	8.246	1.895	4.082	8 794	8325	1 9169
6 9	0 145	47 61	328 5	2.627	8 307	1.904	4.102	8.837	8388	1 9315
7.0	0.143	49.00	343 0	2.646	8.367	1.913	4.121	8 879	8451	1 9459
7.1	0.141	50.41	357.9	2 665	8 426	1.922	4.141	8 921	8513	1.9601
7.2	0.139	51.84	373.2	2.683	8.485	1.931	4.160	8.963	8573	1.9741
7.3	0 137	53.29	389.0	2.702	8.544	1.940	4.179	9.004	8633	1.9879
7.4	0.135	54 76	405.2	2.720	8 602	1.949	4.198	9.045	8692	2.0015
7.5	0.133	56.25	421.9	2.739	8.660	1.957	4.217	9.086	8751	2.0149
7.6	0.132	57.76	439 0	2 757	8.718	1.966	4.236	9.126	8808	2.0281
7.7	0 130	59 29	456 5	2 775	8.775	1.975	4.254	9.166	8865	2.0412
7.8	0 128	60.84	474 6	2.793	8.832	1.983	4 273	9.205	8921	2.0541
7.9	0 127	62.41	493.0	2.811	8.888	1 992	4 291	9.244	8976	2.0669
8.0	0 125	64.00	512 0	2.828	8.944	2.000	4.309	9.283	9031	2.0794
8.1	0 123	65.61	531.4	2 846	9.000	2 008	4.327	9.322	9085	2.0919
8.2	0.122	67.24	551.4	2.864	9 055	2 017	4.344	9 360	9138	2.1041
8.3	0.120	68.89	571.8	2.881	9.110	2 025	4.362	9.398	9191	2.1163
8 4	0 119	70.56	592.7	2.898	9.165	2.033	4 380	9.435	9243	2.1282
8.5	0.118	72.25	614.1	2.915	9.220	2.041	4.397	9.473	9294	2.1401
8.6	0.116	73.96	636.1	2 933	9.274	2 049	4.414	9 510	9345	2.1518
8.7	0.115	75.69	658.5	2.950	9.327	2.057	4.431	9.546	9395	2.1633
8.8	0.114	77.44	681.5	2.966	9.381	2.065	4.448	9.583	9445	2.1748
8 9	0 112	79 21	705.0	2 983	9.434	2 072	4.465	9 619	9494	2.1861
9.0	0.111	81.00	729.0	3.000	9.487	2.080	4.481	9.655	9542	2 1972
9.1	0.110	82.81	753.6	3.017	9.539	2.088	4.498	9.691	9590	2.2083
9.2	0.109	84.64	778.7	3.033	9.592	2.095	4.514	9.726	9638	2.2192
9.3	0.108	86.49	804.4	3.050	9.644	2.103	4.531	9.761	9685	2.2300
9.4	0.106	88.36	830 6	3 066	9.695	2.110	4.547	9.796	9731	2 2407
9.5	0.105	90.25	857.4	3.082	9.747	2.118	4.563	9.830	9777	2.2513
9.6	0.104	92.16	884.7	3.098	9.798	2.125	4.579	9 865	9823	2.2618
9.7	0.103	94.09	912 7	3.114	9.849	2.133	4.595	9 899	9868	2 2721
9.8	0.102	96.04	941.2	3.130	9.899	2.140	4.610	9.933	9912	2.2824
9.9	0.101	98.01	970.3	3.146	9.950	2.147	4.626	9.967	9956	2.2925
10.0	0.100	100.00	1000.0	3.162	10.000	2.154	4.642	10.000	0000	2.3026

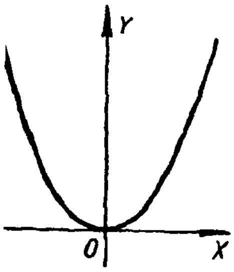
IV. Trigonometric Functions

$x°$	x (radians)	sin x	tan x	cot x	cos x		
0	0.0000	0.0000	0.0000	∞	1.0000	1.5708	90
1	0.0175	0.0175	0.0175	57 29	0.9998	1.5533	89
2	0.0349	0.0349	0.0349	28.64	0.9994	1 5359	88
3	0.0524	0.0523	0.0524	19.08	0.9986	1.5184	87
4	0.0698	0.0698	0.0699	14 30	0.9976	1.5010	86
5	0.0873	0.0872	0.0875	11.43	0.9962	1.4835	85
6	0.1047	0.1045	0.1051	9.514	0.9945	1.4661	84
7	0.1222	0.1219	0.1228	8.144	0.9925	1.4486	83
8	0.1396	0.1392	0.1405	7.115	0.9903	1 4312	82
9	0.1571	0.1564	0 1584	6 314	0.9877	1 4137	81
10	0.1745	0.1736	0.1763	5.671	0.9848	1 3963	80
11	0.1920	0.1908	0.1944	5.145	0.9816	1.3788	79
12	0.2094	0.2079	0.2126	4.705	0 9781	1 3614	78
13	0.2269	0.2250	0.2309	4.331	0.9744	1.3439	77
14	0.2443	0.2419	0.2493	4.011	0.9703	1.3265	76
15	0.2618	0.2588	0.2679	3.732	0.9659	1.3090	75
16	0.2793	0.2756	0.2867	3.487	0.9613	1.2915	74
17	0.2967	0.2924	0.3057	3.271	0.9563	1.2741	73
18	0.3142	0.3090	0.3249	3.078	0.9511	1 2566	72
19	0.3316	0.3256	0.3443	2.904	0.9455	1 2392	71
20	0.3491	0.3420	0.3640	2.747	0 9397	1 2217	70
21	0.3665	0.3584	0.3839	2.605	0.9336	1.2043	69
22	0.3840	0.3746	0.4040	2.475	0.9272	1.1868	68
23	0.4014	0.3907	0.4245	2.356	0.9205	1.1694	67
24	0.4189	0.4067	0.4452	2.246	0.9135	1.1519	66
25	0.4363	0.4226	0.4663	2.145	0.9063	1 1345	65
26	0.4538	0 4384	0 4877	2.050	0.8988	1.1170	64
27	0.4712	0.4540	0.5095	1.963	0.8910	1.0996	63
28	0.4887	0.4695	0.5317	1.881	0.8829	1.0821	62
29	0.5061	0 4848	0.5543	1.804	0.8746	1.0647	61
30	0.5236	0.5000	0.5774	1 732	0.8660	1.0472	60
31	0.5411	0.5150	0.6009	1.6643	0.8572	1.0297	59
32	0.5585	0.5299	0.6249	1.6003	0.8480	1.0123	58
33	0.5760	0.5446	0 6494	1.5399	0.8387	0.9948	57
34	0.5934	0.5592	0.6745	1.4826	0 8290	0.9774	56
35	0.6109	0.5736	0.7002	1.4281	0.8192	0.9599	55
36	0.6283	0.5878	0.7265	1.3764	0.8090	0.9425	54
37	0.6458	0.6018	0.7536	1.3270	0.7986	0.9250	53
38	0.6632	0.6157	0.7813	1.2799	0.7880	0.9076	52
39	0.6807	0.6293	0.8098	1.2349	0.7771	0.8901	51
40	0.6981	0.6428	0.8391	1.1918	0.7660	0.8727	50
41	0.7156	0.6561	0.8693	1.1504	0.7547	0.8552	49
42	0.7330	0.6691	0.9004	1.1106	0.7431	0.8378	48
43	0.7505	0.6820	0 9325	1.0724	0.7314	0.8203	47
44	0.7679	0.6947	0.9657	1.0355	0.7193	0.8029	46
45	0.7854	0.7071	1.0000	1.0000	0.7071	0.7854	45
		cos x	cot x	tan x	sin x	x (radians)	$x°$

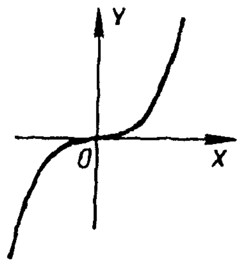
V. Exponential, Hyperbolic and Trigonometric Functions

x	e^x	e^{-x}	sinh x	cosh x	tanh x	sin x	cos x
0.0	1.0000	1.0000	0.0000	1.0000	0.0000	0.0000	1.0000
0.1	1.1052	0.9048	0.1002	1.0050	0.0997	0.0998	0.9950
0.2	1.2214	0.8187	0.2013	1.0201	0.1974	0.1987	0.9801
0.3	1.3499	0.7408	0.3045	1.0453	0.2913	0.2955	0.9553
0.4	1.4918	0.6703	0.4108	1.0811	0.3799	0.3894	0.9211
0.5	1.6487	0.6065	0.5211	1.1276	0.4621	0.4794	0.8776
0.6	1.8221	0.5488	0.6367	1.1855	0.5370	0.5646	0.8253
0.7	2.0138	0.4966	0.7586	1.2552	0.6044	0.6442	0.7648
0.8	2.2255	0.4493	0.8881	1.3374	0.6640	0.7174	0.6967
0.9	2.4596	0.4066	1.0265	1.4331	0.7163	0.7833	0.6216
1.0	2.7183	0.3679	1.1752	1.5431	0.7616	0.8415	0.5403
1.1	3.0042	0.3329	1.3356	1.6685	0.8005	0.8912	0.4536
1.2	3.3201	0.3012	1.5095	1.8107	0.8337	0.9320	0.3624
1.3	3.66[illegible]3	0.2725	1.6981	1.9709	0.8617	0.9636	0.2675
1.4	4.0552	0.2466	1.9043	2.1509	0.8854	0.9854	0.1700
1.5	4.4817	0.2231	2.1293	2.3524	0.9051	0.9975	0.0707
1.6	4.9530	0.2019	2.3756	2.5775	0.9217	0.9996	-0.0292
1.7	5.4739	0.1827	2.6456	2.8283	0.9354	0.9917	-0.1288
1.8	6.0496	0.1653	2.9422	3.1075	0.9468	0.9738	-0.2272
1.9	6.6859	0.1496	3.2682	3.4177	0.9562	0.9463	-0.3233
2.0	7.3891	0.1353	3.6269	3.7622	0.9640	0.9093	-0.4161
2.1	8.1662	0.1225	4.0219	4.1443	0.9704	0.8632	-0.5048
2.2	9.0250	0.1108	4.4571	4.5679	0.9757	0.8085	-0.5885
2.3	9.9742	0.1003	4.9370	5.0372	0.9801	0.7457	-0.6663
2.4	11.0232	0.0907	5.4662	5.5569	0.9837	0.6755	-0.7374
2.5	12.1825	0.0821	6.0502	6.1323	0.9866	0.5985	-0.8011
2.6	13.4637	0.0743	6.6947	6.7690	0.9890	0.5155	-0.8569
2.7	14.8797	0.0672	7.4063	7.4735	0.9910	0.4274	-0.9041
2.8	16.4446	0.0608	8.1919	8.2527	0.9926	0.3350	-0.9422
2.9	18.1741	0.0550	9.0596	9.1146	0.9940	0.2392	-0.9710
3.0	20.0855	0.0498	10.0179	10.0677	0.9950	0.1411	-0.9900
3.1	22.1979	0.0450	11.0764	11.1215	0.9959	0.0416	-0.9991
3.2	24.5325	0.0408	12.2459	12.2366	0.9967	-0.0584	-0.9983
3.3	27.1126	0.0369	13.5379	13.5748	0.9973	-0.1577	-0.9875
3.4	29.9641	0.0334	14.9654	14.9987	0.9978	-0.2555	-0.9668
3.5	33.1154	0.0302	16.5426	16.5728	0.9982	-0.3508	-0.9365

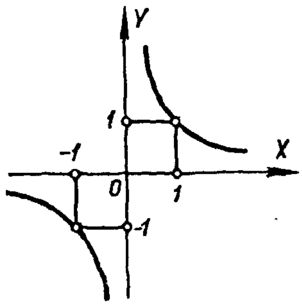
VI. Some Curves (for Reference)



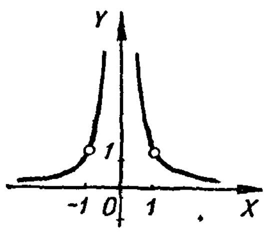
1. Parabola, $y = x^2$.



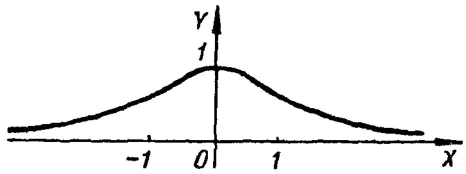
2. Cubic parabola, $y = x^3$.



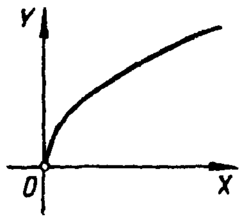
3. Rectangular hyperbola, $y = \frac{1}{x}$.



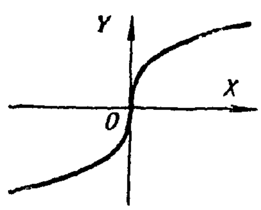
4. Graph of a fractional function, $y = \frac{1}{x^2}$.



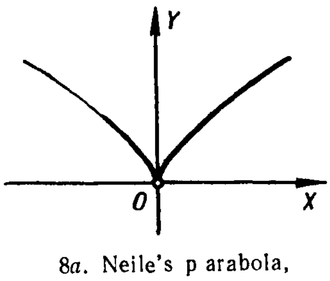
5. The witch of Agnesi, $y = \frac{1}{1+x^2}$.



6. Parabola (upper branch), $y = \sqrt{x}$.

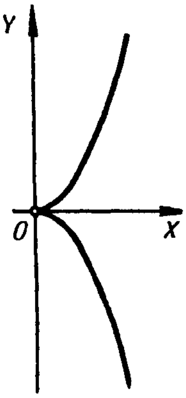


7. Cubic parabola, $y = \sqrt[3]{x}$.



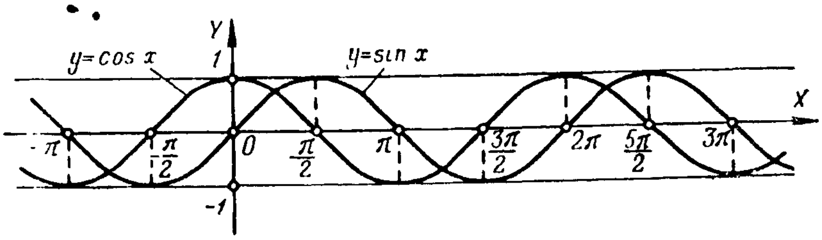
8*a*. Neile's parabola,

$y = x^{\frac{2}{3}}$ or $\begin{cases} x = t^3, \\ y = t^2. \end{cases}$

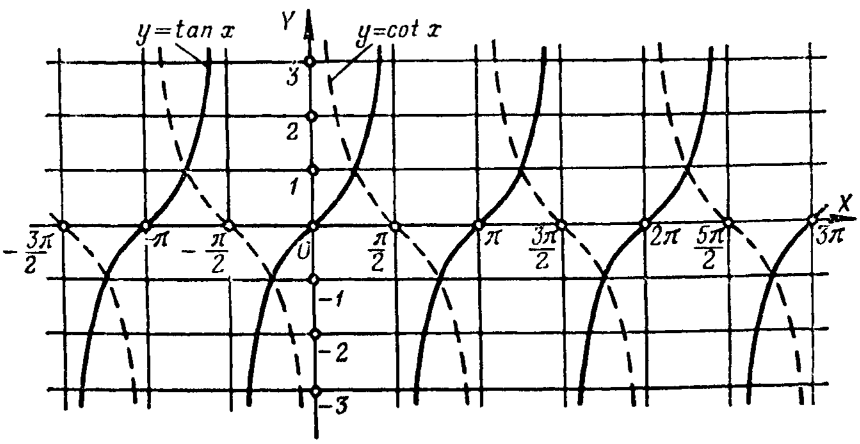


8*b* Semicubical parabola,

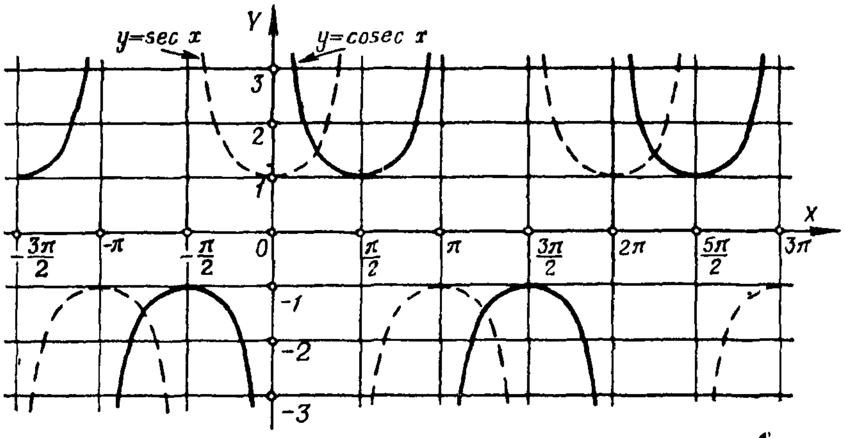
$y^2 = x^3$ or $\begin{cases} x = t^2, \\ y = t^3. \end{cases}$



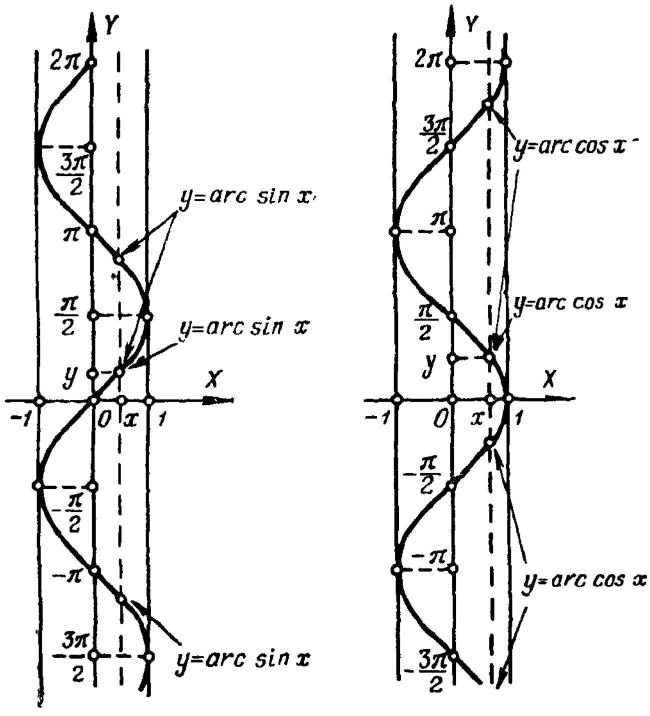
9. Sine curve and cosine curve, $y = \sin x$ and $y = \cos x$.



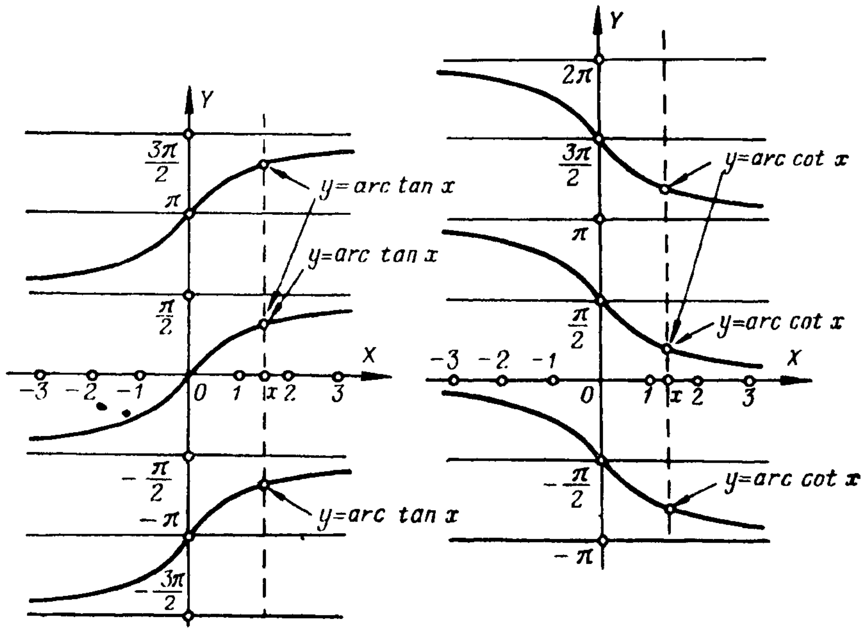
10. Tangent curve and cotangent curve, $y = \tan x$ and $y = \cot x$.



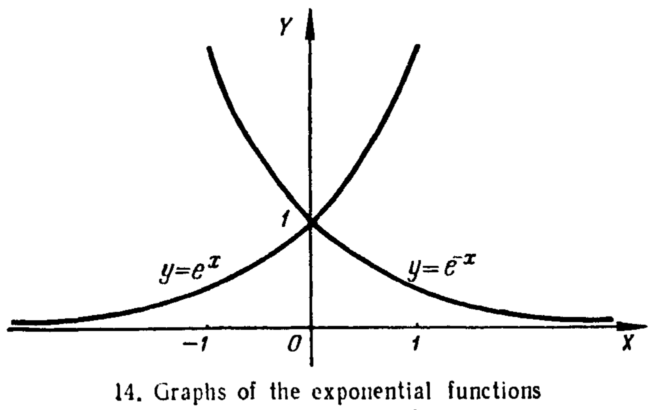
11. Graphs of the functions
$y = \sec x$ and $y = \operatorname{cosec} x$.



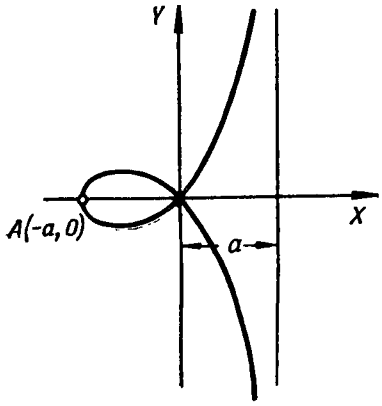
12. Graphs of the inverse trigonometric functions
$y = \arcsin x$ and $y = \arccos x$.



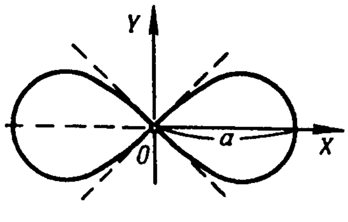
13. Graphs of the inverse trigonometric functions $y = \operatorname{arc} \tan x$ and $y = \operatorname{arc} \cot x$.



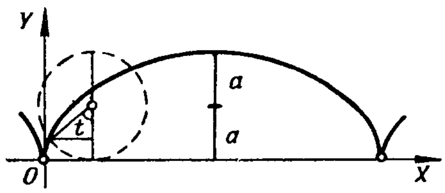
14. Graphs of the exponential functions $y = e^x$ and $y = e^{-x}$.



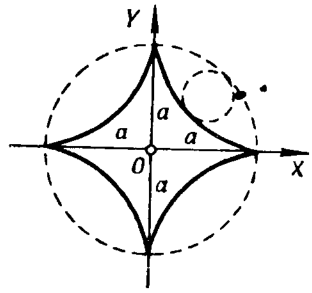
24. Strophoid,
$$y^2 = x^2 \frac{a+x}{a-x}.$$



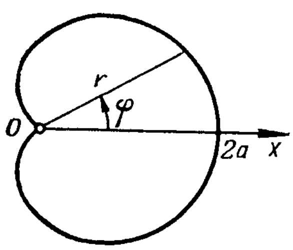
25. Bernoulli's lemniscate,
$$(x^2+y^2)^2 = a^2(x^2-y^2)$$
or $r^2 = a^2 \cos 2\varphi$.



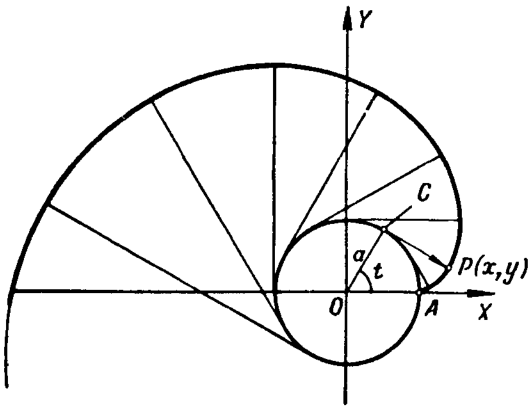
26. Cycloid,
$$\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t). \end{cases}$$



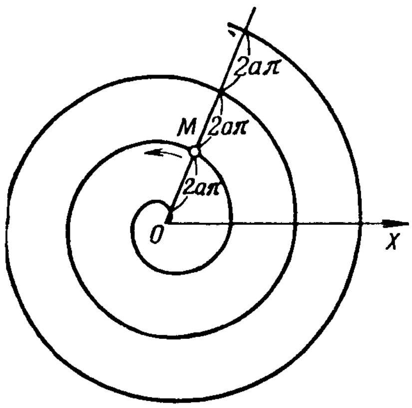
27. Hypocycloid (astroid),
$$\begin{cases} x = a\cos^3 t, \\ y = a\sin^3 t \end{cases}$$
or $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.



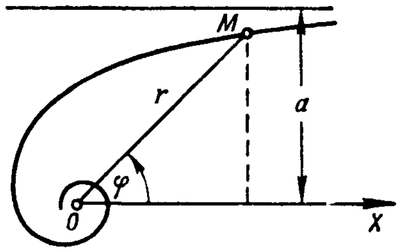
28. Cardioid,
$r = a(1 + \cos\varphi)$.



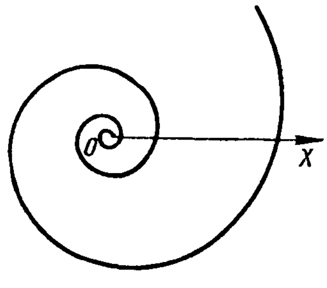
29. Evolvent (involute) of the circle
$$\begin{cases} x = a(\cos t + t\sin t), \\ y = a(\sin t - t\cos t). \end{cases}$$



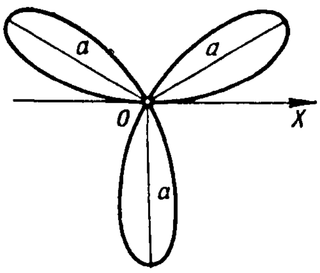
30. Spiral of Archimedes,
$r = a\varphi$.



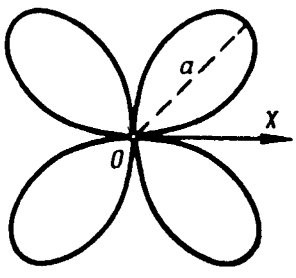
31. Hyperbolic spiral,
$r = \frac{a}{\varphi}$.



32. Logarithmic spiral,
$r = e^{a\varphi}$.



33. Three-leafed rose,
$r = a \sin 3\varphi$.



34. Four-leafed rose,
$r = a \sin 2\varphi$.

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